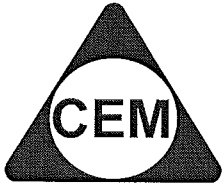


NAME :



Centre of Excellence in Mathematics  
S201 / 414 GARDENERS RD. ROSEBERY 2018  
[www.cemtuition.com.au](http://www.cemtuition.com.au)

MOBILE 04412088475



PHONE 919963311

## YEAR 11 – EXT.1 MATHS

### REVIEW TOPIC : AUXILIARY ANGLE METHOD – BOOK 1

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**Question 1:**

Find all values of  $\theta$  between  $0^\circ$  and  $360^\circ$  satisfying the equation

$$3 \cos \theta + 4 \sin \theta = 1$$

by first expressing  $3 \cos \theta + 4 \sin \theta$  in the form  $R \cos(\theta - \alpha)$ .

Give your answers to the nearest degree.

[6]

$5 \cos(\theta - 53.1^\circ); \theta = 132^\circ, 335^\circ$ (to the nearest deg)
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**Question 2:**

Find the value of the acute angle  $\alpha$  for which

$$7 \cos x - 3 \sin x = \sqrt{58} \cos(x + \alpha).$$

for all values of  $x$ .

[4]

$$\alpha = 23.2^\circ \text{ (to 1 d.p.)}$$

Solve each of the following equations, giving your answers correct to one decimal place.

(a)  $7 \cos x - 3 \sin x = 5$

$0 \leq x \leq 360^\circ$

[5]

$$x = 25.8^\circ, 287.8^\circ \text{ (to 1 d.p.)}$$

(b)  $7 \cos 2x - 3 \sin 2x = 5$

$0 \leq x \leq 360^\circ$

[5]

$$x = 12.9^\circ, 143.9^\circ, 192.9^\circ, 323.9^\circ \text{ (to 1 d.p.)}$$

**Question 3:**

Find all values of  $x$  between  $0^\circ$  and  $360^\circ$  which satisfy the equation

$$4 \cos x - 3 \sin x = 1.$$

Give your answers correct to 1 decimal place.

[6]

$$x = 41.6^\circ, 244.7^\circ \text{ (to 1 d.p.)}$$

**Question 4:**

The parametric equations of a curve are

$$x = 4 \cos \theta, \quad y = 5 \sin \theta, \quad 0 \leq \theta \leq 2\pi .$$

- (a) Use the parametric equations to calculate  $\frac{dy}{dx}$  in terms of  $\theta$ . [3]

$$\boxed{\frac{dy}{dx} = -\frac{5 \cos \theta}{4 \sin \theta}}$$

- (b) Prove that the equation of the tangent to the curve at  $(4 \cos \theta, 5 \sin \theta)$  is  $5x \cos \theta + 4y \sin \theta = 20$ . [3]

- (c) If the tangents to the curve pass through the point  $(2, 5)$ , show that  
$$\cos \theta + 2 \sin \theta = 2. \quad [1]$$

- (d) By solving the equation in (c) find two values of  $\theta$  (in radians to 2 decimal places) which correspond to the two tangents which pass through the point  $(2, 5)$ . [5]

$x = 0.64^c, 1.57^c; \text{ or } 36.67^0, 89.95^0 \text{ (to 2 d.p.)}$
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**Question 5:**

A curve has parametric equations given by

$$x = 2 \sin \theta, \quad y = \cos \theta.$$

- (a) Find  $\frac{dy}{dx}$  in terms of  $\theta$ . [3]

$$\frac{dy}{dx} = -\frac{\sin \theta}{2 \cos \theta}$$

- (b) Find the equation of the normal at the point,  $P$ , on the curve for which  
 $\sin \theta = \frac{4}{5}$  and  $\cos \theta = \frac{3}{5}$ . [3]

$$15x - 10y - 18 = 0$$

- (c) Show that the values of  $\theta$  where this normal meets the curve again are given by  $5 \cos \theta - 15 \sin \theta + 9 = 0$ .  
Solve this equation for values of  $\theta$  in the range  $0 \leq \theta \leq 2\pi$  and hence find the co-ordinates of  $Q$ , the point where the normal meets the curve again. [6]

