C.E.M.TUITION

Student Name :________________________________

Review Topic : Series & Applications

(Paper 1)

Year 11 - 2 Unit

1. A geometric sequence has third term 12 and seventh term 192. Find the fourth, fifth and sixth terms of the sequence.

2. An arithmetic sequence has first term 5. If $T_n = 105$ and $S_n = 495$, find the value of n and the common difference of the sequence.

3. Evaluate $\sum_{k=1}^{8} 4.\left(\frac{1}{2}\right)^{k-1}$.

4. Consider the arithmetic sequence 48, 44, 40, ... If $S_n = 300$, find the value of n.

5. Express $0.\dot{1}\dot{2}$ as a geometric series, and hence as a fraction in its simplest form.

6. A teacher buys a new computer for \$4000. After each year it depreciates in value by 20% of its value at the start of the year. Find its value after 5 years.

7. A clerk is employed at an initial salary of \$18 000 per year. At the end of each year he receives an increase of \$$x$ per year. At the end of 10 years he found that during the second 5 years he earned \$20 000 more than he did during the first 5 years. Find the value of x.

8. A loan of \$10 000 is to be repaid over a period of time with fixed repayments of \$300 being made at the end of each month. Interest is charged on the current amount owed at the beginning of each month at a rate of 12% p.a. reducible monthly. Find the time taken to repay the loan.

1. GP with first term a, common ratio r.

$T_3 = 12 \Rightarrow ar^2 = 12 \quad (1)$

$T_7 = 192 \Rightarrow ar^6 = 192 \quad (2)$

$(2) \div (1) \Rightarrow r^4 = 16$

$r = \pm 2$

Substituting $r^2 = 4$ in (1) $\Rightarrow a = 3$

$$\left.\begin{matrix} r=2 \\ a=3 \end{matrix}\right\} \Rightarrow \begin{matrix} T_4 = 2\times T_3 = 24 \\ T_5 = 2\times T_4 = 48 \\ T_6 = 2\times T_5 = 96 \end{matrix}$$

or

$$\left.\begin{matrix} r=-2 \\ a=3 \end{matrix}\right\} \Rightarrow \begin{matrix} T_4 = -2\times T_3 = -24 \\ T_5 = -2\times T_4 = 48 \\ T_6 = -2\times T_5 = -96 \end{matrix}$$

2. AP with first term $a = 5$ and common difference d.

$S_n = \frac{n}{2}(a + T_n) \Rightarrow 495 = \frac{n}{2}(5 + 105)$

$495 = 55n$

$n = 9$

$T_9 = 105 \Rightarrow 5 + 8d = 105$

$8d = 100$

$d = 12.5$

3. $\sum_{k=1}^{8} 4.\left(\frac{1}{2}\right)^{k-1}$

$= 4 + 4.\frac{1}{2} + 4.\left(\frac{1}{2}\right)^2 + 4.\left(\frac{1}{2}\right)^3 + ... + 4.\left(\frac{1}{2}\right)^7$

This is the sum of eight terms where each term is $\frac{1}{2}$ the term before. The terms form a GP with first term $a = 4$ and common ratio $r = \frac{1}{2}$.

$S_8 = \dfrac{4\left(1 - \left(\frac{1}{2}\right)^8\right)}{1 - \frac{1}{2}}$

$= \dfrac{4 \times \left(\frac{255}{256}\right)}{\left(\frac{1}{2}\right)}$

$= 4 \times \frac{255}{256} \times 2$

$= \frac{255}{32}$

Hence $\sum_{k=1}^{8} 4.\left(\frac{1}{2}\right)^{k-1} = 7\frac{31}{32}$

4. AP 48, 44, 40, ... has first term $a = 48$ and common difference $d = -4$.

$S_n = 300 \Rightarrow \frac{1}{2}n\{96 - 4(n-1)\} = 300$

$\frac{1}{2}n(100 - 4n) = 300$

$2n(25 - n) = 300$

$n(25 - n) = 150$

$n^2 - 25n + 150 = 0$

$(n - 10)(n - 15) = 0$

$n = 10$ or $n = 15$

5. $0.\dot{1}\dot{2} = 0.12121212...$

$= 0.12 + 0.0012 + 0.000012 + ...$

$= \frac{12}{100} + \frac{12}{100} \times \frac{1}{100} + \frac{12}{100} \times \left(\frac{1}{100}\right)^2 + ...$

