C.E.M. TUITION

Student Name : ______________________________

Review Topic : Differentiation & Applications

(Paper 1)

Year 11 - 2 Unit

1. Sketch the graph of the function $f(x)=|x|$. Is the function differentiable at $x=0$? Give a reason for your answer.

2. A curve has equation $y=2x^3+ax+b$. At the point $P(1,1)$ on the curve, the tangent to the curve is parallel to the line $y=x$. Find the values of a and b.

3. $P(1,-3)$ is a point on the curve $y = x^2 - 4$. The tangent to the curve at P meets the x-axis at A and the y-axis at B.

(i) Show that the tangent at P has equation $2x - y - 5 = 0$.

(ii) Find the area of ΔAOB.

4. $P(1,0)$ is a point on the curve $y = 1 - x^2$. The normal to the curve at P meets the curve again at Q.

(i) Show that the normal at P has equation $x - 2y - 1 = 0$.

(ii) Find the coordinates of Q.

5. If $y = \dfrac{x}{2x+1}$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

6. The cost $C(v)$ dollars per hour of running a truck at an average speed of $v\,\text{kmh}^{-1}$ is given by $C(v) = av + \frac{b}{v}$, for some constants a and b. At an average speed of $20\,\text{kmh}^{-1}$ the cost is \$85 per hour, and at an average speed of $40\,\text{kmh}^{-1}$ the cost is \$50 per hour.

(i) Show that $a = \frac{1}{4}$ and $b = 1600$.

(ii) Find the most economical average speed of running the truck.

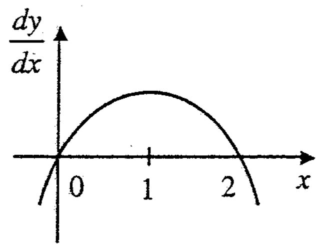
7. Consider the function $f(x)=(x-3)\sqrt{x}$.
   - **(i)** Sketch the graph of the function showing clearly the important features.
   - **(ii)** Find the domain and range of the function.

8. The curve $y = f(x)$ has gradient function $\frac{dy}{dx}$. The graph of $\frac{dy}{dx}$ as a function of x is shown in Figure 9.19. Find where the curve $y = f(x)$

(i) has a maximum turning point

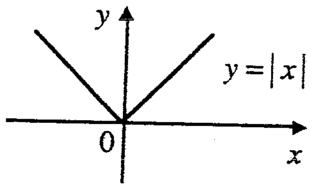
(ii) is concave down.

Figure 9.19



1.

Figure 14.43



f is not differentiable at $x=0$.
The gradient of the tangent is -1 as x approaches 0 from the left, and $+1$ as x approaches 0 from the right. Hence there is no limiting tangent at $x=0$ and the derivative is not defined. Note that the graph is not a smooth curve at $x=0$ but has sharp corners.

2. $y=2x^3+ax+b$

$\frac{dy}{dx}=6x^2+a$

$\frac{dy}{dx}=6+a$ when $x=1$

At $P(1,1)$ the tangent is parallel to the line $y=x$ and has gradient 1.

$\therefore 6+a=1 \Rightarrow a=-5$

$P(1,1)$ lies on the curve $\Rightarrow 1=2+a+b$

$\therefore a=-5$ and $b=4$.

3. (i) $y=x^2-4 \Rightarrow \frac{dy}{dx}=2x$

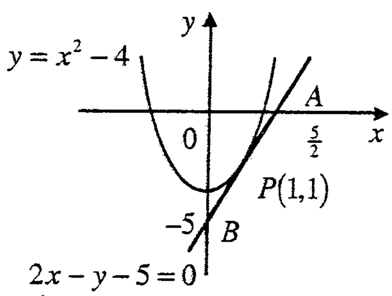
When $x=1$, $\frac{dy}{dx}=2$

Tangent at $P(1,-3)$ has gradient 2 and equation

$$y+3=2(x-1)$$
$$y+3=2x-2$$
$$2x-y-5=0$$

(ii) At A, $y=0 \Rightarrow x=\frac{5}{2}$
At B, $x=0 \Rightarrow y=-5$

Figure 14.44



Area $\Delta AOB=\frac{1}{2}\times5\times\frac{5}{2}=\frac{25}{4}$ sq. units.

4. (i) $y=1-x^2 \Rightarrow \frac{dy}{dx}=-2x$

When $x=1$, $\frac{dy}{dx}=-2$

Hence normal at $P(1,0)$ has gradient $\frac{1}{2}$ and equation

$$y-0=\frac{1}{2}(x-1)$$
$$2y=x-1$$
$$x-2y-1=0$$

(ii) Q lies on both the normal at P and the curve. Hence at Q

$$\left.\begin{aligned}2y&=x-1\\2y&=2-2x^2\end{aligned}\right\}\Rightarrow \quad x-1=2-2x^2$$
$$2x^2+x-3=0$$
$$(2x+3)(x-1)=0$$
$$x=-\tfrac{3}{2} \text{ or } x=1$$

$x=1$ gives the point P. Hence at Q $x=-\frac{3}{2}$ and $y=1-\left(-\frac{3}{2}\right)^2=-\frac{5}{4}$.

