TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 1994

MATHEMATICS

2/3 UNIT
COMMON PAPER

Total time allowed - THREE hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES :

- All questions may be attempted.
- All questions are of equal value.
- All necessary working should be shown in every question.
- Full marks may not be awarded for badly arranged work.
- Standard integrals are provided.
- Approved silent calculators may be used.
- Start each question on a new page.
- You must hand in a blank page if a question is unanswered.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a>0, \quad -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x>0$

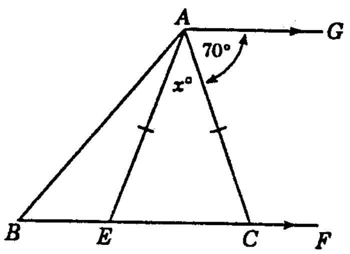
QUESTION ONE

(a) Evaluate $\sqrt{\dfrac{4 \cdot 5^2}{13 \cdot 8 - 7 \cdot 21}}$ correct to 2 decimal places.

(b) Factorise fully $5x^2 - 20y^2$.

(c) Solve $\dfrac{x-6}{3} - \dfrac{x-1}{2} = 1$.

(d) NOT TO SCALE



In the diagram, $AE = AC$, AG is parallel to BF. $G\hat{A}C = 70°$. Find the value of x, giving reasons.

(e) (i) Express $\dfrac{6}{\sqrt{5}+\sqrt{2}}$ with a rational denominator.

(ii) Find the value of x and y such that $\dfrac{6}{\sqrt{5}+\sqrt{2}} = \sqrt{x} - \sqrt{y}$.

(f) Mark on the number line the values of x for which $|4x - 2| \le 14$.

QUESTION TWO

(a) A man earns \$41 500 in 1994 and invests 15% of his earnings in an account earning 10% interest per annum, compounded annually. How much interest does he earn on his investment at the end of 5 years?

(b) (i) On a number plane, plot the points $A(1, -1)$, $B(5, 1)$ and $C(7, -3)$. Join A to B and B to C.

(ii) Find the midpoint of the interval AC.

(iii) Show that the coordinates of D the fourth vertex of the parallelogram $ABCD$ are $(3, -5)$.

(iv) Show that the equation of line AB is $x - 2y = 3$.

(v) Calculate the perpendicular distance from the point D to the line AB.

(vi) Find the area of the parallelogram $ABCD$.

QUESTION THREE

(a) Write down the derivatives of:

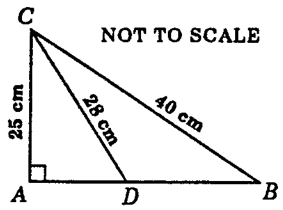
(i) $\ln(3-2x)$ (ii) $x\sin x$ (iii) $\dfrac{1}{2x+1}$

(b) Find: (i) $\int (3x-2)^8 \, dx$ (ii) $\int e^{\frac{x}{2}} \, dx$

(c) Find the equation of the normal to the parabola $y = x^2 - 2x + 1$ at the point where $x = 2$.

QUESTION FOUR

(a)



In the diagram, $AC = 25$ cm, $CD = 28$ cm and $BC = 40$ cm.

(i) Calculate the lengths AD and DB. (Give your answers correct to 2 decimal places.)

(ii) Using the Cosine Rule, calculate the size of angle CDB.

(iii) Find the area of triangle CDB.

(b) A bottle of solvent is open and the solvent evaporates in such a way that the amount remaining, V mL, in the bottle is given by $V = 2000e^{-0\cdot005t}$, where t is time in hours.

(i) How much solvent is in the bottle initially?

(ii) How much solvent has evaporated out of the bottle after 30 hours?

(iii) How long is it before half the initial amount of solvent has evaporated from the bottle?

(iv) If the solvent continues to evaporate will the bottle ever become empty? Explain.

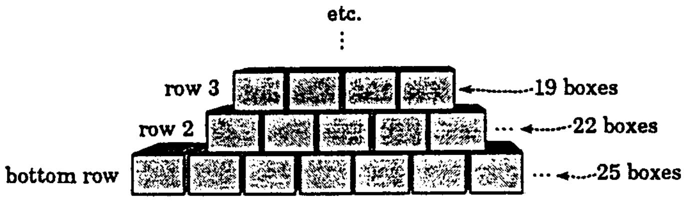
QUESTION FIVE

(a) (i) For what values of r does the geometric series $a + ar + ar^2 + \cdots$ have a limiting sum?

(ii) Write down the formula for the limiting sum of a geometric series.

(iii) By expressing $1\cdot2\dot{5}$ (i.e. 1·2555 ···) as the sum of an infinite geometric series, find the simple equivalent fraction for $1\cdot2\dot{5}$.

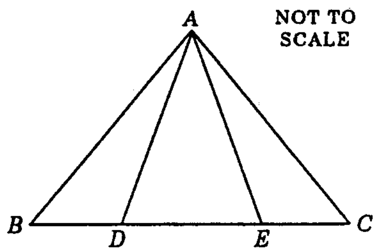
(b) Boxes in a storeroom are stacked in a pile such that there are 25 on the bottom row, 22 on the next, 19 on the next, and so on until 117 boxes are on the pile altogether.



(i) How many rows of boxes are there?

(ii) How many boxes are there on the top row?

(c)



In ΔABC, $AB = AC$ and $A\hat{D}E = D\hat{E}A$.

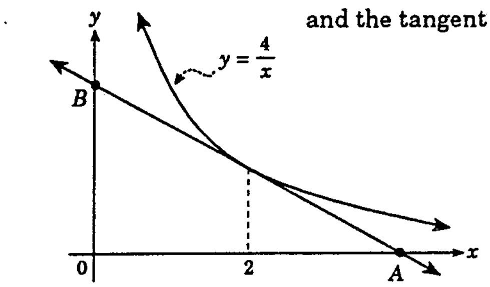
(i) Copy the diagram into your book and mark on it all the given information. Show that $D\hat{A}B = C\hat{A}E$.

(ii) Prove that triangle ABD and CAE are congruent.

(iii) Deduce that $BD = CE$.

QUESTION SIX

(a) The diagram shows a graph of the curve $y = \frac{4}{x}$ and the tangent to the curve at $x = 2$.



(i) Find the equation of the tangent at $x = 2$.

(ii) Find the area of triangle OAB.

(b) For the function defined by $f(x) = 4\sin 2x$:

(i) sketch the graph of $y = f(x)$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$;

(ii) what is the range of the function?

