C.E.M.TUITION

FINAL TRIAL HSC EXAMINATION 1995

MATHEMATICS

2/3 UNIT

COMMON PAPER

Total time allowed - THREE hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES :

- Attempt ALL questions.
- All questions are of equal value.
- All necessary working should be shown in every question.
- Full marks may not be awarded for badly arranged work.
- Standard integrals are on the inside cover.
- Approved silent calculators may be used.
- You must hand in a blank page if a question is unanswered.

Question 1 **Marks**

(a) Find the value of $\dfrac{x+y}{\sqrt{x+y}}$ if $x = 6.53$ and $y = 2.05$. **2**

Give your answer to 3 significant figures.

(b) Evaluate $\dfrac{5^{\frac{1}{3}}.5^{0}.25^{\frac{2}{3}}}{125^{\frac{1}{3}}}$ **2**

(c) Find the average of the roots of the quadratic equation **2**

$3x^2 + 10x - 1 = 0$

(d) Expand and simplify $(\sqrt{2} - 1)^2 - (\sqrt{2} + 1)^2$ **3**

(e) Solve the equation $\dfrac{2x+1}{x-1} = \dfrac{2}{3}$ **3**

Question 2 **Marks**

The points A and B have co-ordinates $(3,-6)$ and $(5,2)$ respectively.

(a) Show that the equation of the line l joining the points A and B is $4x-y-18=0$ **2**

(b) The line k is drawn through B with a gradient of 2. **2**

Show that the equation of k is $2x-y-8=0$.

(c) Show that the line k passes through $P(2,-4)$. **1**

(d) Draw a neat sketch showing A, B, P, k, l. **2**

(e) Find the perpendicular distance between P and l. **2**

(f) Find the exact area of ΔABP. **3**

Question 3 **Marks**

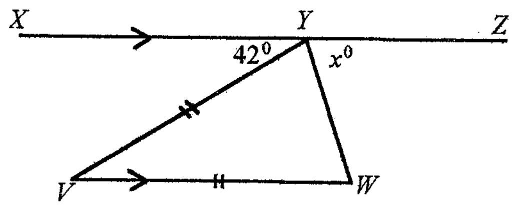
(a) Differentiate, with respect to x :

(i) $x^5 - \frac{1}{x^2}$ **2**

(ii) $x \ln 2x$ **2**

(iii) $\frac{x-2}{2x+1}$ **2**

(b) **3**



Given that $XZ \parallel VW$ and $VY = VW$, find

the value of x (giving reasons).

(c) Find :

(i) $\int \frac{3x}{x^2+1}\, dx$ **1**

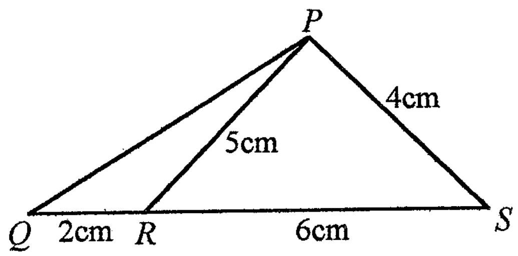
(ii) $\int_0^1 e^{2x}\, dx$ **2**

Question 4 **Marks**

(a) Without the use of the calculator, evaluate **4**

(i) $\cos \angle PRS$

(ii) PQ^2



(b) For the parabola : $(x-3)^2 = 16(y+1)$, find

(i) the coordinates of its vertex, **1**

(ii) the coordinates of the focus and **2**

(iii) the equation of the directrix **1**

(c) Elections are held in towns A and B of a certain state The results are as :

	Liberals	Democrats	Labor
Town A	40%	30%	30%
Town B	20%	70%	10%

A political news reporter chooses one town at random, then interviews two voters at random in that town. Find the probabilities that

(i) neither vote Labor **2**

(ii) at least one votes Democrats. **2**

Question 5 **Marks**

(a) Calculate the smallest positive integer p for which the equation $3x^2 + px + 7 = 0$ has real roots. **3**

(b) Prove the identity : $\frac{1+\sin x}{\cos x} + \frac{\cos x}{1+\sin x} = 2\sec x$ **3**

(c) The first term of a geometric progression is a, and the common ratio, r, is positive.

