C.E.M.TUITION

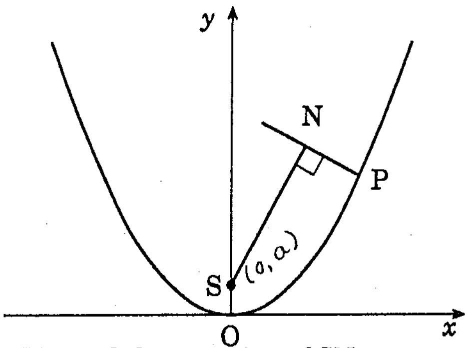
Student Name :______________________________

Review Topic : Parabola $x^2 = 4ay$

(Preliminary - Paper 1)

Year 12 - 3 Unit

1.



P$(2ap, ap^2)$ is a point on the parabola $x^2 = 4ay$. SN is perpendicular to the normal at P, where S is the focus of the parabola.

(a) Show that the equation of the normal at P is

$$x + py = 2ap + ap^3$$

(b) Find the equation of SN.

(c) Find the co-ordinates (X, Y) of the point N.

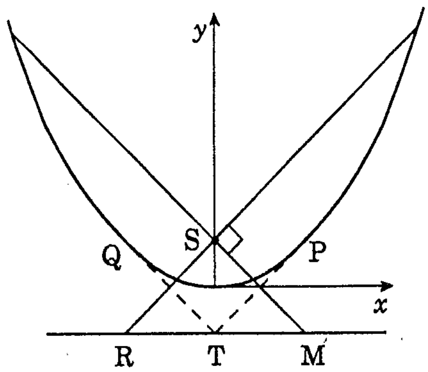
(d) Show that the locus of N is the parabola $x^2 = a(y - a)$.

2. Two perpendicular lines $x = my$ and $y = -mx$ are drawn through the origin O to meet the parabola $x^2 = 4ay$ in the points A and B.

(a) Find the co-ordinates of the points A and B.

(b) Show that the tangents at the points A and B intersect on the line $y = -4a$.

3.



Two perpendicular chords through the focus S(0, a) of the parabola $x^2 = 4ay$ meet the directrix in R and M respectively. P($2ap$, ap^2) and Q($2aq$, aq^2) are two points on the parabola such that the tangents at P and Q are parallel to the two chords.

(a) Write down the equations of the tangents at P and Q.

(b) Show that the equations of the focal chords RS and MS are $y - a = px$ and $y = qx + a$ respectively.

(c) Hence, show that $pq = -1$.

(d) Show that the coordinates of T is $(a(p+q), -a)$.

(e) Show that T is also the midpoint of MR.

1. Using the given diagram: $S(0, a)$, $P(2ap, ap^2)$

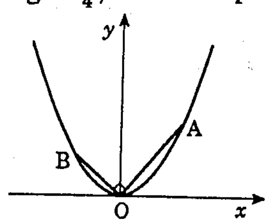
(a) The normal at P is
$x + py = 2ap + ap^3$
(Theory)

(b) SN ⊥ the normal.
Gradient of SN is p.
The equation of SN is
$y - a = px$
$y = px + a$

(c) Solve
$y = px + a$ ... (1)
and
$x + py = 2ap + ap^3$ ... (2)
to find $N(X, Y)$.
Put $y = px + a$ into (2)
$\therefore\ x + p(px + a) = 2ap + ap^3$
$x(1 + p^2) = ap(1 + p^2)$
$x = ap$
Then $y = px + a$
$= a(1 + p^2)$
$\therefore$ N is $(ap, a(1 + p^2))$

(d) To find the locus of N, eliminate the parameter p.
$X = ap,\ Y = a + ap^2$
$\therefore X^2 = a^2p^2 = a \cdot ap^2$
But $ap^2 = Y - a$
$\therefore X^2 = a(Y - a)$
Hence the locus of N is a parabola $x^2 = a(y - a)$, vertex $(0, a)$, focal length $\frac{a}{4}$, concave up.

2.



(a)
Solve $x = my$ and $x^2 = 4ay$ to find the coordinates of A.
$\therefore\ m^2y^2 = 4ay$
$y(m^2y - 4a) = 0$

At A, $y \neq 0$, $y = \dfrac{4a}{m^2}$
$x = m \cdot \dfrac{4a}{m^2} = \dfrac{4a}{m}$
A is $\left(\dfrac{4a}{m}, \dfrac{4a}{m^2}\right)$
Solve $y = -mx$ and $x^2 = 4ay$ to find the coordinates of B.
B is $(-4am, 4am^2)$

(b) From $x^2 = 4ay$, $y = \dfrac{x^2}{4a}$
$$\frac{dy}{dx} = \frac{x}{2a}$$
m_1 = gradient of the tangent at A
$m_1 = \dfrac{4a}{m} \cdot \dfrac{1}{2a} = \dfrac{2}{m}$
m_2 = gradient of the tangent at B
$m_2 = -4am \cdot \dfrac{1}{2a} = -2m$
The tangents at A and B are: respectively:
$y - \dfrac{4a}{m^2} = \dfrac{2}{m}\left(x - \dfrac{4a}{m}\right)$
and
$y - 4am^2 = -2m(x + 4am)$
i.e. $m^2y = 2mx - 4a$ ... (1)
and
$y = -2mx - 4am^2$ ... (2)
Adding
$y(1 + m^2) = -4a(1 + m^2)$
$y = -4a$

Hence the tangent at A and B intersect on the line $y = -4a$.
[Students note that it is not necessary to find the x-coordinate.]

3. (a) $y = px - ap^2$ ... (1)
$y = qx - aq^2$ ... (2)

(b) The gradient of line (1) is $m = p$. The focal chord parallel to (1) is
$y - a = px$ or $y = px + a$.
Similarly the other focal chord is $y = qx + a$.

(c) The chords are perpendicular to each other. Using $m_1m_2 = -1$, with $m_1 = p,\ m_2 = q$, we have $pq = -1$.

(d) Solving (1) and (2):
Subtract (1) − (2), then:
$0 = x(p - q) - a(p^2 - q^2)$
$p \neq q$, divide by $p - q$.
$x = a(p + q)$
Then
$y = p(ap + aq) - ap^2$
$= apq$
$\therefore$ T is $(ap + aq, apq)$
But $pq = -1$
$\therefore$ T is $(ap + aq, -a)$

(e) $y = px + a$
At R, $y = -a$, so $x = \dfrac{-2a}{p}$
$\therefore$ R is $\left(\dfrac{-2a}{p}, -a\right)$
Similarly M is $\left(\dfrac{-2a}{q}, -a\right)$
The mid-point of RM is, say $N(X, Y)$, then:
$X = \frac{1}{2}\left(-\dfrac{2a}{p} - \dfrac{2a}{q}\right)$
$= -a\left(\dfrac{1}{p} + \dfrac{1}{q}\right)$
$X = \dfrac{-a(p + q)}{pq}$, $pq = -1$
$X = a(p + q)$
$Y = \frac{1}{2}(-a - a) = -a$
$\therefore$ N is $(ap + aq, -a)$
N and T have the same coordinates, hence T is the mid-point of RM.