(iii) how many values of x exist in the range $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ such that $4\sin 2x = 3$? (Do not calculate the values.)

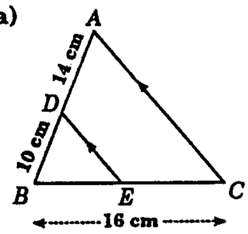
(iv) without evaluating the integral, state the value of

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\sin 2x \, dx;$$

(v) find the area enclosed by the curve $y = 4\sin 2x$, the x axis and the ordinates $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$.

QUESTION SEVEN

(a)



In the figure $AC \parallel DE$.
$BD = 10$ cm, $EC = 16$ cm and $DA = 14$ cm.

(i) Draw a sketch of the diagram in your book and prove that $\Delta ABC \parallel \Delta DBE$.

(ii) Calculate the length of BE.

(b) The gradient of a curve is $3x^2 - 2x + 1$ and the curve passes through the point (−1, 2). Find the equation of the curve.

(c) Consider the function defined by $f(x) = \sqrt{1 + x^2}$, where $-1 \le x \le 4$.

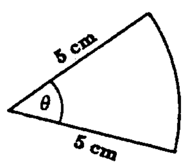
(i) Copy and complete the table of values in your book. Give answers correct to 2 decimal places.

x	−1	0	1	2	3	4
$f(x)$						

(ii) Using the Trapezoidal Rule with six function values, approximate the area under the curve $y = f(x)$ for $-1 \le x \le 4$. (Give your answer correct to one decimal place.)

QUESTION EIGHT

(a)

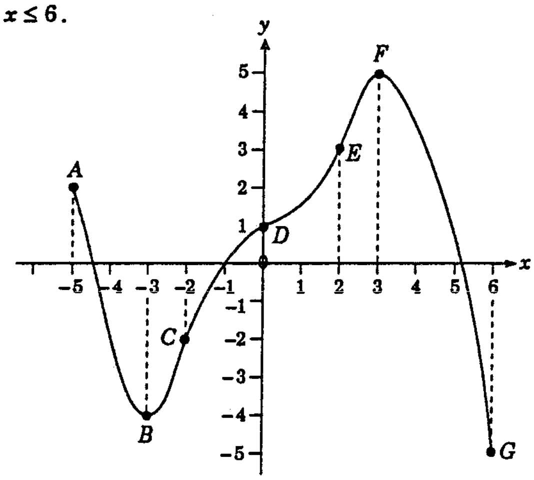


NOT TO SCALE

The diagram shows the sector of a circle of radius 5 cm. If the area of this sector is 15 cm^2, calculate the size of the angle θ.

Give your answer to the nearest degree.

(b) The diagram shows the graph of $y = f(x)$ defined in the domain $-5 \le x \le 6$.



Use the graph to answer the following questions.

(i) Name by using a letter:

(α) a point where $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$;

(β) a point where $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$.

(ii) For what values of x in the domain $-5 \le x \le 6$ is:

(α) $\dfrac{dy}{dx} > 0$ and $\dfrac{d^2y}{dx^2} < 0$;

(β) $\dfrac{dy}{dx} > 0$ and $\dfrac{d^2y}{dx^2} > 0$.

(iii) Where is the minimum value of $y = f(x)$ in the domain $-5 \le x \le 6$ and what is its value?

(c) The probability that septuagenarians will live 5 more years after 70 is $\frac{2}{5}$ for men and for women is $\frac{3}{4}$. What is the probability that for a given husband and wife:

(i) both will live 5 more years;

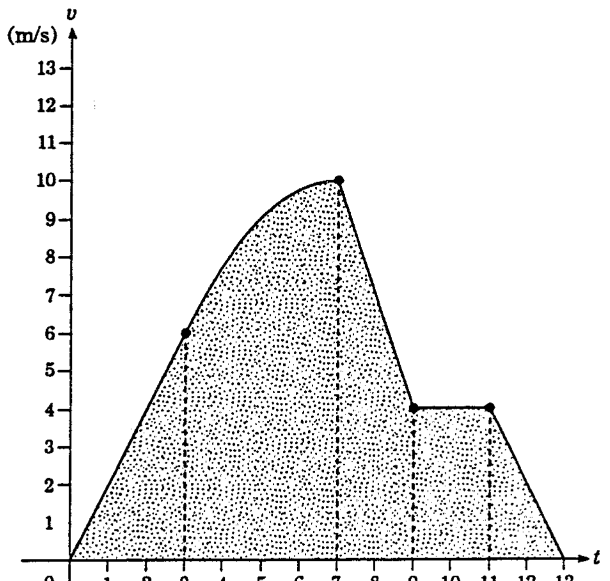
(ii) at least one of them will live 5 more years;

(iii) only the wife will live 5 more years.

QUESTION NINE

(a) Solve the equation $2\ln x = \ln(6 - x)$.

(b) A particle moves in a straight line and the graph below shows the velocity v metres per second of the particle from a fixed point at time t seconds.



(i) Is the acceleration of the particle greater at $t = 5$ or at $t = 8$?

(ii) Between what times is the acceleration zero?

(iii) What does the shaded area represent?

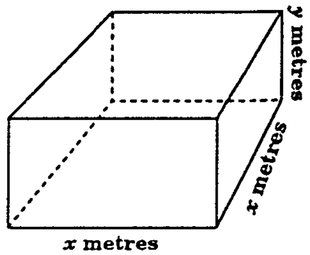
(c) Eleni invests \$1050 at the beginning of each year in a superannuation fund. Compound interest is paid at 7·5% per annum on the investment. The first \$1050 was invested at the beginning of 1985 and the last will be invested at the beginning of 2014. Calculate to the nearest dollar:

(i) the amount to which the 1985 investment will have grown by the beginning of 2015;

(ii) the amount to which the total investment will have grown by the beginning of 2015.

QUESTION TEN

(a) A closed tank with square base of length x metres and height y metres is to have a surface area of 96 square metres.

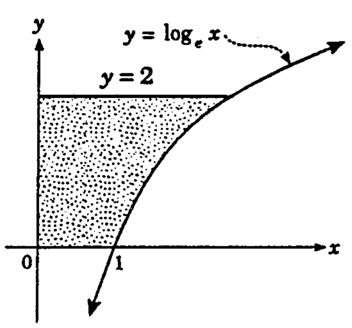


(i) Show that the volume, $V(x)$ cubic metres, of the tank is given by $V(x) = 24x - \frac{1}{2}x^3$.

(ii) Show that $0 < x \le 4\sqrt{3}$.