(i) Given that the sum of the second and third terms is $\frac{10a}{9}$, calculate the value of r. **3**

(ii) Find its limiting sum. **1**

(d) Solve the equation for $0^0 \le \theta \le 360^0$ where **2**

$$2\cos\theta - 5 = 14\cos\theta$$

Question 6 **Marks**

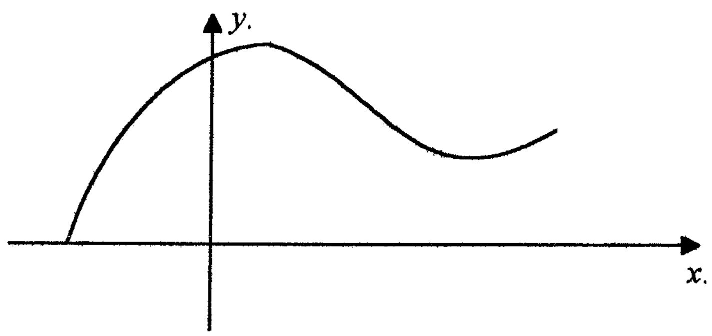
(a) A function is given by $y = 3x^4 - 4x^3 - 12x^2 + 1$

(i) Find the coordinates of the stationary points and determine their nature. 4

(ii) Find the point(s) of inflection. 1

(iii) Draw a sketch of the function showing all the important features. 2

(b) The diagram shows the graph of a certain function $y = f(x)$.



(i) Copy the graph into your Writing Booklet. 1

(ii) On the same set of axes, draw a sketch of the derivative $f'(x)$ of the function. 2

(c) Given that the area of a sector is 0.6 cm^2, and its radius is 2 cm, find the value of the angle subtended at its centre in degrees and minutes. 2

Question 7 **Marks**

(a) The intensity, I, of radiation passing through a wall of a certain material of thickness of t cm, is given by $I = I_0 e^{-\lambda t}$, where λ is the coefficient of radiation-absorption of the material.

(i) Show that it satisfies the equation $\frac{dI}{dt} = -\lambda I$ **1**

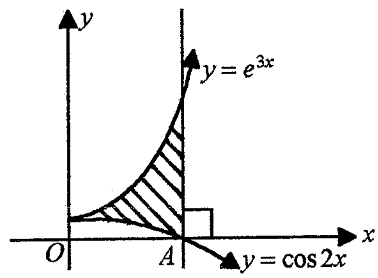
(ii) Find λ if the thickness of 20 cm reduces the intensity by 7%. **2**

(iii) How thick would the wall have to be to reduce the radiation by 75% ? **2**

(b) The diagram shows part of the graphs of $y = e^{3x}$ and $y = \cos 2x$. Find **4**

(i) the x–coordinates of A,

(ii) the area of the shaded region. (Answer to 2 decimal places).

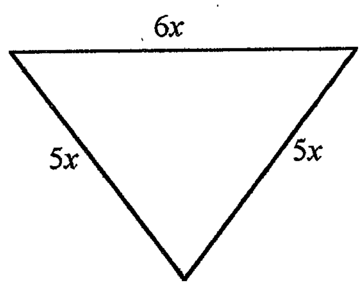


(c) Express y in terms of x, without $\log_e$ symbols, given that **3**

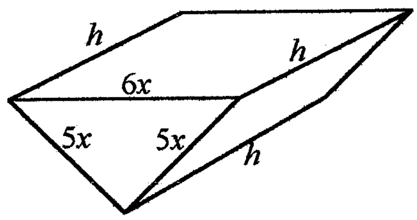
$$x + \log_e y = \log_e 12$$

Question 8 **Marks**

(a) Show that the area of the triangle below is $12x^2$ units^2. **2**



(b) A container with an open rectangular top is constructed from four pieces of cardboard sheet. The two end pieces are isosceles triangles with sides $6x$ cm, $5x$ cm and $5x$ cm as shown below.



The two side pieces are rectangles of length h cm and width $5x$ cm.
The total amount of cardboard sheet used is 450 cm^2.

