C.E.M. TUITION

Student Name : ____________________

Review Topic : Integration

(HSC - PAPER 3)

Year 12 - 2 Unit

13. (a) Find correct to one decimal place the value of

$$\int_0^1 (2x^3+1)(2x^3-1)\,dx.$$

(b) **(i)** Find the points of intersection of the curve $y = 4\sqrt{x} - 4$ and the x and y axes.

(ii) Calculate the exact area contained by the curve $y = 4\sqrt{x} - 4$ and the coordinate axes.

14. (a) Find the value of m such that $\int_1^{2m} 3x^2 \, dx = 215$.

(b) **(i)** Find the points of intersection of the curves $y = \frac{1}{8}x^2$ and $y = \sqrt{x}$.

(ii) Calculate the area contained between the two curves.

15. (a) Evaluate $\int_1^{16} \frac{dy}{\sqrt{y}}$.

(b) **(i)** Sketch the curve $y = \sqrt{16 - x^2}$, for $-4 \le x \le 4$.

(ii) The area between this curve and the x axis is rotated about the x axis. Calculate the volume so formed in its simplest form (leave in terms of π).

(iii) Name the solid formed by this rotation.

(iv) Use the formula $V = \frac{4}{3}\pi r^3$ with an appropriate value of r to check the answer in part (ii).

16. (a) (i) Sketch $y = \sqrt{x}$ for $1 \le x \le 9$.

(ii) Give reasons why $\sqrt{x} + \dfrac{1}{\sqrt{x}} > \sqrt{x}$ for all $x > 0$.

(iii) By using selected points, sketch the curve

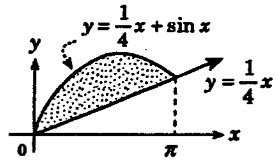
$y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$, $1 \le x \le 9$ on the same graph as $y = \sqrt{x}$.

(iv) Calculate the area between the curves $y = \sqrt{x}$ and

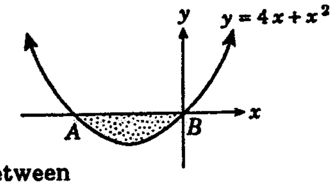
$y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$ between $x = 1$ and $x = 9$.

(v) If this area is rotated about the x axis calculate the volume of revolution so formed.

17. This diagram illustrates the area between the curve $y = \frac{1}{4}x + \sin x$ and the line $y = \frac{1}{4}x$ between $x = 0$ and $x = \pi$. Calculate this area.



18. The curve $y = 4x + x^2$ cuts the x axis at A and B, with B being the origin.



(a) Find the coordinates of A.

(b) (i) Calculate the area contained between this curve and the x axis (shaded).

(ii) If this area is rotated about the x axis, calculate the volume so formed.

13. (a) $\int_0^1 (2x^3+1)(2x^3-1)\,dx$

$= \int_0^1 (4x^6-1)\,dx$

$= \left[\frac{4}{7}x^7 - x\right]_0^1$

$= \left(\frac{4}{7}-1\right) - 0$

$= -0{\cdot}428\,571\,4$

$= -0{\cdot}4$ (one dec. place).

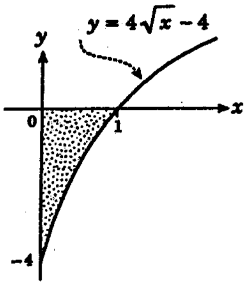
(b) (i) $y = 4\sqrt{x} - 4$

$x=0,\ y=-4$

$y=0,\ x=1$

Curve cuts x axis at $(1, 0)$, y axis at $(0, -4)$.

(ii)



$A = \left|\int_0^1 \left(4\sqrt{x} - 4\right)dx\right|$

$= \left|4\int_0^1 \left(x^{\frac{1}{2}} - 1\right)dx\right|$

$= \left|4\left[\frac{2}{3}x^{\frac{3}{2}} - x\right]_0^1\right|$

$= \left|4\left[\frac{2}{3}-1\right]\right|$

$= \frac{4}{3}$

Area is $\frac{4}{3}$ units^2.