(iii) By graphing $V(x)$ against x, find the value of x for which the volume is greatest. What is the greatest volume?

(b)



The shaded region in the diagram is bounded by the curve $y = \log_e x$, the y axis, and the line $y = 2$.

Calculate the volume of the solid of revolution formed when this region is rotated about the y axis.

QUESTION ONE

(a) 1·75 (2dp)

(b) $5x^2 - 20y^2$

$= 5(x^2 - 4y^2)$ (difference of two squares)

$= 5(x + 2y)(x - 2y)$

(c) $\overset{2}{\not 6}\left(\frac{x-6}{\not 3_1}\right) - \overset{3}{\not 6}\left(\frac{x-1}{\not 2_1}\right) = (1) \times 6$

(multiply both sides by 6)

$2(x-6) - 3(x-1) = 6$

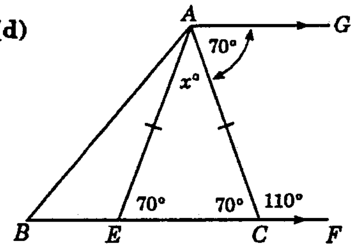
$2x - 12 - 3x + 3 = 6$

$-x - 9 = 6$

$-x = 15$

$x = -15$

(d)



$E\hat{C}A = 70°$ (alternate to $\angle GAC$, $AG \parallel BF$)

$A\hat{E}C = E\hat{C}A = 70°$ (base angles of isosceles triangle AEC)

$C\hat{A}E = 40°$ (angle sum of ΔCAE)

$\therefore \quad x = 40$.

(e) (i) $\frac{6}{\sqrt{5}+\sqrt{2}}$

$= \frac{6}{\sqrt{5}+\sqrt{2}} \times \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}}$

$= \frac{6(\sqrt{5}-\sqrt{2})}{5-2}$

$= \frac{6(\sqrt{5}-\sqrt{2})}{3}$

$= 2(\sqrt{5}-\sqrt{2})$.

(ii) $\sqrt{x} - \sqrt{y}$

$= \frac{6}{\sqrt{5}+\sqrt{2}}$

$= 2(\sqrt{5}-\sqrt{2})$ [from (i)]

$= 2\sqrt{5} - 2\sqrt{2}$

$= \sqrt{20} - \sqrt{8}$

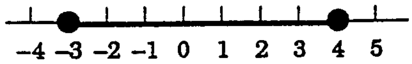
$\therefore \; x = 20, \; y = 8$.

(f) $|4x - 2| \le 14$

$-14 \le 4x - 2 \le 14$

$-12 \le 4x \le 16$

$-3 \le x \le 4$



QUESTION TWO

(a) First calculate the amount invested $= 15\%$ of \$41 500

$= \frac{15}{100} \times \$41\,500$

$= \$6225$.

$A = P\left(1 + \frac{r}{100}\right)^n$

| where $P = 6225$, $r = 10$, and $n = 5$ |

$= 6225\left(1 + \frac{10}{100}\right)^5$

$= 6225(1 \cdot 1)^5$

$= 10\,025 \cdot 43$ (2dp).

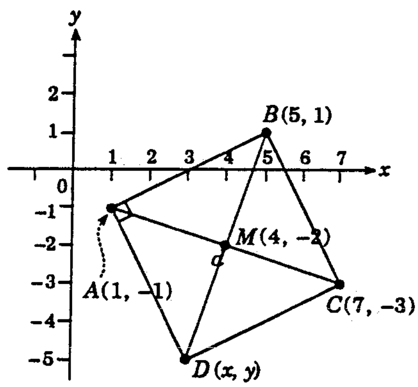
Interest $= A - P$

$= 10\,025 \cdot 43 - 6225$

$= 3800 \cdot 43$.

Therefore, the investment earns the man \$3800·43 interest.

(b) (i)



(ii) The midpoint of AC has coordinates

$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$x_1 \; y_1$ $\quad$ $x_2 \; y_2$
$A(1,-1)$ $\quad$ $C(7,-3)$

$= \left(\frac{1+7}{2}, \frac{-1-3}{2}\right)$

$= \left(\frac{8}{2}, \frac{-4}{2}\right)$

$= (4, -2)$.

Therefore, coordinates of the midpoint of AC are $(4, -2)$.

(iii) An alternative method

Let the coordinates of D be (x, y). On the diagram in part (i) join point C to D so that $ABCD$ is a parallelogram. Since the diagonals of a parallelogram bisect each other, then the midpoint of AC must have the same coordinates as the midpoint of BD.

From part (ii), we know

$d = \left|\frac{1(3) + -2(-5) + (-3)}{\sqrt{1^2 + (-2)^2}}\right|$

$= \left|\frac{3 + 10 - 3}{\sqrt{1+4}}\right|$

$= \frac{10}{\sqrt{5}}$ units.

(vi) Area of the parallelogram

= base × perpendicular height

$= AB \times d$,

where AB = distance between points A and B

$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= \sqrt{(5-1)^2 + (1 - -1)^2}$

$= \sqrt{4^2 + 2^2}$

$= \sqrt{16 + 4}$

$= \sqrt{20}$

$= 2\sqrt{5}$.

Therefore, area of parallelogram $= AB \times d$

$= 2\sqrt{5} \times \frac{10}{\sqrt{5}}$

$= 20$ unit^2.

QUESTION THREE

(a) (i) $\frac{d}{dx}[\log(3 - 2x)]$

$= \frac{1}{3-2x} \times -2$

$= \frac{-2}{3-2x}$.

(ii) $\frac{d}{dx}(x\sin x) = u\frac{dv}{dx} + v\frac{du}{dx}$ [Product Rule]

where $u = x$, $\frac{du}{dx} = 1$,

$v = \sin x$, $\frac{dv}{dx} = \cos x$

$\therefore \frac{d}{dx}(x\sin x)$

$= x \times \cos x + \sin x \times 1$

$= x\cos x + \sin x$.

(iii) $\frac{d}{dx}\left[\frac{1}{2x+1}\right]$ [$\frac{1}{a} = a^{-1}$]

$= \frac{d}{dx}\left[(2x+1)^{-1}\right]$

(Use the chain Rule to differentiate)

$= -1(2x+1)^{-2} \times 2$

$= -2(2x+1)^{-2}$

$= \frac{-2}{(2x+1)^2}$.