(i) Using the result in part (a) or otherwise, show that $h = \dfrac{45-2\cdot4x^2}{x}$. **2**

(ii) Show that the volume of the container, V cm^3, is given by **2**

$$V = 540x - 28\cdot8x^3.$$

(iii) Find the value of x for which V has a stationary value. Find this value of V and determine whether it is a maximum or a minimum. **6**

Question 9 **Marks**

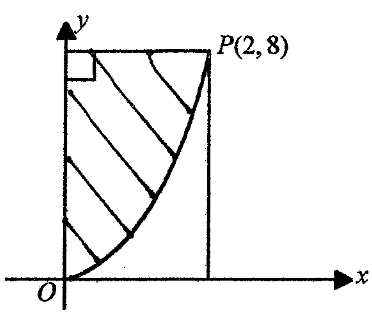
(a) A **sinking fund** is a fund into which periodic payments are made in order to accumulate a specified sum at some time in the future. Find how much should be invested by a company in a sinking fund at the end of each quarter for 5 years to accumulated \$200 000 if interest is earned at 5% per annum compounded quarterly. **4**

(You may assume the formula).

(b) Fill in the table below and find the approximation to $\int_0^{\frac{\pi}{2}} \sin^2 2x \, dx$ by Simpson's Rule. **4**

x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
$f(x)$			1.00		

(c) **4**



The sketch shows the arc of the curve $y = x^3$ from the origin, O, to the point $P(2, 8)$.

Calculate the volume of the solid formed when the shaded region is rotated about the y–axis.

Question 10 **Marks**

(a) A particle P starts from a point O and moves in a straight line so that its velocity, v ms^{-1}, is given by $v = 8 + 2t - t^2$, where t is the time in seconds after leaving O. Calculate

(i) the values of t at the instants when the magnitude of the acceleration is 1 ms^{-2}, **3**

(ii) the distance of P from O when $t = 5$, **2**

(iii) the distance of P from O when P comes to rest. **2**

(iv) the total distance travelled by P in the first 5 seconds. **1**

(b) Liquid is poured into a container at a rate of 12 cm^3/s. The volume of the liquid in the container is V cm^3, where $V = \frac{1}{2}(h^2 + 4h)$ and h cm is the height of liquid in the container. Find, when $V = 16$,

(i) the value of h, **2**

(ii) the rate at which h is increasing. **2**

Question 1 — **Marks**

(a) 2.93 — 2

(b) $\dfrac{5^{\frac{1}{3}}.5^{0}.5^{\frac{4}{3}}}{5^{\frac{3}{3}}}$ — 1

$= 5^{\frac{5}{3}-\frac{3}{3}} = 5^{\frac{2}{3}}$ — 1

(c) $3x^2 + 10x - 1 = 0$

If α, β are the roots, then

$\alpha + \beta = -\frac{b}{a} = -\frac{10}{3}$ — 1

Average of roots $= \frac{\alpha+\beta}{2} = -\frac{10}{6}$ — 1

(d) $(\sqrt{2} - 1)^2 - (\sqrt{2} + 1)^2$

$= (2 - 2\sqrt{2} + 1) - (2 + 2\sqrt{2} + 1)$ — 1

$= 3 - 2\sqrt{2} - 3 - 2\sqrt{2}$ — 1

$= -4\sqrt{2}$ — 1

(e) $\frac{2x+1}{x-1} = \frac{2}{3}$

$3(2x+1) = 2(x-1)$ — 1

$6x + 3 = 2x - 2$

$4x = -5$ — 1

$x = -\frac{5}{4}$ — 1

Question 2

(a) $m_{AB} = 4$ — 1

$l: y - 2 = 4(x-5)$

$y - 2 = 4x - 20$ — 1

$4x - y - 18 = 0$

(b) $k: y - 2 = 2(x-5)$ — 1

$y - 2 = 2x - 10$ — 1

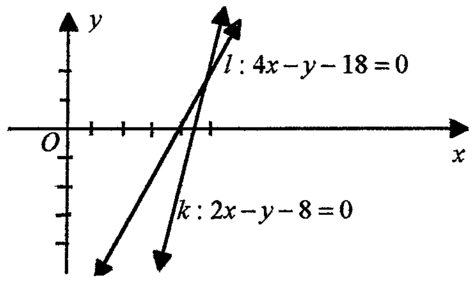
$2x - y - 8 = 0$

(c) Sub. $x = 2, y = -4$ into

L.H.S $= 2(2) - (-4) - 8$ — 1

$= 4 + 4 - 8 = 0 =$ R.H.S.

(d) — 0.5 ea



(e) $d = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2+B^2}}$

$= \dfrac{|4.2 + (-1)(-4) - 18|}{\sqrt{4^2 + (-1)^2}}$ — 1

$= \dfrac{6\sqrt{17}}{17}$ — 1

(f) Area of ΔABP.

