CSSA
CATHOLIC SECONDARY SCHOOLS ASSOCIATION OF NSW

Centre Number

Student Number

2014
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

Morning Session
Friday 8 August 2014

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided on a SEPARATE sheet
- In Questions 11–14, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and the Student Number at the top of this page

Total marks – 70

Section I Pages 2–5

10 marks
- Attempt Questions 1–10
- Allow 15 minutes for this section

Section II Pages 6–10

60 marks
- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

STUDENT NUMBER/NAME: ....

STANDARD INTEGRALS

$\int x^n\,dx$	$=\frac{1}{n+1}x^{n+1},\quad n\neq -1;\quad x\neq 0,\ \text{if } n<0$
$\int \frac{1}{x}\,dx$	$=\ln x,\quad x>0$
$\int e^{ax}\,dx$	$=\frac{1}{a}e^{ax},\quad a\neq 0$
$\int \cos ax\,dx$	$=\frac{1}{a}\sin ax,\quad a\neq 0$
$\int \sin ax\,dx$	$=-\frac{1}{a}\cos ax,\quad a\neq 0$
$\int \sec^2 ax\,dx$	$=\frac{1}{a}\tan ax,\quad a\neq 0$
$\int \sec ax\,\tan ax\,dx$	$=\frac{1}{a}\sec ax,\quad a\neq 0$
$\int \frac{1}{a^2+x^2}\,dx$	$=\frac{1}{a}\tan^{-1}\frac{x}{a},\quad a\neq 0$
$\int \frac{1}{\sqrt{a^2-x^2}}\,dx$	$=\sin^{-1}\frac{x}{a},\quad a>0,\ -a<x<a$
$\int \frac{1}{\sqrt{x^2-a^2}}\,dx$	$=\ln\left(x+\sqrt{x^2-a^2}\right),\quad x>a>0$
$\int \frac{1}{\sqrt{x^2+a^2}}\,dx$	$=\ln\left(x+\sqrt{x^2+a^2}\right)$

NOTE: $\ln x=\log_e x,\quad x>0$

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 The point A has coordinates $(-6, 4)$ and the point B has coordinates $(5, 1)$.

Find the coordinates of the point which divides AB internally in the ratio 3:4.

(A) $(-39, 13)$

(B) $\left(-\frac{12}{7}, \frac{23}{7}\right)$

(C) $\left(-\frac{9}{7}, \frac{19}{7}\right)$

(D) $\left(\frac{2}{7}, \frac{16}{7}\right)$

2 What is the remainder when the polynomial $P(x) = 5x^3 - 17x^2 - x + 11$ is divided by $x - 2$?

(A) -147

(B) -95

(C) -19

(D) 11

3 A function is represented by the parametric equations

$$x = 2t + 1$$
$$y = t - 2.$$

Find the Cartesian equation of the function.

(A) $x - 2y + 3 = 0$

(B) $x - 2y - 3 = 0$

(C) $x + 2y + 5 = 0$

(D) $x - 2y - 5 = 0$

4 What is the solution to the inequality $\frac{2}{3-x} < 1$?

(A) $x < 1$

(B) $x > 1$

(C) $1 < x < 3$

(D) $x < 1$ or $x > 3$

5 Find $\int \frac{dx}{1+9x^2}$.

(A) $\tan^{-1} 3x + C$

(B) $\frac{1}{3}\tan^{-1} 3x + C$

(C) $\frac{1}{9}\tan^{-1} 3x + C$

(D) $\frac{1}{27}\tan^{-1} 3x + C$

6 A particle is moving in Simple Harmonic Motion and its displacement, x units, at time t seconds is given by the equation

$$x = A\cos(nt) + 2.$$

The period of the motion is 4π seconds and the particle is initially at rest, 12 units to the right of the origin.

Find the values of A and n.

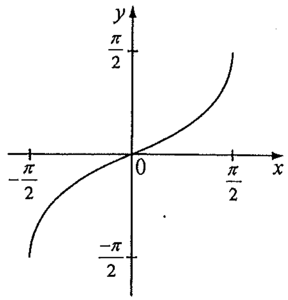
(A) $A = 10,\ n = \frac{1}{2}$

(B) $A = 10,\ n = 2$

(C) $A = 12,\ n = \frac{1}{2}$

(D) $A = 12,\ n = 2$

7



Which of the following could be the equation of the graph shown above?