(b) (i) $\int (3x-2)^8 \, dx$

$= \frac{(3x-2)^9}{9 \times 3} + c$

$= \frac{1}{27}(3x-2)^9 + c$

> *Note* $\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a \times (n+1)} + c$

(ii) $\int e^{\frac{x}{2}} \, dx = \int e^{\frac{1}{2}x} \, dx$

$= \frac{e^{\frac{1}{2}x}}{\frac{1}{2}} + c$

$= 2e^{\frac{x}{2}} + c$

> *Note* $\frac{x}{2} = \frac{1}{2}x$

(c) When $x = 2$, $y = x^2 - 2x + 1$

$= 2^2 - 2 \times 2 + 1$

$= 4 - 4 + 1$

$= 1$.

Therefore, we need to find the equation of the normal at (2, 1).

$y = x^2 - 2x + 1$

$\frac{dy}{dx} = 2x - 2$

At $x = 2$, $\frac{dy}{dx} = 2 \times 2 - 2 = 2$

$\therefore$ Gradient of tangent = 2.

Gradient of normal

$= \frac{-1}{\text{Gradient of tangent}}$

$= \frac{-1}{2} \quad = -\frac{1}{2}$.

Equation is of form

$y - y_1 = m(x - x_1)$

$y - 1 = -\frac{1}{2}(x-2)$

$2(y-1) = -1(x-2)$

$2y - 2 = -x + 2$

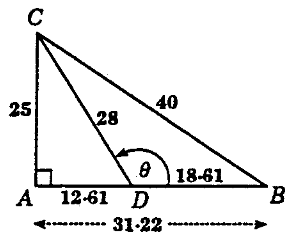
$\therefore x + 2y - 4 = 0$.

Therefore, the gradient of the normal to curve $y = x^2 - 2x + 1$ at $x = 2$ is $x + 2y - 4 = 0$.

> *Note* The normal and tangent are always perpendicular lines, i.e. $m_1 m_2 = -1$.

QUESTION FOUR

(a)



(i) In ΔADC and using Pythagoras' Theorem:

$(AD)^2 + 25^2 = 28^2$

$(AD)^2 = 28^2 - 25^2$

$AD = \sqrt{28^2 - 25^2}$

$= 12{\cdot}61$ (2dp)

In ΔABC and using Pythagoras' Theorem:

$(AB)^2 + (AC)^2 = (CB)^2$

i.e. $(AB)^2 + 25^2 = 40^2$

$(AB)^2 = 40^2 - 25^2$

$AB = \sqrt{40^2 - 25^2}$

$\approx 31{\cdot}22$ (2dp).

$AB = AD + DB$

$\therefore 31{\cdot}22 = 12{\cdot}61 + DB$

$\therefore DB = 18{\cdot}61$.

Therefore the length of AD is 12·61 cm and the length of DB = 18·61 cm.

(ii) Let $C\hat{D}B = \theta°$

Using Cosine Rule in ΔBCB we get:

$\cos\theta = \frac{28^2 + 18{\cdot}61^2 - 40^2}{2 \times 28 \times 18{\cdot}61}$

$\therefore \theta = 116°47'$.

Therefore, the size of angle CDB is $116°47'$.

(iii) Area of ΔCDB

$= \frac{1}{2} \times CD \times DB \times \sin\theta$

$= \frac{1}{2} \times 28 \times 18{\cdot}61 \times \sin 116°47'$

$\approx 232{\cdot}6$ (1 decimal place).

Therefore, the area of ΔCDB is 232·6 square centimetres.

(b) (i) Initially $\Rightarrow$ find V when $t = 0$;

$t = 0$, $V = 2000e^{-0{\cdot}005t}$

$= 2000e^{-0{\cdot}005 \times 0}$

$= 2000e^0$ [$e^0 = 1$]

$= 2000$.

Therefore, there is 2000 mL of solvent initially in the bottle.

(ii) When $t = 30$,

$V = 2000e^{-0{\cdot}005 \times 30}$

≈ 1721 (nearest unit).

Therefore, after 30 minutes there will be 1721 mL of solvent in the bottle.

Amount of solvent evaporated out of the bottle after 30 hours

= initial amount − amount left after 30 hours

= 2000 − 1721

= 279.

Therefore, after 30 hours 279 mL of solvent has evaporated.

(iii) The question is asking us to find t when $V = \frac{1}{2}$ of 2000 $= 1000$.

$V = 2000e^{-0{\cdot}005t}$

$\therefore 2000e^{-0{\cdot}005t} = 1000$

$e^{-0{\cdot}005t} = \frac{1000}{2000}$

$-0{\cdot}005t = \ln\left(\frac{1}{2}\right)$

$t = \frac{\ln\left(\frac{1}{2}\right)}{-0{\cdot}005}$

$\approx 138{\cdot}6$

> *Note* $e^x = a \Rightarrow x = \ln a$

Therefore, it will take 138·6 hours before half the initial amount of solvent has evaporated from the bottle.

(iv) $V = 2000e^{-0\cdot005t}$

$\therefore\ V = \dfrac{2000}{e^{0\cdot005t}}$

> Note $a^{-m} = \dfrac{1}{a^m}$

as t becomes very large, ($t \to \infty$), $V \to 0$ (is nearly zero).

Theoretically the bottle will never become empty, since $e^{-0\cdot005t} \neq 0$. It does, however, approach zero.

QUESTION FIVE

(a) (i) The geometric series $a + ar + ar^2 + \cdots$ has a limiting sum (S_∞) for $|r| < 1,\ r \neq 0$

i.e. $-1 < r < 1,\ r \neq 0$.

(ii) The formula for the limiting sum (S_∞) of a geometric series is

$$S_\infty = \frac{a}{1-r}$$

where a is the first term, and r is the common ratio.

(iii) $1\cdot2\dot{5} = 1\cdot2555\cdots)$

$$= 1\cdot2 + \underbrace{\frac{5}{10^2} + \frac{5}{10^3} + \frac{5}{10^4} + \cdots}_{\text{infinite sum of a geometric series with } a = \frac{5}{100},\ r = \frac{1}{10}}$$

$= 1\cdot2 + S_\infty$

$= 1\cdot2 + \dfrac{a}{1-r}$

$= 1\cdot2 + \dfrac{\frac{5}{100}}{1-\frac{1}{10}}$

$= 1\cdot2 + \dfrac{5}{90}$

$= 1\dfrac{23}{90}$.

> Check by calculator: $\dfrac{113}{90} = 1\cdot2\dot{5}$

Therefore, $1\cdot2\dot{5} = 1\dfrac{23}{90}$.