$$Q(0.56, -0.96)$$



**SOLUTIONS TO Q1 TO 5:**
**Question 1:**

$$3 \cos \theta + 4 \sin \theta \equiv R \cos(\theta - \alpha)$$

$$\equiv R \cos \theta \cos \alpha + R \sin \theta \sin \alpha$$

 Compare coefficients of  $\cos \theta$ 

$$3 = R \cos \alpha \quad -\{1\}$$

 Compare coefficients of  $\sin \theta$ 

$$4 = R \sin \alpha \quad -\{2\}$$

 Squaring  $\{1\}$  and  $\{2\}$  and adding gives

$$3^2 + 4^2 = R^2(\cos^2 \alpha + \sin^2 \alpha)$$

$$= R^2$$

$$\Rightarrow R^2 = 25$$

$$R = \sqrt{25}$$

$$R = 5$$

$$\frac{\{2\}}{\{1\}} \quad \frac{R \sin \alpha}{R \cos \alpha} = \frac{4}{3}$$

$$\Rightarrow \tan \alpha = \frac{4}{3}$$

$$\alpha = \tan^{-1}\left(\frac{4}{3}\right)$$

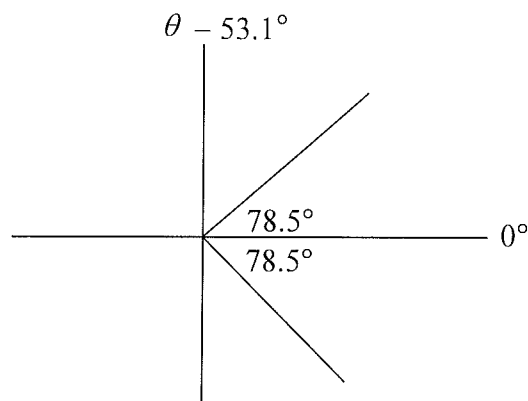
$$\alpha = 53.1^\circ$$

$$\Rightarrow 3 \cos \theta + 4 \sin \theta \equiv 5 \cos(\theta - 53.1^\circ)$$

$$\Rightarrow 5 \cos(\theta - 53.1^\circ) = 1$$

$$\Rightarrow \cos(\theta - 53.1^\circ) = \frac{1}{5}$$

$$\cos^{-1}\left(\frac{1}{5}\right) = 78.5^\circ$$



$$\Rightarrow \theta - 53.1^\circ = 78.5^\circ \quad \text{or} \quad \theta - 53.1^\circ = 360^\circ - 78.5^\circ$$

$$\Rightarrow \theta = 131.6^\circ \quad \theta = 334.6^\circ$$

$$\Rightarrow \theta = 132^\circ, 335^\circ \text{ to nearest degree}$$

**Question 2:**

$$\begin{aligned}
 7 \cos x - 3 \sin x &\equiv \sqrt{58} \cos(x + \alpha) \\
 &\equiv \sqrt{58} (\cos x \cos \alpha - \sin x \sin \alpha) \\
 &\equiv \sqrt{58} \cos \alpha \cos x - \sqrt{58} \sin \alpha \sin x
 \end{aligned}$$

Compare coefficients of  $\cos x$

$$7 = \sqrt{58} \cos \alpha \quad -\{1\}$$

Compare coefficients of  $\sin x$

$$-3 = -\sqrt{58} \sin \alpha$$

$\Rightarrow$

$$3 = \sqrt{58} \sin \alpha \quad -\{2\}$$

$$\frac{\{2\}}{\{1\}}$$

$$\frac{\sqrt{58} \sin \alpha}{\sqrt{58} \cos \alpha} = \frac{3}{7}$$

$\Rightarrow$

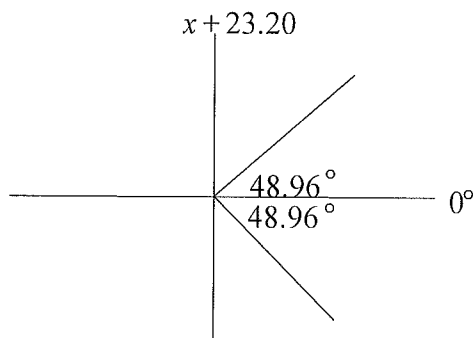
$$\tan \alpha = \frac{3}{7}$$

$$\alpha = \tan^{-1}\left(\frac{3}{7}\right)$$

$$\alpha = 23.20^\circ \text{ correct to 2 decimal places}$$

(a)