14. (a) $\int_1^{2m} 3x^2\,dx = 215$

$\therefore\ \left[x^3\right]_1^{2m} = 215$

$\therefore\ (2m)^3 - 1 = 215$

$\therefore\ 8m^3 = 216$

$\therefore\ m^3 = 27$

$\therefore\ m = 3.$

(b) (i) $y = \frac{1}{8}x^2$ —①

$y = \sqrt{x}$ —②

Substitute ① into ②:

$\therefore\ \frac{1}{8}x^2 = \sqrt{x}$

i.e., $\frac{1}{64}x^4 = x$

$\therefore\ x^4 - 64x = 0$

$\therefore\ x(x^3 - 64) = 0$

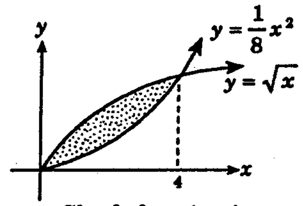
i.e., $x=0$ or $x^3 = 64$

$x = 0$ or $x = 4$

In ②, when $x=0,\ y=0$

$x=4,\ y=2.$

(ii) Curves intersect at (0, 0) and (4, 2).



Shaded region is area required.

$A = \int_0^4 \sqrt{x}\ dx - \int_0^4 \frac{1}{8}x^2\ dx$

$= \int_0^4 x^{\frac{1}{2}}\ dx - \frac{1}{8}\int_0^4 x^2\ dx$

$= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^4 - \frac{1}{8}\left[\frac{1}{3}x^3\right]_0^4$

$= \frac{16}{3} - \frac{1}{8}\left[\frac{64}{3}\right]$

$= \frac{16}{3} - \frac{8}{3}$

$= \frac{8}{3}.$

Area is $\frac{8}{3}$ units^2.

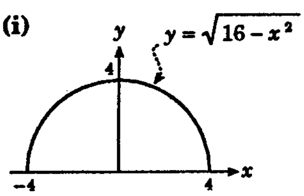
15. (a) $\int_1^{16} y^{-\frac{1}{2}}\ dy = \left[2y^{\frac{1}{2}}\right]_1^{16}$

$= \left[2\sqrt{16} - 2\sqrt{1}\right]$

$= 8 - 2$

$= 6.$

(b) (i)



[Top half of $x^2 + y^2 = 16$.]

Note: $y^2 = 16 - x^2$

(ii) $V = \pi\int_{-4}^{4} (16 - x^2)\,dx$

$V = \pi\int y^2\ dx$

$= 2\pi\int_0^4 (16 - x^2)\,dx$

$= 2\pi\left[16x - \frac{1}{3}x^3\right]_0^4$

$= 2\pi\left[64 - \frac{64}{3}\right]$

$= \frac{256}{3}\pi.$

Volume is $\frac{256}{3}\pi$ units^3

(iii) Sphere radius 4 units.

(iv) $V = \frac{4}{3}\pi r^3$

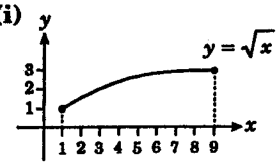
$= \frac{4}{3}\pi \times 4^3$

$= \frac{4}{3}\pi \times 64$

$= \frac{256}{3}\pi.$

Volume is again $\frac{256}{3}\pi$ units^3.

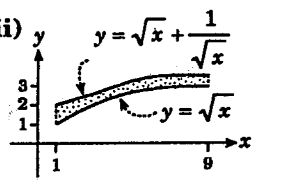
16. (a) (i)



(ii)

$\sqrt{x} + \frac{1}{\sqrt{x}} > \sqrt{x}$ because $\frac{1}{\sqrt{x}}$ must always be greater than zero when $x > 0$. So $\sqrt{x}$ + (positive quantity) must be bigger than $\sqrt{x}$.