(b)

⋮	⋮	
Row 3	19 boxes	Arithmetic series
Row 2	22 boxes	
Row 1	25 boxes	

The rows of boxes form an arithmetic series

$25 + 22 + 19 + \cdots + T_n = 117$

where T_n is the number of bricks on the top row.

(i) To find the number of rows of boxes we need to find n such that $S_n = 117$.

$S_n = 117$,
where $a = 25,\ d = -3$

$S_n = \dfrac{n}{2}[2a + (n-1)d]$

$\therefore\ \dfrac{n}{2}[2a + (n-1)d] = 117$

> Formula for the sum of n terms of an arithmetic series.

$\dfrac{n}{2}[50 + (n-1)\times -3] = 117$

$\dfrac{n}{2}(50 - 3n + 3) = 117$

$\dfrac{n}{2}(53 - 3n) = 117$

$n(53 - 3n) = 234$

> Multiply both sides by 2.

$53n - 3n^2 = 234$

$3n^2 - 53n + 234 = 0$

> Solve this quadratic equation by using the formula.

$a = 3,\ b = -53,\ c = 234$

$n = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{--53 \pm \sqrt{(-53)^2 - 4\times3\times234}}{2\times3}$

$= \dfrac{53 \pm 1}{6}$

$= \dfrac{53+1}{6}$ or $\dfrac{53-1}{6}$

$= \dfrac{54}{6}$ or $\dfrac{52}{6}$

$= 9$ or $8\dfrac{2}{3}$.

But since n has to be a positive integer, then $n = 9$. Therefore, there are 9 rows of boxes.

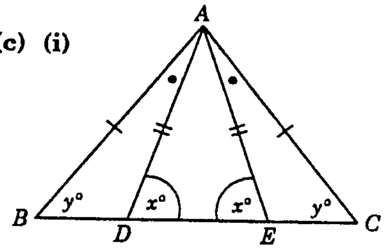
(ii) The top row has T_n boxes, where $T_n = a + (n-1)d$

> $a = 25, d = -3, n = 9$

$\therefore\ T_9 = 25 + (9-1)\times -3$
$= 25 + 8 \times -3$
$= 25 - 24$
$= 1$.

Therefore the top row has 1 box.

(c) (i)



Let $A\hat{D}E = D\hat{E}A = x°$

Let $A\hat{B}C = y°$

$\therefore\ A\hat{B}C = B\hat{C}A = y°$ (base angles of isosceles ΔABC)

$D\hat{A}B + y° = x°$ ($A\hat{D}E$ is exterior angle of ΔABD)

$\therefore\ D\hat{A}B = (x - y)°$

$C\hat{A}E + y° = x°$ ($D\hat{E}A$ is exterior angle of ΔECA)

$\therefore\ C\hat{A}E = (x - y)°$.

Therefore, $D\hat{A}B = C\hat{A}E$ [both $(x - y)°$]

(ii) In triangles ABD and CAE,

$AB = AC$ (S) (given in data)

$D\hat{A}B = C\hat{A}E$ (A) [proved in part (i)]

$AD = AE$ (S) (triangle ADE is isosceles)

$\therefore\ \Delta ABD \equiv \Delta CAE$ (SAS).

(iii) $BD = CE$ (corresponding sides in congruent Δs ABD and CAE)

QUESTION SIX

(a) (i) When $x = 2$,

$$y = \frac{4}{x} = \frac{4}{2} = 2.$$

Therefore we need to find the equation of the tangent to the curve $y = \dfrac{4}{x}$ at the point (2, 2).

$y = \dfrac{4}{x}$

$y = 4x^{-1}$

$\dfrac{dy}{dx} = -4x^{-2}$

$= \dfrac{-4}{x^2}$

> Note $\dfrac{1}{a} = a^{-1}$

when $x = 2$,

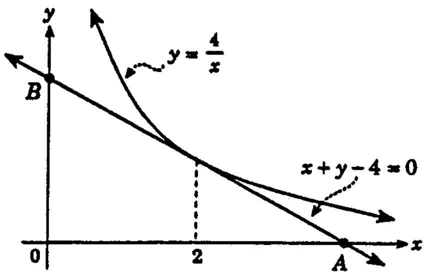
$\dfrac{dy}{dx} = \dfrac{-4}{2^2} = \dfrac{-4}{4} = -1$

$\therefore$ the gradient of the tangent is -1.

Then $y - y_1 = m(x - x_1)$
$y - 2 = -1(x - 2)$
$y - 2 = -x + 2$

$\therefore\ x + y - 4 = 0$ is equation of tangent.

(ii)

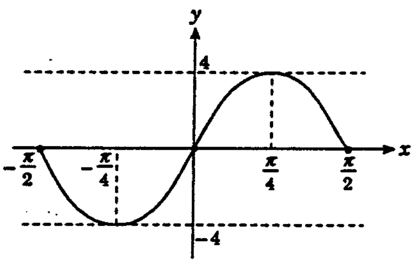


At A (x intercept), $y = 0$
subst. $y = 0$ in $x + y - 4 = 0$
we get $x = 4$, $\therefore$ $OA = 4$ units.

At B (y intercept), $x = 0$
subst. $y = 0$ in $x + y - 4 = 0$
we get $y = 4$, $\therefore$ $OB = 4$.

Area of $\Delta OAB = \frac{1}{2} \times OA \times OB$
$= \frac{1}{2} \times 4 \times 4$
$= 8$ unit^2.

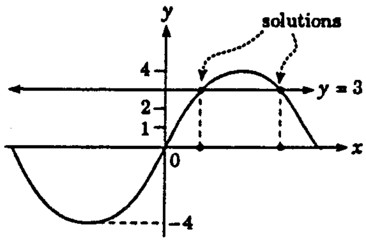
(b) (i) $f(x) = 4\sin 2x$
For $y = f(x)$:
the amplitude = 4, and
the period $= \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$.



(ii) The range of the function $y = f(x)$ is $-4 \le y \le 4$.

> *Note* The range of a function is the possible y values.

(iii) To solve $4\sin 2x = 3$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, we need to graph $y = 4\sin 2x$ and $y = 3$ on the same cartesian plane. The point(s) where the two curves intersect are the solutions of the equation $4\sin 2x = 3$.