This is a geometric series with first term $a = \frac{12}{100}$ and common ratio $\frac{1}{100}$. Since $|r| < 1$, the limiting sum S exists and is given by $S = \dfrac{a}{1-r}$.

$\therefore 0.\dot{1}\dot{2} = \dfrac{\left(\frac{12}{100}\right)}{1 - \frac{1}{100}} = \dfrac{\left(\frac{12}{100}\right)}{\left(\frac{99}{100}\right)}$

$\therefore 0.\dot{1}\dot{2} = \frac{12}{99} = \frac{4}{33}$

6. Let the original value be P and let r be the annual rate of depreciation expressed as a decimal. Then the value after n years is given by $A = PR^n$ where $R = 1 - r$.

$$\left.\begin{matrix} P = 4000 \\ r = 20\% = 0.2 \\ R = 1 - r = 0.8 \\ n = 5 \end{matrix}\right\} \Rightarrow \begin{matrix} A = 4000 \times (0.8)^5 \\ = 1310.72 \end{matrix}$$

Hence the value after 5 years is \$1310.72.

7. His annual salary in successive years forms an AP with first term $a = 18\,000$ and common difference x.

Total salary over first 5 years is
$S_5 = \frac{5}{2}(36\,000 + 4x) = 90\,000 + 10x$

Total salary over first 10 years is
$S_{10} = \frac{10}{2}(36\,000 + 9x) = 180\,000 + 45x$

Total salary over second 5 years is
$S_{10} - S_5 = 90\,000 + 35x$

$$90\,000 + 35x = 20\,000 + 90\,000 + 10x$$
$$25x = 20\,000$$
$$x = 800$$

8. Let A_n be amount owed at end of nth month after instalment paid.
Monthly rate : $r = 1\% = 0.01$
$R = 1 + r = 1.01$
Monthly instalment : $P = 300$

If no repayments had been made amount owed after n months
$\Rightarrow 10\,000 \times (1.01)^n$

1st P not owed for $(n-1)$ months $\Rightarrow -300 \times (1.01)^{n-1}$

2nd P not owed for $(n-2)$ months $\Rightarrow -300 \times (1.01)^{n-2}$

....

$(n-1)$st P not owed for 1 month $\Rightarrow -300 \times 1.01$

nth P not owed $\Rightarrow -300$

$\therefore A_n = 10000 \times (1.01)^n - 300 \times S_n$ where

$$S_n = 1 + (1.01) + \ldots + (1.01)^{n-2} + (1.01)^{n-1}$$

This is the sum to n terms of a GP with first term 1 and common ratio 1.01.

$$S_n = \frac{(1.01)^n - 1}{1.01 - 1} = 100\{(1.01)^n - 1\}$$

Loan is repaid when $A_n = 0$. Then

$$300 \times S_n = 10\,000 \times (1.01)^n$$
$$30\,000 \times \{(1.01)^n - 1\} = 10\,000 \times (1.01)^n$$
$$3 \times \{(1.01)^n - 1\} = (1.01)^n$$
$$3 \times (1.01)^n - 3 = (1.01)^n$$
$$2 \times (1.01)^n = 3$$
$$(1.01)^n = 1.5$$

Taking logarithms of both sides

$$n \log(1.01) = \log(1.5)$$
$$n = \frac{\log(1.5)}{\log(1.01)} \approx 40.75$$

Hence loan is repaid after 3 years and 5 months.