(A) $y = \sin^{-1}\frac{\pi x}{2}$

(B) $y = \sin^{-1}\frac{2x}{\pi}$

(C) $y = \frac{\pi}{2}\sin^{-1}x$

(D) $y = \frac{2}{\pi}\sin^{-1}x$

8 A Mathematics department consists of 5 female and 5 male teachers.

How many committees of 3 teachers can be chosen which contain at least one female and at least one male?

(A) 100

(B) 120

(C) 200

(D) 2500

9 A particle is acted on by short bursts of radiation. After each short burst the particle moves forward one unit, backward one unit or does not move.

The probability that the particle moves forward one unit after a single short burst is p where $0 < p < \frac{1}{2}$. The probability that the particle moves backward one unit after a single short burst is also p.

What is the probability that after two short bursts the particle is at its starting point?

(A) p^2

(B) $2p^2$

(C) $5p^2 - 4p + 1$

(D) $6p^2 - 4p + 1$

10 What is the term independent of x in the expansion of $(1+2x)^2\left(2x+\frac{1}{x}\right)^{12}$?

(A) $\binom{12}{6}\times 2^6$

(B) $\binom{12}{6}\times 2^6 + \binom{12}{7}\times 2^6$

(C) $\binom{12}{6}\times 2^6 + \binom{12}{7}\times 2^7$

(D) $\binom{12}{6}\times 2^6 + \binom{12}{7}\times 2^9$

Section II

60 marks
Attempt Questions 11– 14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11– 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Use the substitution $u = 1 + \ln x$ to evaluate $\displaystyle\int_1^e \frac{1}{x(1+\ln x)^5}\,dx$. **3**

(b) Find $\displaystyle\int 4\sin^2 x\,dx$. **2**

(c) Newton's law of cooling states that the rate at which an object loses heat is proportional to the difference in temperatures between the object and its surroundings. That is,

$$\frac{dT}{dt} = k(T - S)$$

where T is the temperature of the object in degrees Celsius after t minutes, k is a constant and S is the temperature of the surrounding in degrees Celsius.

(i) Verify that $T = S + Ae^{kt}$ satisfies the equation $\dfrac{dT}{dt} = k(T-S)$. **1**

(ii) A liquid is cooling in a room which has a constant temperature of 20°C. The initial temperature of the liquid is 80°C and it cools to 50°C after 15 minutes. **3**

State the value of S and find the values of A and k.

(iii) Draw a clearly labelled graph showing the temperature of the liquid, T, against time, t. **2**

(d) (i) Find $\dfrac{d}{dx}\cos^{-1}\left(\dfrac{x-10}{10}\right)$. **2**

(ii) Hence, or otherwise, evaluate $\displaystyle\int_5^{10} \frac{1}{\sqrt{20x - x^2}}\,dx$. **2**

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Let α, β and γ be the roots of the equation $x^3 - 3x^2 - 6x - 1 = 0$.

(i) Find $2\alpha + 2\beta + 2\gamma$. **1**

(ii) Find $\alpha^2 + \beta^2 + \gamma^2$. **2**

(b) The function $f(x) = e^{-x} - x$ has a zero near $x = 0.5$. **3**

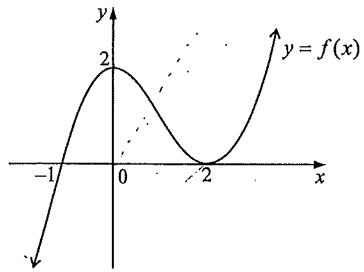
Use one application of Newton's method to obtain another approximation to this zero. Give your answer correct to two decimal places.

(c) A standard die with faces numbered 1 to 6 is tossed 7 times. Find, correct to 3 decimal places,

(i) the probability that exactly 5 sixes are tossed, **2**

(ii) the probability that exactly three sixes are tossed, with the third six occurring on the seventh toss. **2**

(d) The diagram below shows a sketch of the cubic polynomial function $y = f(x)$.