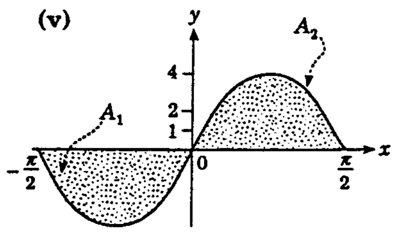


Therefore, 2 values of x exist such that $4\sin 2x = 3$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

(iv) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\sin 2x \, dx = 0$

because, as it can be seen from the graph of $y = f(x)$ in (i), $f(x) = 4\sin 2x$ is an *odd* function [integrated over the interval $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$], which has point symmetry about the origin.

(v)



$A_1 = \left| \int_{-\frac{\pi}{2}}^{0} 4\sin 2x \, dx \right|$

> *Note* the absolute value. This is because A_1 lies below the x axis.

$= \left| \left[\frac{-4\cos 2x}{2} \right]_{-\frac{\pi}{2}}^{0} \right|$

$= \left| \left[-2\cos 2x \right]_{-\frac{\pi}{2}}^{0} \right|$

> $\cos 0 = 1$
> $\cos \pi = 1,$
> $\cos(-\pi) = -1$

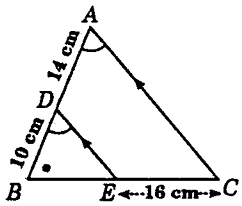
$= \left| (-2\cos 0) - (-2\cos - \pi) \right|$
$= \left| (-2) - (-2 \times -1) \right|$
$= \left| -2 - 2 \right|$
$= \left| -4 \right|$
$= 4.$

Therefore, $A_1 = 4$ square units.
$A_2 = A_1$ (function is odd, areas are equal in size).
Therefore,
area $= A_1 + A_2$
$= (4 + 4)$
$= 8$ square units.

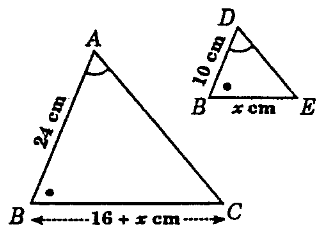
> *Note* You could have started with A_2, then calculated the required area.

QUESTION SEVEN

(a) (i)



Separate the two triangles for (ii).



In Δs ABC and DBE,
$A\hat{B}C = D\hat{B}E$ (common angle)
$C\hat{A}B = E\hat{D}B$ (corresponding angles, $AC \parallel DE$)
$\therefore$ $\Delta ABC \parallel \Delta DBE$ (equiangular)

(ii) Let the length of interval $BE = x$ cm

$\therefore BC = (16 + x)$ cm.

$\dfrac{BC}{BE} = \dfrac{AB}{DB}$ (from diagram)

(The ratio of the corresponding sides of similar triangles are equal.)

$\therefore \quad \dfrac{16 + x}{x} = \dfrac{24}{10}$ ☒

$10(16 + x) = 24x$

$160 + 10x = 24x$

$160 = 14x$

$\therefore \quad x = \dfrac{160}{14} = 11\dfrac{3}{7}.$

Therefore, the length of BE is $11\dfrac{3}{7}$ cm $\approx 11 \cdot 4$ cm (1 decimal place).

(b) The gradient of any curve $= \dfrac{dy}{dx}$.

Therefore, to find the equation of the curve we need to use

$\dfrac{dy}{dx} = 3x^2 - 2x + 1$

$\therefore \; y = \int (3x^2 - 2x + 1)\, dx$

$y = x^3 - x^2 + x + c$

> Substitute initial conditions to calculate c, i.e. passes through the point $(-1, 2)$
> $\Rightarrow$ when $x = -1, y = 2$

When $x = -1, y = 2$,

$\Rightarrow 2 = (-1)^3 - (-1)^2 + (-1) + c$

$\Rightarrow 2 = -1 - 1 - 1 + c$

$\Rightarrow 2 = -3 + c$

$\Rightarrow c = 5.$

Therefore, the equation of the curve is $x^3 - x^2 + x + 5$.

(c) **(i)** Using the calculator, substitute the given x values into $f(x) = \sqrt{1 + x^2}$ to find the $y = f(x)$ values.

x	−1	0	1	2	3	4
$f(x)$	1·41	1·00	1·41	2·24	3·16	4·12
	y_1	y_2	y_3	y_4	y_5	y_6

(ii)

Area $= \dfrac{h}{2}\big[(y_1 + y_6) + 2(y_2 + y_3 + y_4 + y_5)\big]$

> *Note* $h = 1$

$= \dfrac{1}{2}\big[(1 \cdot 41 + 4 \cdot 12) + 2(1 \cdot 00 + 1 \cdot 41 + 2 \cdot 24 + 3 \cdot 16)\big]$

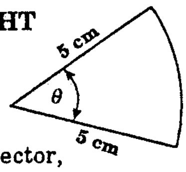
$= 10 \cdot 575$

$\approx 10 \cdot 6$ (1dp).

Therefore, the required area is 10·6 unit².

QUESTION EIGHT

(a)



The area of a sector, A, is given by the formula

$$A = \frac{1}{2} r^2 \theta$$

where r is the radius and θ is the size of the angle in radians.

> $r = 5, \;\; A = 15$

$\therefore \; \dfrac{1}{2} \times 5^2 \times \theta = 15$

$\dfrac{25}{2}\theta = 15$

$\theta = 1 \cdot 2.$

> *Note* θ is in radians. Need to convert it to degrees:
> 1 rad. $= \dfrac{180^\circ}{\pi}$

Therefore,

$\theta = 1 \cdot 2 \times \dfrac{180^\circ}{\pi}$

$= 68 \cdot 754\,935^\circ$

$\approx 69^\circ$ (nearest degree).

Therefore, the angle of the sector is 69°.

(b) **(i)** (α) The point where $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2} > 0$ is a minimum turning point. Therefore, from the diagram, it is point B.

(β) The point where $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2} < 0$ is a minimum turning point. Therefore, from the diagram, it is point F.

(ii) (α) $\dfrac{dy}{dx} > 0$ means that the function is increasing, and $\dfrac{d^2y}{dx^2} < 0$ means the function is concaving down.

From the diagram, this occurs between points C and D, i.e. for $-2 < x < 0$ and between points E and F, i.e. for $2 < x < 3$.

(β) $\dfrac{d^2y}{dx^2} > 0$ means the function is concaving up and $\dfrac{dy}{dx} > 0$ means it is increasing.