Hence Q has coordinates $\left(-\frac{3}{2},-\frac{5}{4}\right)$.

5. $y=\dfrac{x}{2x+1}$

$$\frac{dy}{dx}=\frac{1.(2x+1)-x.2}{(2x+1)^2}=\frac{1}{(2x+1)^2}$$

$$\frac{dy}{dx}=(2x+1)^{-2}$$

$$\frac{d^2y}{dx^2}=-2(2x+1)^{-3}\cdot2=\frac{-4}{(2x+1)^3}$$

6. **(i)** $C(v) = av + \frac{b}{v}$

$C(20) = 85 \Rightarrow 20a + \frac{b}{20} = 85 \quad (1)$

$C(40) = 50 \Rightarrow 40a + \frac{b}{40} = 50 \quad (2)$

$2 \times (2) \Rightarrow 80a + \frac{b}{20} = 100 \quad (3)$

$(3) - (1) \Rightarrow 60a = 15$

$a = \frac{1}{4}$

Substituting in (1) gives $\frac{b}{20} = 80$.

Hence $a = \frac{1}{4}$ and $b = 1600$.

(ii) $C(v) = \frac{1}{4}v + 1600v^{-1}$

$C'(v) = \frac{1}{4} - 1600v^{-2}$

$= \frac{v^2 - 6400}{4v^2}$

$\therefore C'(v) = 0 \Rightarrow v^2 = 6400$

$v = 80$

$C''(v) = 3200v^{-3} \Rightarrow C''(80) > 0$

Hence minimum cost C occurs when $v = 80$. Most economical average speed is $80\,\text{kmh}^{-1}$.

7. **(i)** $y = (x-3)\sqrt{x}$

Domain: $\{x : x \geq 0\}$

y-intercept: $x = 0 \Rightarrow y = 0$

x-intercepts: $y = 0 \Rightarrow x = 0,\ x = 3$

$y = (x-3)x^{\frac{1}{2}}$

$y = x^{\frac{3}{2}} - 3x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} - \frac{3}{2}x^{-\frac{1}{2}}$

$= \frac{3}{2}x^{-\frac{1}{2}}(x-1)$

$\frac{dy}{dx} = \frac{3(x-1)}{2\sqrt{x}}$

$\frac{d^2y}{dx^2} = \frac{3}{4}x^{-\frac{1}{2}} + \frac{3}{4}x^{-\frac{3}{2}}$

$= \frac{3}{4}x^{-\frac{3}{2}}(x+1)$

$\frac{d^2y}{dx^2} = \frac{3(x+1)}{4x\sqrt{x}}$

Stationary points:

$\frac{dy}{dx} = 0 \Rightarrow x = 1$

$x = 1 \Rightarrow \frac{d^2y}{dx^2} > 0$

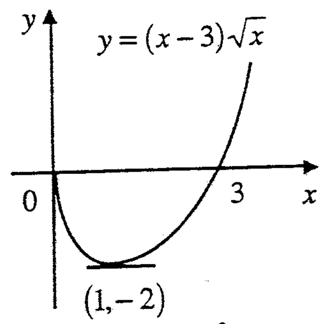
When $x = 1$, $y = (1-3)\sqrt{1} = -2$

$\therefore (1, -2)$ is a minimum turning point.

Points of inflexion:

Since $\frac{d^2y}{dx^2} > 0$ for all x in the domain, the curve has no points of inflexion and is always concave up.

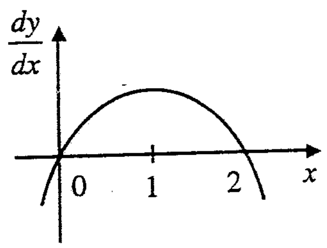
Figure 14.45



(ii) Domain: $\{x : x \geq 0\}$

Range: $\{y : y \geq -2\}$

8.



(i) $\frac{dy}{dx} = 0$ and changes sign at $x = 0,\ x = 2$.

$\therefore$ turning points occur at $x = 0,\ x = 2$.

$x < 0 \Rightarrow \frac{dy}{dx} < 0$ and f is decreasing.

$0 < x < 2 \Rightarrow \frac{dy}{dx} > 0$ and f is increasing.

$x > 2 \Rightarrow \frac{dy}{dx} < 0$ and f is decreasing.

Hence as x increases, f decreases to a minimum value at $x = 0$, then increases to a maximum value at $x = 2$, then decreases again.

$\therefore\ y = f(x)$ has a maximum turning point at $x = 2$.

(ii) $\frac{d^2y}{dx^2}$ is the derivative with respect to x of the function $\frac{dy}{dx}$. Hence $\frac{d^2y}{dx^2} < 0$ when $\frac{dy}{dx}$ is decreasing, that is for $x > 1$. Hence the curve $y = f(x)$ is concave down for $x > 1$.