The graph intersects the y-axis at $y = 2$ and the x-axis at $x = -1$ and $x = 2$. The point $(2, 0)$ is a stationary point.

(i) Find an expression for the cubic polynomial $f(x)$. **2**

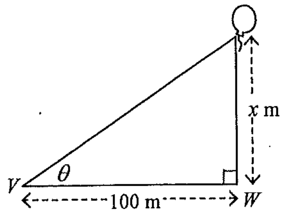
(ii) Let $g(x) = f(x)$ for $x \le 0$. Sketch a clearly labelled graph of the inverse function $g^{-1}(x)$. **1**

(iii) Find the gradient of the inverse function $y = g^{-1}(x)$ at the point where the curve intersects the y-axis. **2**

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Victor is standing at a point V watching a weather balloon being released from a point W which is 100 metres away on the horizontal ground. The weather balloon rises vertically at a constant velocity of 5 ms^{-1}. **3**

Let θ radians be the angle of elevation of the weather balloon at time t seconds and let x metres be the distance the weather balloon has travelled in that time.



Find the rate of change of the angle of elevation of the weather balloon, when $\theta = \dfrac{\pi}{4}$.

(b) Given that $\cos\theta = \dfrac{1}{5}$ and $0 < \theta < \dfrac{\pi}{2}$, find the exact value of $\tan\dfrac{\theta}{2}$. **2**

(c) (i) Use the binomial theorem to obtain an expression for $\dfrac{(1+x)^n}{x}$. **1**

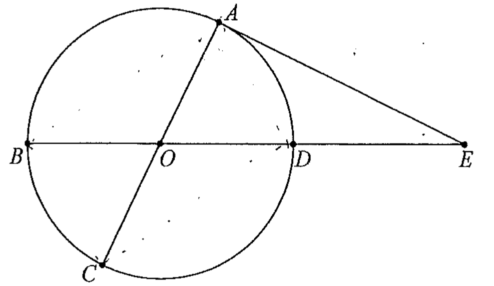
(ii) By differentiating both sides of the identity obtained in part (i), show that for $n \geq 2$, **2**

$$\binom{n}{2} - 2\binom{n}{3} + 3\binom{n}{4} - 4\binom{n}{5} + \cdots + (-1)^n(n-1)\binom{n}{n} = 1.$$

Question 13 continues on page 9

Question 13 (continued)

(d) The diagram below shows the diameters AC and BD of a circle with centre O. The tangent to the circle at A meets BD produced at E. **3**



Prove that $\angle OBC = \angle AED + \angle BAO$.

(e) Prove by mathematical induction that, for all positive integers n, **4**

$$\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}.$$

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

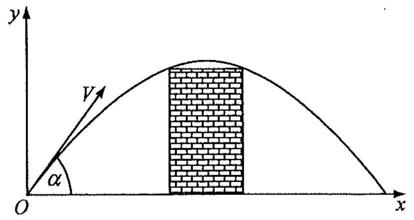
(a) Find the general solution to the equation $\cos 2\theta + \cos\theta + 1 = 0$. **3**

(b) A particle is moving along the x-axis, starting from a position 1 metre to the right of the origin (that is, $x = 1$ when $t = 0$) with an initial velocity of $2\ \text{ms}^{-1}$ and an acceleration given by $\ddot{x} = x$.

(i) Prove that $v = \sqrt{x^2 + 3}$, where v is the velocity of the particle. **2**

(ii) Using part (i) and the standard integrals given, find an expression for x in terms of t. **3**

(c) A projectile is fired from the origin towards the wall of a fort with initial velocity $V\ \text{ms}^{-1}$ at an angle α to the horizontal.



On its ascent, the projectile just clears one edge of the wall and on its descent it just clears the other edge of the wall, as shown in the diagram.