$= \frac{1}{2} \times AB \times h$ — 1

$AB = \sqrt{(3-5)^2 + (-6-2)^2}$

$= \sqrt{68} = 2\sqrt{17}$ — 1

Area $= \frac{1}{2} \times 2\sqrt{17} \times \frac{6}{\sqrt{17}}$

$= 6$ units^2 — 1

Question 3

(a) (i) $y = x^5 - x^{-2}$ — 1

$y' = 5x^4 + 2x^{-3}$

$= 5x^4 + \frac{2}{x^3}$ — 1

(ii) $y = x \ln 2x$

$y' = x.\frac{2}{2x} + \ln 2x.1$ — 1

$= 1 + \ln 2x$ — 1

(iii) $y = \frac{u}{v}$

$y = \dfrac{vu' - uv'}{v^2}$

$= \dfrac{(2x+1).1 - (x-2).2}{(2x+1)^2}$ — 1

$= \frac{5}{(2x+1)^2}$ 1

(b) $\angle YVW = 42^0$ (Alternate $\angle$, $XZ \| VW$) 1

$\angle VYW = (180 - 42)^0 \div 2 = 69^0$
(Angle sum of isosceles Δ) 1

$x = 180^0 - 42^0 - 69^0 = 69^0$
(Straight angle) 1

(c) (i) $\frac{3}{2} \int \frac{2x}{x^2+1}\, dx$

$= \frac{3}{2}\ln(x^2+1) + c$ 1

(ii) $\left[\frac{e^{2x}}{2}\right]_0^1 = \frac{1}{2}[e^1 - e^0]$ 1

$= \frac{1}{2}[e^2 - 1]$ 1

Question 4

(a) (i) Using cosine rule,

$a^2 = b^2 + c^2 - 2bc\cos A$ 1

$\cos \angle PRS = \frac{6^2+5^2-4^2}{2\times6\times5} = \frac{3}{4}$ 1

(ii) In ΔPQR, using cosine rule

$r^2 = p^2 + q^2 - 2pq\cos \angle PRQ$

Note : $\angle PRQ$ =Obtuse $\angle PRS$ 1

$PQ^2 = 25 + 4 - 2.5.2(-\frac{3}{4})$ 1

$= 44$ units.

(b) (i) $V(3,-1)$ 1

(ii) focal length , $a = 4$ 1

$S(3,3)$ 1

(iii) $y = -5$ 1

(c) (i) P(neither Labor)= $0.4 \times 0.3 + 0.2 \times 0.7$

$= 0.26$ 2

(ii) P(at least one Democrats)

$= 0.3 \times 0.4 + 0.3 \times 0.3 + 0.5 \times 0.3 \times 0.3$

$+ 0.7 \times 0.2 + 0.7 \times 0.1 + 0.5 \times 0.7 \times 0.7$

$= 0.71$ 2

Question 5

(a) For real roots, $\Delta \geq 0$

i.e. $p^2 - 4.3.7 \geq 0$ 1

$(p + \sqrt{84})(p - \sqrt{84}) \geq 0$

Test with $p = 0$

$p \leq -\sqrt{84}$ or $p \geq \sqrt{84}$ 1

$p = 10$ (positive integer) 1

(b) L.H.S. $= \frac{(1+\sin x)^2+\cos^2 x}{\cos x(1+\sin x)}$ 1

$= \frac{1+2\sin x+\sin^2 x+\cos^2 x}{\cos x(1+\sin x)}$

$= \frac{2+2\sin x}{\cos x(1+\sin x)}$ 1

$= \frac{2(1+\sin x)}{\cos x(1+\sin x)}$

$= \frac{2}{\cos x}$ =R.H.S. 1

(c) (i) $\frac{10a}{9} = T_2 + T_3 = ar^1 + ar^2$

$10 = 9r + 9r^2$ 1

$(3r+5)(3r-2) = 0$ 1

$r = -\frac{5}{3}$ or $\frac{2}{3}$

Therefore $r = \frac{2}{3}$ for positive r. 1

(ii) $S_\infty = \frac{a}{1-\frac{2}{3}} = 3a$ 1

(d) $14\cos\theta - 2\cos\theta = -5$

$12\cos\theta = -5$ 1

$\cos\theta = -\frac{5}{12}$

$\theta = 114^0 37', 245^0 23'$ 1

Question 6

(a) $y = 3x^4 - 4x^3 - 12x^2 + 1$
$y' = 12x^3 - 12x^2 - 24x$
$y'' = 36x^2 - 24x - 24$

For Stationary points, $y' = 0$
$12x(x-2)(x+1) = 0$
$x = 0, 2$ or -1

Testing nature of:

$x = 0, y = 1, f''(0) < 0$
$(0, 1)$ is a local max.

