C.E.M.TUITION

Name :________________________________

Review Paper No. 1

Quadratic Polynomial & Parabola

Year 11 - 2 Unit

1. The parabola $y = ax^2 + bx$ has vertex $(1, -1)$. Find the values of a and b.

2. One of the roots of the quadratic equation $x^2 + bx + 18 = 0$ is twice the other root. Find both roots and the value of b.

3. Solve $4^x - 9(2^x) + 8 = 0$.

4. Consider the quadratic equation
$x^2 + (k-2)x + (1-2k) = 0$. k real.

(i) Find an expression in k for the sum of the roots of the equation. Hence find the value of k for which the roots are equal in magnitude but opposite in sign.

(ii) Find an expression in k for the product of the roots of the equation. Hence find the set of values of k for which the roots are opposite in sign.

5. Find the values of k such that the quadratic equation $x^2 + (k - 2)x + 4 = 0$ has equal roots.

6. Find the set of values of k such that
$x^2 + (k+1)x + (2k-1) > 0$ for all x.

7. For the parabola $8y = x^2 - 6x + 17$

(i) find the vertex and focal length

(ii) find the focus and directrix.

8. $P(2,2)$ is a point on the parabola $x^2 = 2y$.
F is the focus of the parabola. The tangent to the parabola at P meets the axis of symmetry at T.

(i) Show that the tangent at P has equation $2x - y - 2 = 0$.

(ii) Show that $FP = FT$.

1. $y = ax^2 + bx$ is parabola with axis of symmetry $x = -\dfrac{b}{2a}$.

$\therefore$ vertex $(1,-1) \Rightarrow -\dfrac{b}{2a} = 1$

$b = -2a \quad (1)$

$(1,-1)$ on curve $\Rightarrow \quad -1 = a + b \quad (2)$

By substitution $\quad -1 = a - 2a$

$-1 = -a$

$\therefore a = 1$ and $b = -2$

2. Let the roots of $x^2 + bx + 18 = 0$ be α and 2α.

Product of roots is 18 $\Rightarrow 2\alpha^2 = 18$

$\alpha^2 = 9$

$\alpha = \pm 3$

Sum of roots is $-b \quad \Rightarrow \quad 3\alpha = -b$

$b = -3\alpha$

Hence roots are 3 and 6 with $b = -9$,
or –3 and –6 with $b = 9$.

3. $4^x - 9(2^x) + 8 = 0$. Let $X = 2^x$.

Then using $4^x = (2^2)^x = 2^{2x} = (2^x)^2$,

$X^2 - 9X + 8 = 0$

$(X-8)(X-1) = 0$

$X = 8 \quad$ or $\quad X = 1$

$2^x = 2^3 \qquad 2^x = 2^0$

$\therefore x = 3 \quad$ or $\quad x = 0$

4. $x^2 + (k-2)x + (1-2k) = 0$

(i) Sum of roots is $-(k-2)$.
$\therefore$ roots $\alpha, -\alpha \Rightarrow$ sum is 0 $\Rightarrow k = 2$

(ii) Product of roots is $(1-2k)$.
$\therefore$ product is negative $\Rightarrow 1 - 2k < 0$

$2k > 1$

$k > \frac{1}{2}$

Required set is $\{k : k > \frac{1}{2}\}$.

5. Let $x^2 + (k-2)x + 4 = 0$ have equal roots α, α.

Product of roots is 4 $\Rightarrow \alpha^2 = 4$

$\alpha = \pm 2$

Sum of roots is $(k-2) \Rightarrow 2\alpha = (k-2)$

$k = 2\alpha + 2$

Hence roots are 2, 2 with $k = 6$
or –2, –2 with $k = -2$.

Alternatively, we can use the condition '$\Delta = 0$ for equal roots' to find k.

$$\left.\begin{array}{l} a = 1 \\ b = (k-2) \\ c = 4 \end{array}\right\} \Rightarrow \begin{array}{l} \Delta = b^2 - 4ac \\ \quad = (k-2)^2 - 16 \end{array}$$

Then $\Delta = 0 \Rightarrow (k-2)^2 = 16$

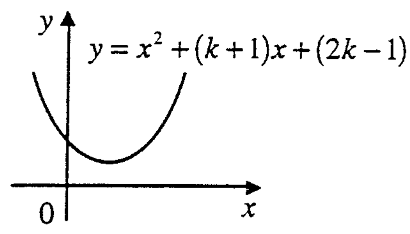
$(k-2) = \pm 4$

$k = 6$ or $k = -2$

6. $x^2 + (k+1)x + (2k-1) > 0$ for all x.