The equations of motion of the projectile are

$$x = Vt\cos\alpha \quad \text{and} \quad y = Vt\sin\alpha - \frac{g}{2}t^2. \text{ (Do NOT prove this.)}$$

(i) Show that the horizontal range R of the projectile is $\dfrac{V^2\sin 2\alpha}{g}$. **2**

(ii) Hence show that the equation of the path of the projectile is **2**

$$y = x\left(1 - \frac{x}{R}\right)\tan\alpha.$$

(iii) The projectile is fired at 45° and the wall of the fort is 10 metres high. Show that the x coordinates of the edges of the wall are the roots of the equation **1**

$$x^2 - Rx + 10R = 0.$$

(iv) If the wall of the fort is 4.5 metres thick, find the value of R. **2**

End of Paper

CATHOLIC SECONDARY SCHOOLS ASSOCIATION OF NSW
2014 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION
MATHEMATICS EXTENSION 1 – MARKING GUIDELINES

Section I
10 marks

Questions 1-10 (1 mark each)

Question 1 (1 mark)
Outcomes Assessed: PE2
Targeted Performance Bands: E2

Solution	Answer	Mark
The point dividing $A(-6,4)$ to $B(5,1)$ in the ratio 3:4 is $\left(\frac{4\times -6+3\times 5}{3+4},\frac{4\times 4+3\times 1}{3+4}\right)$ $=\left(-\frac{9}{7},\frac{19}{7}\right)$	C	1

Question 2 (1 mark)
Outcomes Assessed: PE3
Targeted Performance Bands: E2

Solution	Answer	Mark
The remainder when $P(x)$ is divided by $(x-2)$ is $P(2)=5\times 2^3-17\times 2^2-2+11$ $=-19$	C	1

Question 3 (1 mark)
Outcomes Assessed: PE3
Targeted Performance Bands: E2

Solution	Answer	Mark
$t=y+2$ $\therefore x=2(y+2)+1$ $x-2y-5=0$	D	1

6300-2

Question 4 (1 mark)
Outcomes Assessed: PE3
Targeted Performance Bands: E2-E3

Solution	Answer	Mark
$\frac{2}{3-x}<1$ $2(3-x)<(3-x)^2$ $6-2x<9-6x+x^2$ $x^2-4x+3>0$ $(x-3)(x-1)>0$ $x<1, x>3$ $\therefore x<1$ or $x>3$	D	1

Question 5 (1 mark)
Outcomes Assessed: HE4
Targeted Performance Bands: E2-E3

Solution	Answer	Mark
$\int\frac{dx}{1+9x^2}=\int\frac{dx}{1+(3x)^2}$ $=\frac{1}{3}\tan^{-1}3x+C$	B	1

Question 6 (1 mark)
Outcomes Assessed: HE3
Targeted Performance Bands: E3

Solution	Answer	Mark
The centre of the motion is $x=2$ and an extreme of the motion is $x=12$. This gives the amplitude, $A=10$. Given the period of the motion is 4π, $\frac{2\pi}{n}=4\pi$, $\therefore n=\frac{1}{2}$.	A	1

Question 7 (1 mark)
Outcomes Assessed: HE4
Targeted Performance Bands: E3

Solution	Answer	Mark
$y=\sin^{-1}\left(\frac{2x}{\pi}\right)$ gives an inverse sine graph with a range $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$ and a domain $-1\le\frac{2x}{\pi}\le 1$, i.e., $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$.	B	1

Question 8 (1 mark)
Outcomes Assessed: PE3
Targeted Performance Bands: E3

Solution	Answer	Mark
If the committee of 3 is to contain at least one female and at least one male, it can either have 2 females and 1 male, or 1 female and 2 males. The number of ways of doing this is $\binom{5}{2}\binom{5}{1}+\binom{5}{1}\binom{5}{2}=100$.	A	1

Question 9 (1 mark)
Outcomes Assessed: HE3, HE7
Targeted Performance Bands: E4

Solution	Answer	Mark
To be at its starting point after two short bursts, the particle could have: • moved forward one unit then backward one unit; this would have happened with probability $p\times p=p^2$. • moved backward one unit then forward one unit; this would have also happened with probability $p\times p=p^2$. • not moved for either of the short bursts; this would have happened with probability $(1-2p)\times(1-2p)=1-4p+4p^2$. Therefore, the probability that after two short bursts the particle is at its starting point is $(p^2)+(p^2)+(1-4p+4p^2)=6p^2-4p+1$.	D	1