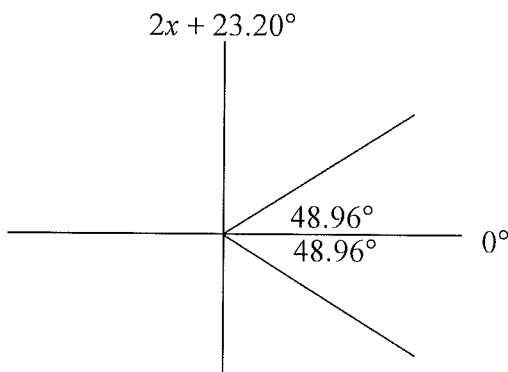
$$\begin{aligned}
 \Rightarrow \quad 7 \cos \alpha - 3 \sin \alpha &= 5 \\
 \sqrt{58} \cos(x + 23.20) &= 5 \\
 \cos(x + 23.20) &= \frac{5}{\sqrt{58}} \\
 \cos^{-1}\left(\frac{5}{\sqrt{58}}\right) &= 48.96^\circ
 \end{aligned}$$



$$\begin{aligned}
 \Rightarrow \quad x + 23.20 &= 48.96^\circ \quad \text{or} \quad x + 23.20 = 360 - 48.96 \\
 \Rightarrow \quad x &= 25.76 \quad \quad \quad x = 287.84 \\
 \Rightarrow \quad x &= 25.8^\circ, 287.8^\circ \text{ correct to 1 decimal place}
 \end{aligned}$$

(b)  $7 \cos 2x - 3 \sin 2x = 5$

To solve this equation we can replace  $x$  with  $2x$  in the quadrant diagram above



$$\begin{aligned} \Rightarrow 2x + 23.20 &= 48.96 & \text{or} & \quad 2x + 23.20 = 360 - 48.96 \\ \Rightarrow 2x &= 25.76 & & \quad 2x = 287.84 \\ \Rightarrow x &= 12.88^\circ & & \quad x = 143.92^\circ \end{aligned}$$

$$\begin{aligned} \text{or } 2x + 23.20 &= 360 + 48.96 & \text{or} & \quad 2x + 23.20 = 720 - 48.96 \\ \Rightarrow 2x &= 385.76 & & \quad 2x = 647.84 \\ \Rightarrow x &= 192.88^\circ & & \quad x = 323.92^\circ \end{aligned}$$

Solution to the equation is  
 $x = 12.9^\circ, 143.9^\circ, 192.9^\circ, 323.9^\circ$  correct to 1 decimal place

**Question 3:**

$$\begin{aligned} 4 \cos x - 3 \sin x & \equiv R \cos(x + \alpha) \\ & \equiv R (\cos x \cos \alpha - \sin x \sin \alpha) \\ & \equiv R \cos \alpha \cos x - R \sin \alpha \sin x \end{aligned}$$