The description 'for all x' prompts us to think of the graph of the corresponding quadratic function
$y = x^2 + (k+1)x + (2k-1)$
Since the quadratic expression is always positive, all the y values satisfy $y > 0$, and the graph lies above the x-axis.

Figure 14.39



concave up $\Rightarrow a > 0$

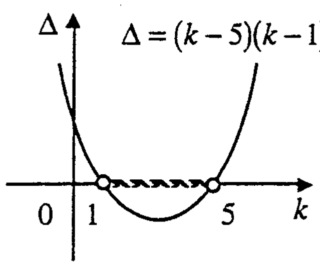
no x-intercepts $\Rightarrow \begin{cases} \text{equation has no} \\ \text{real roots and } \Delta < 0. \end{cases}$

$$\left.\begin{array}{l} a = 1 \\ b = (k+1) \\ c = (2k-1) \end{array}\right\} \Rightarrow \begin{array}{l} \Delta = b^2 - 4ac \\ \quad = (k+1)^2 - 4(2k-1) \\ \quad = k^2 - 6k + 5 \\ \quad = (k-5)(k-1) \end{array}$$

Clearly $a = 1 \Rightarrow a > 0$.
$\Delta < 0 \Rightarrow (k-5)(k-1) < 0$

This quadratic inequality in k is best solved by inspection of the graph of Δ as a function of k, a concave up parabola with k-intercepts $k = 1$, $k = 5$. (Note that this graph is not the same parabola as the graph of the original expression in x.)

Figure 14.40



$\therefore \Delta < 0$ for $1 < k < 5$.
Hence the required set is $\{k:\ 1 < k < 5\}$.

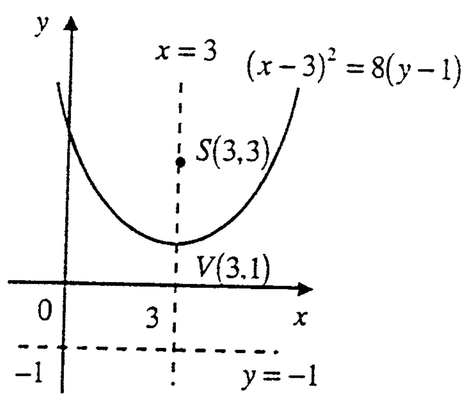
7.

$$8y = x^2 - 6x + 17$$
$$8y - 17 = x^2 - 6x$$
$$8y - 17 + 9 = x^2 - 6x + 9$$
$$8y - 8 = (x-3)^2$$
$$(x-3)^2 = 4 \times 2(y-1)$$

(i) Vertex $(3, 1)$, focal length 2.

(ii) The parabola is concave up with the vertical line $x = 3$ as its axis of symmetry.

Figure 14.41



The focus lies on the line $x = 3$ and is two units vertically above the vertex with coordinates $(3, 3)$. The directrix is a horizontal line two units below the vertex with equation $y = -1$.

8. (i) $y = \frac{1}{2}x^2 \Rightarrow \dfrac{dy}{dx} = x$

At $P(2, 2)$, $\dfrac{dy}{dx} = 2$

Hence tangent at P has gradient 2 and equation

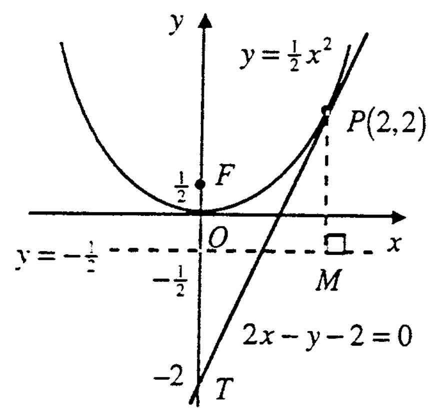
$$y - 2 = 2(x-2)$$
$$y - 2 = 2x - 4$$
$$2x - y - 2 = 0$$

(ii) T is the point where the tangent at P meets the y-axis.
tangent: $2x - y - 2 = 0$
y-intercept: $x = 0 \Rightarrow y = -2$
T has coordinates $(0, -2)$.

$x^2 = 2y \Rightarrow x^2 = 4 \times \frac{1}{2}y$
Hence the parabola has vertex at the origin and focal length $\frac{1}{2}$. The focus is $F(0, \frac{1}{2})$ and the directrix has equation $y = -\frac{1}{2}$.

Let M be the foot of the perpendicular from P to the directrix.

Figure 14.42



Using the locus definition of the parabola, $FP = PM = 2\frac{1}{2}$. But $FT = 2\frac{1}{2}$.
Hence $FP = FT$.