Question 10 (1 mark)
Outcomes Assessed: HE3
Targeted Performance Bands: E4

Solution	Answer	Mark
$(1+2x)^2\left(2x+\frac{1}{x}\right)^{12}$ $=(1+4x+4x^2)\left((2x)^{12}+\binom{12}{1}(2x)^{11}\left(\frac{1}{x}\right)+\ldots+\binom{12}{6}(2x)^6\left(\frac{1}{x}\right)^6+\binom{12}{7}(2x)^5\left(\frac{1}{x}\right)^7+\ldots.+\left(\frac{1}{x}\right)^{12}\right)$ $=(1+4x+4x^2)\left(2^{12}x^{12}+\binom{12}{1}2^{11}x^{10}+..+\binom{12}{6}2^6+\binom{12}{7}2^5\frac{1}{x^2}+..+\frac{1}{x^{12}}\right)$ The term independent of x is $\binom{12}{6}2^6+4\times\binom{12}{7}2^5=\binom{12}{6}2^6+\binom{12}{7}2^7$	C	1

Section II

60 marks

Question 11 (15 marks)

(a) (3 Marks)

Sample answer:

$u=1+\ln x \Rightarrow \dfrac{du}{dx}=\dfrac{1}{x} \Rightarrow du=\dfrac{dx}{x}$

$x=1 \Rightarrow u=1$

$x=e \Rightarrow u=2$

$$\int_1^e \frac{dx}{x(1+\ln x)^5}=\int_1^2 \frac{du}{u^5}$$
$$=\left[-\frac{1}{4u^4}\right]_1^2$$
$$=\left(-\frac{1}{64}\right)-\left(-\frac{1}{4}\right)$$
$$=\frac{15}{64}$$

(b) (2 Marks)

Sample answer:

$\cos 2x=1-2\sin^2 x \Rightarrow 4\sin^2 x=2-2\cos 2x$

$$\int 4\sin^2 x\,dx = \int 2-2\cos 2x\,dx$$
$$=2x-\sin 2x+C$$

(c) (i) (1 Mark)

Sample answer:

$T=S+Ae^{kt}$

$\dfrac{dT}{dt}=kAe^{kt}$

$\dfrac{dT}{dt}=k(T-S)$, since $Ae^{kt}=T-S$

(c) (ii) (3 Marks)

Sample answer:

Since S is the surrounding temperature, $S=20$. $\quad \therefore T=20+Ae^{kt}$.

When $t=0$, $T=80$

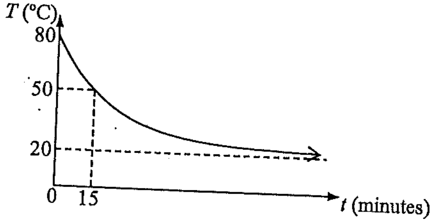
$\therefore\ 80=20+Ae^{k\times 0} \Rightarrow A=60$. $\quad \therefore T=20+60e^{kt}$

When $t=15$, $T=50$

$\therefore\ 50=20+60e^{k\times 15} \Rightarrow k=\dfrac{1}{15}\ln\left(\dfrac{30}{60}\right)=\dfrac{1}{15}\ln\left(\dfrac{1}{2}\right)$

(c) (iii) (2 Marks)

Sample answer:



Question 11 (continued)

(d) (i) (2 Marks)

Sample answer:

$$\frac{d}{dx}\cos^{-1}\left(\frac{x-10}{10}\right)=\frac{-1}{\sqrt{1-\left(\frac{x-10}{10}\right)^2}}\times\frac{1}{10}$$
$$=\frac{-1}{\sqrt{100-(x-10)^2}}$$
$$=\frac{-1}{\sqrt{20x-x^2}}$$

(d) (ii) (2 Marks)

Sample answer:

$$\int_5^{10}\frac{dx}{\sqrt{20x-x^2}}=\left[-\cos^{-1}\left(\frac{x-10}{10}\right)\right]_5^{10}$$
$$=-\cos^{-1}(0)--\cos^{-1}\left(-\frac{1}{2}\right)$$
$$=-\frac{\pi}{2}+\frac{2\pi}{3}$$
$$=\frac{\pi}{6}$$

Question 12 (15 marks)

(a) (i) (1 Mark)