$x = 2, y = -31, f''(2) > 0$
$(2, -31)$ is a local min.

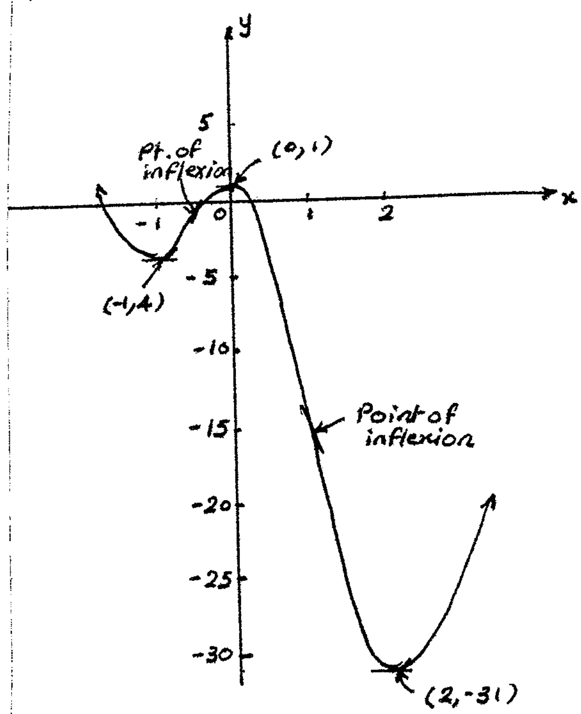
$x = -1, y = -4, f''(-1) > 0$
$(-1, -4)$ is a local min.

(ii) For pts of inflexion, $y'' = 0$
$3x^2 - 2x - 2 = 0$
$x = \frac{2 \pm 2\sqrt{7}}{6} = \frac{1 \pm \sqrt{7}}{3}$

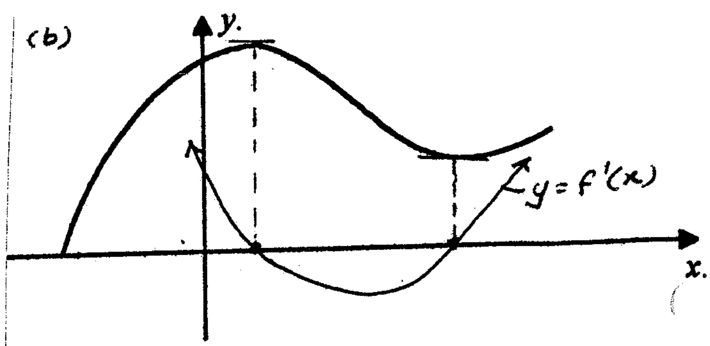
Test for concavity,
$f''(\alpha^-) < 0$ and $f''(\alpha^+) > 0$ then α is an inflexion point.

$f''(\beta^-) < 0$ and $f''(\beta^+) > 0$ then β is also an inflexion point.

(iii)



(b)



(c) $A = \frac{1}{2}r^2\theta$
$0.6 = \frac{1}{2} \times 4\theta$
$\theta = 0.3$ radians 1
$\theta = 0.3 \times \frac{180^0}{\pi} = 17^0 11'$ 1

Question 7

(a) $I = I_0 e^{-\lambda t}$

$\frac{dI}{dt} = -\lambda I_0 e^{-\lambda t} = -\lambda I$ 1

(ii) $0.93 I_0 = I_0 e^{-20t}$ 1

$\lambda = \frac{\ln 0.93}{-20}$

$= 3.63 \times 10^{-3}$ 1

(iii) $0.25 I_0 = I_0 e^{-\lambda t}$ 1

$t = \frac{\ln 0.25}{-0.00363}$

$= 382.05$ cm (2 d.p) 1

(b) (i) $y = 0,\ \cos 2x = 0$ 1

$2x = \frac{\pi}{2},\ x = \frac{\pi}{4}$ 1

$A(\frac{\pi}{4}, 0)$

(ii) Area $= \int_0^{\frac{\pi}{4}} e^{3x} - \cos 2x \, dx$

$= \left[\frac{e^{3x}}{3} - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}}$ 1