Compare coefficients of  $\cos x$

$$4 = R \cos \alpha \quad \text{-\{1\}}$$

Compare coefficients of  $\sin x$

$$-3 = -R \sin \alpha$$

$$\Rightarrow 3 = R \sin \alpha \quad \text{-\{2\}}$$

Squaring {1} and {2} and adding gives

$$4^2 + 3^2 = R^2 (\cos^2 \alpha + \sin^2 \alpha)$$

$$\Rightarrow R^2 = 25$$

$$R = 5$$

$$\frac{\{2\}}{\{1\}} \quad \frac{3}{4} = \frac{R \sin \alpha}{R \cos \alpha}$$

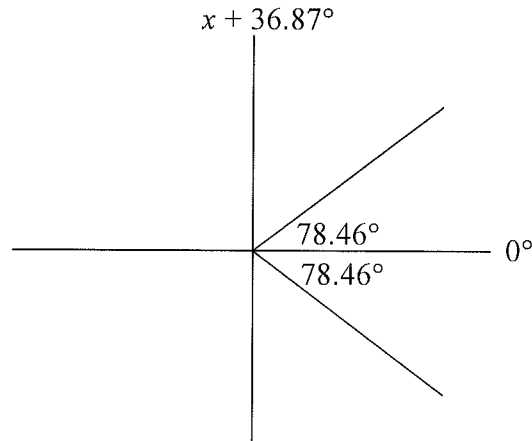
$$\begin{aligned} \Rightarrow \tan \alpha &= 0.75 \\ \alpha &= \tan^{-1} 0.75 \\ &= 36.87^\circ \end{aligned}$$

$$\Rightarrow 4 \cos x - 3 \sin x = 5 \cos(x + 36.87)$$

$$\Rightarrow 5 \cos(x + 36.87) = 1$$

$$\cos(x + 36.87) = \frac{1}{5}$$

$$\cos^{-1}\left(\frac{1}{5}\right) = 78.46^\circ$$



$$\Rightarrow x + 36.87 = 78.46 \quad \text{or} \quad x + 36.87 = 360 - 78.46$$

$$\Rightarrow x = 41.59 \quad \text{or} \quad x = 244.67$$

$$x = 41.6^\circ, 244.7^\circ$$

#### **Question 4:**

(a)

$$x = 4 \cos \theta \quad , \quad y = 5 \sin \theta$$

$$\frac{dx}{d\theta} = -4 \sin \theta \quad \quad \frac{dy}{d\theta} = 5 \cos \theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= 5 \cos \theta \times \frac{1}{-4 \sin \theta}$$

$$\frac{dy}{dx} = -\frac{5 \cos \theta}{4 \sin \theta}$$

- (b) The equation of the tangent with gradient  $m$ , passing through the point  $(x_1, y_1)$  is

$$y - y_1 = m(x - x_1)$$

$\Rightarrow$  Equation of the tangent passing through  $(4 \cos \theta, 5 \sin \theta)$  is

$$y - 5 \sin \theta = -\frac{5 \cos \theta}{4 \sin \theta} (x - 4 \cos \theta)$$

$\times 4 \sin \theta$

$$4y \sin \theta - 20 \sin^2 \theta = -5 \cos \theta (x - 4 \cos \theta)$$

$$4y \sin \theta - 20 \sin^2 \theta = -5x \cos \theta + 20 \cos^2 \theta$$

$$5x \cos \theta + 4y \sin \theta = 20(\cos^2 \theta + \sin^2 \theta)$$

$$\Rightarrow 5x \cos \theta + 4y \sin \theta = 20$$

- (c) If the tangents pass through the point (2, 5) then substituting  $x = 2, y = 5$  gives

$$\begin{aligned} 5(2) \cos \theta + 4(5) \sin \theta &= 20 \\ 10 \cos \theta + 20 \sin \theta &= 20 \\ \div 10 \quad \cos \theta + 2 \sin \theta &= 2 \end{aligned}$$


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- (d)  $\cos \theta + 2 \sin \theta = 2$   
Let  $\cos \theta + 2 \sin \theta \equiv R \cos(\theta - \alpha)$

Where  $R > 0$  and  $0 \leq \alpha \leq \frac{\pi}{2}$

$$\begin{aligned} \cos \theta + 2 \sin \theta &\equiv R (\cos \theta \cos \alpha + \sin \theta \sin \alpha) \\ &\equiv R \cos \alpha \cos \theta + R \sin \alpha \sin \theta \end{aligned}$$