Sample answer:

$$2\alpha+2\beta+2\gamma=2(\alpha+\beta+\gamma)$$
$$=2\left(-\frac{b}{a}\right)$$
$$=2\left(-\frac{-3}{1}\right)$$
$$=6$$

(a) (ii) (2 Marks)

Sample answer:

$$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$$
$$=(3)^2-2(-6)$$
$$=21$$

(b) (3 Marks)

Sample answer:

$f(x)=e^{-x}-x,\ f'(x)=-e^{-x}-1$

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}$$
$$=0.5-\frac{e^{-0.5}-0.5}{-e^{-0.5}-1}$$
$$=0.56631\ldots\approx 0.57 \text{ (2 d.p.)}$$

(c) (i) (2 Marks)

Sample answer:

$P(\text{tossing a six})=\dfrac{1}{6},\quad P(\text{not tossing a six})=\dfrac{5}{6}$

$$P(\text{5 sixes in 7 tosses})=\binom{7}{5}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^2$$
$$=0.001875\ldots\approx 0.002 \text{ (3 d.p.)}$$

(c) (ii) (2 Marks)

Sample answer:

$$P(7^{\text{th}} \text{ toss gives the } 3^{\text{rd}} \text{ six})=P(\text{2 sixes in 6 tosses})\times P(\text{tossing a six})$$
$$=\binom{6}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^4\times\left(\frac{1}{6}\right)$$
$$=0.033489\ldots\approx 0.033$$

Question 12 (continued)

(d) (i) (2 Marks)

Sample answer:

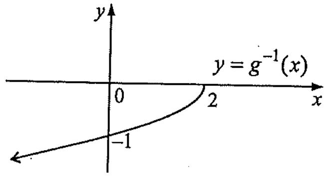
$f(x)=k(x+1)(x-2)^2$

$f(0)=2\Rightarrow 2=k(0+1)(0-2)^2 \quad \therefore k=\frac{1}{2}$

$\therefore f(x)=\frac{1}{2}(x+1)(x-2)^2$

(d) (ii) (1 Mark)

Sample answer:



(d) (iii) (2 Marks)

Sample answer:

Inverse function $y=g^{-1}(x)$ is given by

$$x=\frac{1}{2}(y+1)(y-2)^2$$

$$\frac{dx}{dy}=\frac{1}{2}(y+1)\times 2(y-2)+\frac{1}{2}(y-2)^2$$

$$=\frac{3y}{2}(y-2)$$

$$\frac{dy}{dx}=\frac{2}{3y(y-2)}$$

$\therefore$ gradient of the inverse function at the y-intercept, $y=-1$, is $\frac{2}{3(-1)(-1-2)}=\frac{2}{9}$

Question 13 (15 marks)

(a) (3 Marks)

Sample answer:

The weather balloon rises at $5\text{ms}^{-1} \Rightarrow \frac{dx}{dt}=5$

$$\tan\theta=\frac{x}{100}$$

$$x=100\tan\theta$$

$$\frac{dx}{d\theta}=100\sec^2\theta$$

$$\frac{d\theta}{dx}=\frac{\cos^2\theta}{100}$$

Therefore,

$$\frac{d\theta}{dt}=\frac{d\theta}{dx}\times\frac{dx}{dt}$$

$$=\frac{\cos^2\theta}{100}\times 5$$

$$=\frac{\cos^2\frac{\pi}{4}}{100}\times 5$$

$$=\frac{1}{40}$$

$= 0.025$ radians per second

(b) (2 Marks)

Sample answer:

Let $t=\tan\frac{\theta}{2}$, then $\cos\theta=\frac{1-t^2}{1+t^2}$.

$$\cos\theta=\frac{1}{5}$$

$$\frac{1-t^2}{1+t^2}=\frac{1}{5}$$

$$5-5t^2=1+t^2$$

$$6t^2=4$$

$$t^2=\frac{2}{3}$$

$t=\sqrt{\frac{2}{3}}$ (since θ is acute, $t=\tan\frac{\theta}{2}>0$)

$\theta \quad \sqrt{2}$

Question 13 (continued)

(c) (i) (1 Mark)