(iii)



(iv)

$A = \int_1^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)dx - \int_1^9 \sqrt{x}\ dx$

$= \int_1^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}} - \sqrt{x}\right)dx$

$= \int_1^9 \frac{1}{\sqrt{x}}\ dx$

$\sqrt{x} = x^{\frac{1}{2}}$

$\sqrt{\frac{1}{x}} = x^{-\frac{1}{2}}$

$= \int_1^9 x^{-\frac{1}{2}}\ dx$

$= \left[2x^{\frac{1}{2}}\right]_1^9$

$= \left[2\sqrt{x}\right]_1^9$

$=\left(2\sqrt{9}-2\sqrt{1}\right)$

$=6-2$

$=4.$

Area is 4 units^2.

(v)

$$V=\pi\int_1^9 y^2\,dx$$

$$=\pi\int_1^9\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 dx$$

$$-\pi\int_1^9(\sqrt{x})^2\,dx$$

$$=\pi\int_1^9\left(x+2+\frac{1}{x}\right)dx$$

$$-\pi\int_1^9 x\,dx$$

$$=\pi\int_1^9\left(x+2+\frac{1}{x}-x\right)dx$$

$$=\pi\int_1^9\left(2+\frac{1}{x}\right)dx$$

$$=\pi\left[2x+\log_e x\right]_1^9$$

$$=\pi\left[(18+\log_e 9)-(2+\log_e 1)\right]$$

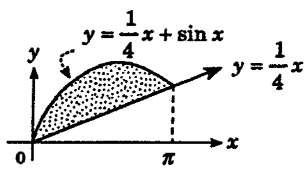
$$=\pi\left[16+\log_e 9\right]$$

$\log_e 9=\log_e 3^2$ $=2\log_e 3$

$$=2\pi\left[8+\log_e 3\right].$$

Volume is $2\pi\left[8+\log_e 3\right]$ units^3.

17.



$$\text{Area}=\int_0^\pi\left(\frac{1}{4}x+\sin x\right)dx-\int_0^\pi\frac{1}{4}x\,dx$$

$$=\int_0^\pi\left(\frac{1}{4}x+\sin x-\frac{1}{4}x\right)dx$$

$$=\int_0^\pi\sin x\,dx$$

$$=\left[-\cos x\right]_0^\pi$$

$$=-\left[\cos\pi-\cos 0\right]$$

$$=-\left[-1-1\right]$$

$$=2.$$

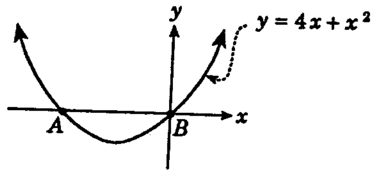
Area is 2 units^2.

18. (a) When $y=0$, $0=4x+x^2$

i.e., $x(x+4)=0$

$\therefore\ x=0$ or -4.

Then A is $(-4, 0)$ as B is $(0, 0)$.



Absolute value is taken as area is a positive quantity and the region lies below the x axis.

(b) (i)

$$A=\left|\int_{-4}^0(4x+x)^2\,dx\right|$$

$$=\left|\left[2x^2+\frac{1}{3}x^3\right]_{-4}^0\right|$$

$$=\left|[0]-\left[32-\frac{64}{3}\right]\right|$$

$$=10\frac{2}{3}.$$

Area is $10\frac{2}{3}$ units^2.

(ii)

$$V=\pi\int_{-4}^0 y^2\,dx$$

$y=4x+x^2$ $\therefore y^2=(4x+x^2)^2$ $=16x^2+8x^3+x^4$

$$=\pi\int_{-4}^0(16x^2+8x^3+x^4)\,dx$$

$$=\pi\left[\frac{16}{3}x^3+2x^4+\frac{1}{5}x^5\right]_{-4}^0$$

$$=\pi\left[(0)-\left(\frac{16}{3}.-64+2.(-4)^4+\frac{1}{5}.(-4)^5\right)\right]$$

$$=\pi\left[\frac{512}{15}\right].$$

Volume is $\frac{512\pi}{15}$ units^3.