$= \left[\frac{e^{\frac{3\pi}{4}}}{3} - \frac{1}{2}\right] - \left[\frac{1}{3}\right]$

$= 2.68$ (to 2 d.p.) 1

(iii) $x = \ln 12 - \ln y$

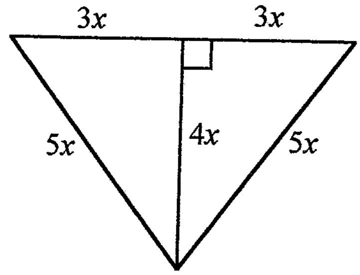
$x = \ln \frac{12}{y}$ 1

$e^x = \frac{12}{y}$ 1

$y = 12e^{-x}$ 1

Question 8

(a)



By Pythagoras' Theorem ,

Height of $\Delta = 4x$ 1

Area $= \frac{1}{2} \times 6x \times 4x = 12x^2$ 1

(b) (i) $S.A. = 2(12x^2 + 5xh)$

$450 = 24x^2 + 10xh$ 1

Therefore, $h = \frac{450 - 24x^2}{10x}$

$= \frac{10(45 - 2.4x^2)}{10x}$ 1

= R.H.S

(ii) $V = 12x^2 h = 12x^2 \left(\frac{45 - 2.4x^2}{x}\right)$ 1

$= \frac{540x^2 - 28.8x^4}{x}$ 1

$= 540x - 28.8x^3$

(iii) $\frac{dV}{dx} = 540 - 86.4x^2$ 1

$\frac{d^2V}{dx^2} = -172.8x$ 1

For stationary values, $V' = 0$
therefore,
$x = \pm 2.5$, for $x = 2.5$ $V'' < 0$ 2
Therefore, Volume is a maximum

$V = 540(2.5) - 28.8(2.5)^3$
$= 900$ units^3 2

Question 9

(a) $A_n = \frac{PR(R^n - 1)}{R - 1}$ 1
(You may assume the formula).

$A = \$200\,000, R = 1.0125, n = 20$ 1

$P = \frac{A_n(R-1)}{R(R^n - 1)}$ 1

$= \$8864.08$ 1

(b) Fill in the table below and find the approximation to

(0.5 ea)

x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
$f(x)$	0	0.5	1.00	0.5	0

$\frac{h}{3}[0 + 0 + 4(0.5 + 0.5) + 2(1)]$ 1

$= \frac{\pi}{4}$ 1

$V = \pi \int_0^8 y^{\frac{2}{3}}\, dy$ 1

$= \pi \left[\frac{3y^{\frac{5}{3}}}{5} \right]_0^8$ 2

$= 19.2\pi$ units^3 1

Question 10

(a) (i) $v = 8 + 2t - t^2$,
$a = \dot{v} = 2 - 2t$ 1
$a = \pm 1$ (because of magnitude) 1
Therefore,
$2 - 2t = \pm 1$
$t = \frac{1}{2}, \frac{3}{2}$ 1

(ii) $x = 8t + t^2 - \frac{t^3}{3} + c$
When $t = 0, x = 0 \Rightarrow c = 0$ 1
$x = 8t + t^2 - \frac{t^3}{3}$
When $t = 5, x = 23\frac{1}{3}$ m 1

(iii) When P is at rest, $v = 0$

$t^2 - 2t - 8 = 0$
$(t-4)(t+2) = 0$
$t = 4$, since $t \geq 0$ 1

At $t = 4, x = 26\frac{2}{3}$ m 1

(iv) When $t = 5, x = 23\frac{1}{3}$ and
from $t = 4$ to $t = 5, x = 3\frac{1}{3}$
Total distance travelled in 5 seconds
$= 26\frac{2}{3} + 3\frac{1}{3} = 30$ m 1

(b) $V = \frac{1}{2}(h^2 + 4h)$ and h cm is the height

When $V = 16$, $h^2 + 4h - 32 = 0$

Therefore, $h = 4$ 1

$V' = \frac{2h+4}{2} = h + 2$ 1

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$ 1

$= \frac{1}{h+2} \times 12$

When $h = 4$, $\frac{dh}{dt} = 2$ cm/s 1