Sample answer:

$$\frac{(1+x)^n}{x}=\frac{\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\binom{n}{3}x^3+\cdots+\binom{n}{n}x^n}{x}$$

$$=\binom{n}{0}\frac{1}{x}+\binom{n}{1}+\binom{n}{2}x+\binom{n}{3}x^2+\cdots+\binom{n}{n}x^{n-1}$$

(c) (ii) (2 Marks)

Sample answer:

Differentiating both sides of the identity in part (i), we get

$$\frac{x\times n(1+x)^{n-1}-(1+x)^n}{x^2}=\binom{n}{0}\frac{-1}{x^2}+\binom{n}{2}+2\binom{n}{3}x+3\binom{n}{4}x^2+\cdots+(n-1)\binom{n}{n}x^{n-2}$$

When $x=-1$

$$0=-\binom{n}{0}+\binom{n}{2}-2\binom{n}{3}+3\binom{n}{4}-\cdots+(n-1)\binom{n}{n}(-1)^{n-2}$$

$$0=-\binom{n}{0}+\binom{n}{2}-2\binom{n}{3}+3\binom{n}{4}-\cdots+(n-1)\binom{n}{n}(-1)^n \quad \text{(since } (-1)^{n-2}=(-1)^n\text{)}$$

$$\binom{n}{2}-2\binom{n}{3}+3\binom{n}{4}-\cdots+(-1)^n(n-1)\binom{n}{n}=1 \quad \text{(since } \binom{n}{0}=1\text{)}$$

(d) (3 Marks)

Sample answer:

Let $\angle AED=\alpha$ and $\angle BAO=\beta$

$\angle OAE=90°$ (radii are perpendicular to tangents at their point of contact)

$\angle ABE=90°-\alpha-\beta$ (angle sum of ΔBAE is 180°)

$\angle ABC=90°$ (angle in a semicircle is 90°)

$\angle OBC=\angle ABC-\angle ABE$

$=\alpha+\beta$

$=\angle AED+\angle BAO$

(e) (4 Marks)

Sample answer:

Let $P(n)$ be the given proposition. $P(1)$ is true since $RHS = 1-\frac{1}{2} = \frac{1}{2} = LHS$.

Assume $P(k)$ is true for some positive integer k.

i.e. $\frac{1}{k+1}+\frac{1}{k+2}+\frac{1}{k+3}+\cdots+\frac{1}{2k} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2k-1}-\frac{1}{2k}$.

Prove $P(k+1)$ is true:

$$\frac{1}{k+2}+\frac{1}{k+3}+\cdots+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2k+2}$$

$$=\left(\frac{1}{k+1}+\frac{1}{k+2}+\frac{1}{k+3}+\cdots+\frac{1}{2k}\right)-\frac{1}{k+1}+\frac{1}{2k+1}+\frac{1}{2k+2}$$

$$=\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2k-1}-\frac{1}{2k}\right)+\frac{1}{2k+1}-\left(\frac{1}{k+1}-\frac{1}{2k+2}\right) \text{ (using the assumption)}$$

$$=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2k-1}-\frac{1}{2k}+\frac{1}{2k+1}-\left(\frac{2-1}{2k+2}\right)$$

$$=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2k-1}-\frac{1}{2k}+\frac{1}{2k+1}-\frac{1}{2k+2}$$

Therefore, by the Principle of Mathematical Induction, $P(n)$ is true for all positive integers n.

Question 14 (15 marks)

(a) (3 Marks)

Sample answer:

$$\cos 2\theta+\cos\theta+1=0$$

$$2\cos^2\theta-1+\cos\theta+1=0$$

$$\cos\theta(2\cos\theta+1)=0$$

$$\cos\theta=0,\ \cos\theta=-\frac{1}{2}$$

$\theta=\pm\frac{\pi}{2}+2\pi n$ or $\pm\frac{2\pi}{3}+2\pi n$, for all integers n.