Compare coefficients of  $\cos \theta$

$$1 = R \cos \alpha \quad \text{-\{1\}}$$

Compare coefficients of  $\sin \theta$

$$2 = R \sin \alpha \quad \text{-\{2\}}$$

Squaring {1} and {2} and adding

$$\begin{aligned} 1^2 + 2^2 &= R^2 (\cos^2 \alpha + \sin^2 \alpha) \\ \Rightarrow R^2 &= 5 \\ R &= \sqrt{5} \end{aligned}$$

$$\frac{\{2\}}{\{1\}} \quad \frac{R \sin \alpha}{R \cos \alpha} = \frac{2}{1}$$

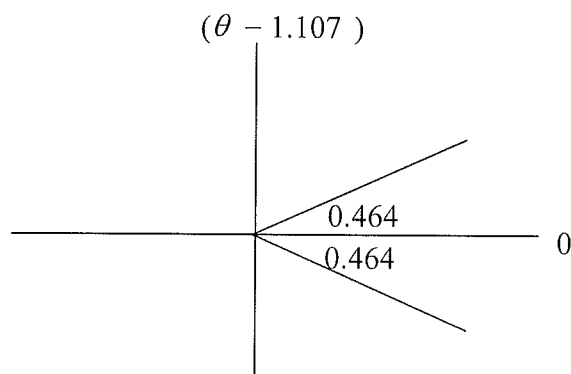
$$\begin{aligned} \Rightarrow \tan \alpha &= 2 \\ \alpha &= \tan^{-1} 2 \\ \alpha &= 1.107 \text{ radians} \end{aligned}$$

$$\Rightarrow \cos \theta + 2 \sin \theta \equiv \sqrt{5} \cos(\theta - 1.107)$$

$$\Rightarrow \sqrt{5} \cos(\theta - 1.107) = 2$$

$$\div \sqrt{5} \quad \cos(\theta - 1.107) = \frac{2}{\sqrt{5}}$$

$$\cos^{-1} \left( \frac{2}{\sqrt{5}} \right) = 0.464 \text{ radians}$$



$$\begin{aligned} \Rightarrow \quad \theta - 1.107 &= 0.464 & \text{or} & \quad \theta - 1.107 = 2\pi - 0.464 \\ \theta &= 1.571 \text{ radians} & & \quad \theta = 6.926 \\ & & & \quad \text{This root is outside the interval} \\ & & & \quad 0 \leq \theta \leq 2\pi . \\ \Rightarrow \quad \theta - 1.107 &= -0.464 \\ \theta &= 0.643 \text{ radians} \\ \Rightarrow \text{The two values of } \theta &\text{ corresponding to the two tangents are} \\ \theta &= 0.64, 1.57 \text{ radians correct to 2 decimal places} \end{aligned}$$

**Question 5:**

$$\begin{aligned} \text{(a)} \quad x &= 2 \sin \theta, & y &= \cos \theta \\ \frac{dx}{d\theta} &= 2 \cos \theta & \frac{dy}{d\theta} &= -\sin \theta \\ \frac{dy}{dx} &= \frac{dy}{d\theta} \times \frac{d\theta}{dx} \\ &= -\sin \theta \times \frac{1}{2 \cos \theta} \\ \frac{dy}{dx} &= -\frac{\sin \theta}{2 \cos \theta} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \text{The gradient of the normal} &= \frac{2 \cos \theta}{\sin \theta} \quad (\text{using } m_1 m_2 = -1) \\ \text{At the point on the curve where } \sin \theta &= \frac{4}{5} \text{ and } \cos \theta = \frac{3}{5} \text{ the gradient} \\ \text{of the normal.} \end{aligned}$$

$$\begin{aligned} m &= \frac{2 \left( \frac{3}{5} \right)}{\frac{4}{5}} \\ &= \frac{3}{2} \end{aligned}$$

$$\begin{aligned} \text{At this point} \quad x &= 2 \sin \theta & y &= \cos \theta \\ &= 2 \times \frac{4}{5} & &= \frac{3}{5} \\ &= \frac{8}{5} & & \end{aligned}$$