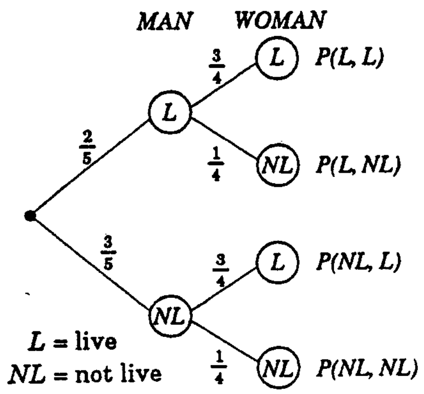
From the diagram, this occurs between points B and C, i.e. for $-3 < x < -2$ and between points D and E, i.e. for $0 < x < 2$.

(iii) The minimum value of $y = f(x)$ in the domain $-5 \le x \le 6$ occurs at G and is −5 (y value as read from diagram at G).

(c) Use a probability tree diagram to answer this question.

Note The probability that a man will live 5 more years is $\frac{2}{5}$, therefore the probability that he *will not* live 5 more years is $(1 - \frac{2}{5}) = \frac{3}{5}$.

Similarly, the probability that his wife will live 5 more years is $\frac{3}{4}$, therefore the probability that she will not live 5 more years is $(1-\frac{3}{4})=\frac{1}{4}$.



(i) $P(\text{both will live}) = P(L, L)$
$$= \frac{2}{5}\times\frac{3}{4}$$
$$= \frac{3}{10}$$

(ii) $P(\text{at least one will live})$
$$= 1 - P(\text{both die})$$
$$= 1 - P(NL, NL)$$
$$= 1-\frac{3}{5}\times\frac{1}{4} = 1-\frac{3}{20}$$
$$= \frac{17}{20}$$

OR $P(\text{at least one alive})$
$$= P(L, L) + P(L, NL) + P(NL, L)$$
$$= \left(\frac{2}{5}\times\frac{3}{4}\right)+\left(\frac{2}{5}\times\frac{1}{4}\right)+\left(\frac{3}{5}\times\frac{3}{4}\right)$$
$$= \frac{3}{10}+\frac{1}{10}+\frac{9}{20}$$
$$= \frac{17}{20}$$

(iii) $P(\text{only wife will live})$
$$= P(NL, L)$$
$$= \frac{3}{5}\times\frac{3}{4}$$
$$= \frac{9}{20}$$

QUESTION NINE

(a) $2\ln x = \ln(6-x)$

Note the rule $a\log b = \log b^a$

$\therefore \quad \ln x^2 = \ln(6-x)$

$\therefore \quad x^2 = 6 - x$

If $\log a = \log b$, then $a = b$

$\therefore \quad x^2 + x - 6 = 0$

$\therefore \quad (x+3)(x-2) = 0$

$\therefore \quad x = -3$ or 2.

Check with original equation:

$x = -3$ LHS $= 2\ln(-3)$ which does not exist (try it on your calculator)

$x = 2$ LHS $= 2\ln 2$
$$= \ln 2^2$$
$$= \ln 4$$
RHS $= \ln(6-2)$
$$= \ln 4$$

$\therefore x = 2$ is the only solution.

(b) (i) At $t = 5$, the gradient of the tangent is positive and at $t = 8$ the gradient of the line is negative. Therefore the acceleration is greater at $t = 5$

Note The gradient of the tangent at any point on the velocity graph is $\frac{dv}{dt}$. But $\frac{dv}{dt}$ = acceleration of the particle at any point t. At $t = 5$, $\frac{dv}{dt} = a = 3$ and at $t = 8$, $\frac{dv}{dt} = a = -3$.

(ii) The acceleration is zero when $\frac{dv}{dt} = 0$, i.e. where the gradient of the tangent to the velocity graph is zero. This occurs in the interval $9 \le t \le 11$ (velocity is constant).

Note v = constant, $\frac{dv}{dt} = 0$, therefore acceleration = 0.

(iii) The shaded area $\int_0^{13} v\,dt$ represents the distance travelled by the particle in the first 13 seconds.

Note The fact $\frac{dx}{dt} = v$ implies $x = \int v\,dt$

(c) (i) The 1985 investment will be invested for 30 years (from the beginning of 1985 to the beginning of 2015, 30 years).

$$A = P\left(1+\frac{r}{100}\right)^n$$

where $r = 7{\cdot}5$, $P = 1050$, $n = 30$

$$= 1050\left(1+\frac{7{\cdot}5}{100}\right)^{30}$$
$$= 1050(1{\cdot}075)^{30}$$
$$= 9192{\cdot}70.$$

Therefore, the 1985 investment grows to \$9193 (nearest dollar) by the beginning of 2015.

(ii) *Note* The interest factor $= 1 + \frac{r}{100} = 1 + \frac{7 \cdot 5}{100} = 1 \cdot 075$.

First \$1050 invested for 30 years amounts to $A_1 = 1050(1 \cdot 075)^{30}$
Second \$1050 invested for 29 years amounts to $A_2 = 1050(1 \cdot 075)^{29}$
Third \$1050 invested for 28 years amounts to $A_3 = 1050(1 \cdot 075)^{28}$
⋮ ⋮ ⋮ ⋮
The 30th (last) \$1050 invested for 1 year amounts to $A_{30} = 1050(1 \cdot 075)^1$

Total amount of superannuation
$= A_{30} + A_{29} + A_{28} + \cdots + A_1$
$= \underbrace{1050(1 \cdot 075) + 1050(1 \cdot 075)^2 + 1050(1 \cdot 075)^3 + \cdots + 1050(1 \cdot 075)^{30}}$
forms a geometric series
with $a = 1050(1 \cdot 075)$, $r = 1 \cdot 075$, $n = 30$

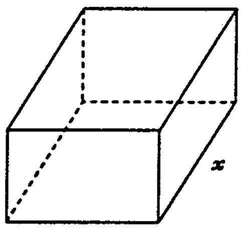
$$= S_{30} = \frac{a(r^n - 1)}{r - 1}$$
$$= \frac{1050(1 \cdot 075)\left[1 \cdot 075^{30} - 1\right]}{1 \cdot 075 - 1}$$
$$= \frac{1050(1 \cdot 075) \times (1 \cdot 075^{30} - 1)}{0 \cdot 075} = 116\,712 \cdot 08.$$

Therefore, the amount to which the total investment will have grown by the beginning of 2015 is \$116 712·08.

QUESTION TEN

(a)

> *Note* The box is closed, therefore it has six faces.