(b) (i) (2 Marks)

Sample answer:

$$\ddot{x}=x$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right)=x$$

$$\frac{1}{2}v^2=\frac{1}{2}x^2+C$$

$$x=1, v=2 \Rightarrow C=\frac{3}{2}$$

$$\therefore v^2=x^2+3$$

Since the particle has a positive displacement, and hence positive acceleration, initially (as well as a positive velocity) the particle will continue to move to the right. The particle's displacement will always be positive, and so its acceleration will always be positive and it will move with increasing velocity to the right.

$$v>0 \Rightarrow v=\sqrt{x^2+3}$$

(b) (ii) (3 Marks)

Sample answer:

$$\frac{dx}{dt}=\sqrt{x^2+3}$$

$$\frac{dt}{dx}=\frac{1}{\sqrt{x^2+3}}$$

$$t=\ln\left(x+\sqrt{x^2+3}\right)+C$$

$$t=0, x=1 \Rightarrow 0=\ln\left(1+\sqrt{1^2+3}\right)+C$$

$$\Rightarrow C=-\ln 3$$

$$t=\ln\left(x+\sqrt{x^2+3}\right)-\ln 3$$

$$t=\ln\left(\frac{x+\sqrt{x^2+3}}{3}\right)$$

$$3e^t=x+\sqrt{x^2+3}$$

$$3e^t-x=\sqrt{x^2+3}$$

$$9e^{2t}-6xe^t+x^2=x^2+3$$

$$6xe^t=9e^{2t}-3$$

$$x=\frac{9e^{2t}-3}{6e^t}$$

$$x=\frac{3e^{2t}-1}{2e^t}$$

$$x=\frac{3}{2}e^t-\frac{1}{2}e^{-t}$$

(c) (i) (2 Marks)

Sample answer:

For the horizontal range R, first solve $y=0$ to find the time of flight

$$0=t\left(V\sin\alpha-\frac{g}{2}t\right)$$

$t=\frac{2V\sin\alpha}{g}$, since $t\neq 0$ for the horizontal range

When $t=\frac{2V\sin\alpha}{g}$,

$$x=V\left(\frac{2V\sin\alpha}{g}\right)\cos\alpha$$

$$=\frac{V^2 2\sin\alpha\cos\alpha}{g}$$

$$=\frac{V^2\sin 2\alpha}{g}$$

Therefore, the horizontal range R is $\frac{V^2\sin 2\alpha}{g}$.

Question 14 (continued)

(c) (ii) (2 Marks)

Sample answer:

$x = Vt\cos\alpha \Rightarrow t = \dfrac{x}{V\cos\alpha}$

Substituting into $y = Vt\sin\alpha - \frac{g}{2}t^2$ gives the equation of the path of the projectile:

$$y = V\left(\frac{x}{V\cos\alpha}\right)\sin\alpha - \frac{g}{2}\left(\frac{x}{V\cos\alpha}\right)^2$$

$$y = x\frac{\sin\alpha}{\cos\alpha} - x^2\frac{\sin\alpha}{\cos\alpha}\left(\frac{g}{2V^2\sin\alpha\cos\alpha}\right)$$

$$y = x\tan\alpha - x^2\tan\alpha\left(\frac{1}{R}\right)$$

$$y = x\left(1 - \frac{x}{R}\right)\tan\alpha$$

(c) (iii) (1 Mark)

Sample answer:
$\alpha = 45°$

For the x coordinates of the edge of the wall, solve $y = 10$:

$$10 = x\left(1 - \frac{x}{R}\right) \times \tan 45°$$

$$10R = x(R - x)$$

$$x^2 - Rx + 10R = 0$$

(c) (iv) (2 Marks)

Sample answer:
Let x_1 and x_2 be the x coordinates of the edge of the wall, with $x_1 > x_2$.

Since x_1, x_2 are the roots of the quadratic $x^2 - Rx + 10R = 0$, using the quadratic formula we have

$$x_1 = \frac{R + \sqrt{R^2 - 40R}}{2} \text{ and } x_2 = \frac{R - \sqrt{R^2 - 40R}}{2}.$$

Since the wall of the fort is 4.5 metres thick, $x_1 - x_2 = 4.5$

$$\therefore \sqrt{R^2 - 40R} = 4.5$$

$$R^2 - 40R - 20\frac{1}{4} = 0$$

$$4R^2 - 160R - 81 = 0$$

$$(2R + 1)(2R - 81) = 0$$

Since R is the horizontal range, $R > 0$, $\therefore R = 40.5$ metres