⇒ Equation of the normal, at the point  $P\left(\frac{8}{5}, \frac{3}{5}\right)$

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - \frac{3}{5} &= \frac{3}{2}\left(x - \frac{8}{5}\right) \\ \times 10 \quad 10y - 6 &= 15x - 24 \\ 10y &= 15x - 18 \end{aligned}$$

(c) In order that the normal cuts the curve again substitute

$$\begin{aligned} x &= 2 \sin \theta, \quad y = \cos \theta \\ 10 \cos \theta &= 15 \times 2 \sin \theta - 18 \\ 10 \cos \theta &= 30 \sin \theta - 18 \\ \div 2 \quad 5 \cos \theta - 15 \sin \theta + 9 &= 0 \end{aligned}$$

Now solve

$$\begin{aligned} 5 \cos \theta - 15 \sin \theta &= -9 \\ 5 \cos \theta - 15 \sin \theta &\equiv R \cos(\theta + \alpha) \end{aligned}$$

Where  $R > 0$  and  $0 \leq \alpha \leq \frac{\pi}{2}$

$$\begin{aligned} 5 \cos \theta - 15 \sin \theta &\equiv R(\cos \alpha \cos \theta - \sin \alpha \sin \theta) \\ &\equiv R \cos \alpha \cos \theta - R \sin \alpha \sin \theta \end{aligned}$$

Compare coefficients of  $\cos \theta$

$$5 = R \cos \alpha \quad -\{1\}$$

Compare coefficients of  $\sin \theta$

$$\begin{aligned} -15 &= -R \sin \alpha \\ 15 &= R \sin \alpha \quad -\{2\} \end{aligned}$$

Squaring {1} and {2} and adding gives

$$\begin{aligned} 5^2 + 15^2 &= R^2(\cos^2 \alpha + \sin^2 \alpha) \\ \Rightarrow R^2 &= 250 \\ R &= \sqrt{250} \end{aligned}$$

$$\frac{\{2\}}{\{1\}} = \frac{R \sin \alpha}{R \cos \alpha} = \frac{15}{5}$$

$$\begin{aligned} \Rightarrow \tan \alpha &= 3 \\ \alpha &= \tan^{-1} 3 \\ \alpha &= 1.249 \text{ radians} \end{aligned}$$

$$\Rightarrow 5 \cos \theta - 15 \sin \theta \equiv \sqrt{250} \cos(\theta + 1.249)$$

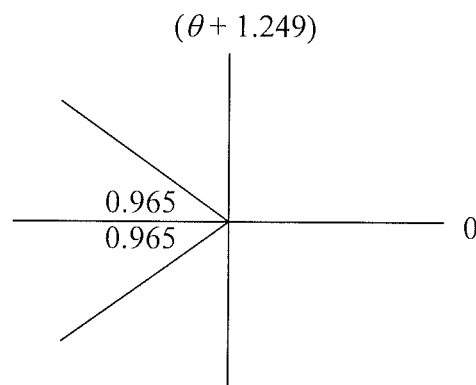
The equation becomes

$$\sqrt{250} \cos(\theta + 1.249) = -9$$

$$\cos(\theta + 1.249) = \frac{-9}{\sqrt{250}}$$

In the quadrant diagram, the acute angle the directions make with the  $x$  axis is

$$\cos^{-1}\left(\frac{9}{\sqrt{250}}\right) = 2.17 \text{ radians}$$



$$\Rightarrow \theta + 1.249 = \pi - 0.965 \quad \text{or} \quad \theta + 1.249 = \pi + 0.965$$

$$\theta = 0.928 \quad \text{or} \quad \theta = 2.858$$

The solution  $\theta = 0.928$  is the value of  $\theta$  at the point  $P$   $\left(\sin^{-1}\frac{4}{5} = 0.93\right)$

$\Rightarrow \theta = 2.858$  is the value of  $\theta$  at the point  $Q$ , the other point where the normal cuts the curve.

$$\text{Co-ordinates of } Q(2\sin 2.858, \cos 2.858) = (0.56, -0.96)$$


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