(i) Given that the surface area, $A = 96 \text{ m}^2$, we get:

$$2x^2 + 4xy = 96$$

> Surface area
> $= x^2 + x^2 + xy + xy + xy + xy$
> $= 2x^2 + 4xy$

$$4xy = 96 - 2x^2$$
$$y = \frac{96 - 2x^2}{4x}$$

Volume $= x \times x \times y$

> $\therefore \quad V = x^2 y$
> Substitute $y = \frac{96 - 2x^2}{4x}$

$$V(x) = x^2 \times \frac{96 - 2x^2}{4x}$$
$$= \frac{x^2(96 - 2x^2)}{4x}$$
$$= \frac{96x - 2x^3}{4}$$
$$= 24x - \frac{1}{2}x^3$$
$$\therefore V(x) = 24x - \frac{1}{2}x^3.$$

(ii) $V(x) \geq 0$
(i.e. volume is positive—cannot have negative volume)

$$24x - \frac{1}{2}x^3 \geq 0$$
$$48x - x^3 \geq 0$$
(multiply both sides by 2)
$$x(48 - x^2) \geq 0,$$
But $x > 0$, (x is side of base)
$\therefore 48 - x^2 \geq 0 \quad (+ \times + = +)$.
That is $x^2 \leq 48$,
$$-\sqrt{48} \leq x \leq \sqrt{48}.$$
But remembering $x > 0$,
then $x \leq \sqrt{48}$,
i.e. $0 < x \leq \sqrt{48}$,
$0 < x \leq 4\sqrt{3}$.

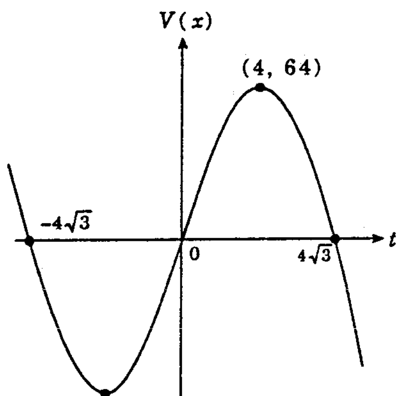
(iii) We need to graph
$$V(x) = 24x - \frac{1}{2}x^3$$
for $0 < x \leq 4\sqrt{3}$.

Find stationary points:
$$V(x) = 24x - \frac{1}{2}x^3$$
$$V'(x) = 24 - \frac{3}{2}x^2$$
$$V''(x) = -3x$$

For stationary points
$V'(x) = 0$:
$$24x - \frac{3}{2}x^2 = 0$$
$\therefore \quad \frac{3}{2}x^2 = 24$
$\therefore \quad 3x^2 = 48$
$\therefore \quad x^2 = 16$
$\therefore \quad x = \pm 4$

When $x = 4$,
$V''(4) = -12 < 0$
$\Rightarrow$ maximum
when $x = -4$,
$V''(-4) = 12 > 0$
$\Rightarrow$ minimum
when $x = 4$,
$$V(4) = 24(4) - \frac{1}{2}(4)^3$$
$$= 64$$
i.e. (4, 64) is maximum;
when $x = -4$,
$$V(-4) = 24(-4) - \frac{1}{2}(-4)^3$$
$$= -64$$
i.e. (−4, −64) is minimum.

Find x intercepts [i.e. find x such that $V(x) = 0$]:
$$24x - \frac{1}{2}x^3 = 0$$



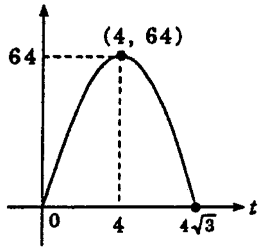
(multiply both sides by 2)

$$48x - x^3 = 0$$
$$x(48 - x^2) = 0$$
$$x\left(\sqrt{48} - x\right)\left(\sqrt{48} + x\right) = 0$$

i.e. $x = 0,\ \sqrt{48}, -\sqrt{48}$
or $x = 0,\ 4\sqrt{3},\ -4\sqrt{3}$.

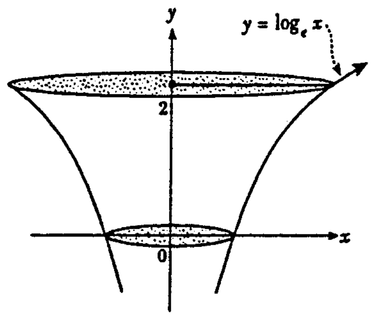
However, we only need to graph $V(x)$ for $0 < x \le 4\sqrt{3}$.

i.e.



Therefore, the value that gives maximum volume is $x = 4$. The maximum volume is 64 cubic metres (from graph).

(b)



To find the volume generated when an area is rotated about the y axis, we use the formula

$$V = \pi \int_a^b x^2\, dy$$

When $y = \log_e x$, then $x = e^y$.

$$= \pi \int_0^2 (e^y)^2\, dy$$
$$= \pi \int_0^2 e^{2y}\, dy$$
$$= \pi \left[\frac{e^{2y}}{2}\right]_0^2$$
$$= \frac{\pi}{2}\left[e^{2y}\right]_0^2$$
$$= \frac{\pi}{2}\left[e^4 - e^0\right]$$
$$= \frac{\pi}{2}\left[e^4 - 1\right] \text{ unit}^3.$$

(2)(iii) (Continued)

the midpoint of AC is the point (4, –2).

Find the midpoint of BD:

$$\begin{matrix} x_1\ y_1 & x_2\ y_2 \\ B(5, 1), & D(x, y) \end{matrix}$$

Therefore,

$$\frac{x+5}{4} = 4 \quad\text{and}\quad \frac{y+1}{2} = -2$$

i.e. $x + 5 = 8$
and $y + 1 = -4$.

Solving the equations we get $x = 3$ and $y = -5$.

Therefore the coordinates of D are (3, – 5).

[*See Chapter 2 for other method.*]

(iv) $\begin{matrix} x_1\ y_1 & x_2\ y_2 \\ A(1, -1), & B(5, 1) \end{matrix}$

$$\text{Gradient of } AB = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - -1}{5 - 1} = \frac{2}{4} = \frac{1}{2}.$$

Equation of AB is of the form $y - y_1 = m(x - x_1)$

$$y - -1 = \frac{1}{2}(x - 1)$$
$$2(y + 1) = 1(x - 1)$$
$$2y + 2 = x - 1$$
$$2y = x - 3.$$

Therefore, line AB has equation $x - 2y = 3$.

(v) To calculate the perpendicular distance from a point to a line, we use the formula:

$$d = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

Line AB has equation
$x - 2y = 3$
i.e. $x - 2y - 3 = 0$
(must be in general form)
$a = 1,\ b = -2, c = -3$

Point D has coordinates (3, –5), i.e. $x_1 = 3,\ y_1 = -5$.