C.E.M.TUITION

Name :________________________________

Review Topic : Finance Mathematics

Year 12 - 2 Unit

7. Paul invests $1500 at the beginning of each year in a superannuation fund. Interest is paid at 10% p.a. on the investment. How much will Paul's investment amount to after 30 years?

8. Barry Olden is 60 years old and about to retire at the beginning of the year 1994. He joined a superannuation scheme at the beginning of 1954. He invested $750 at the beginning of each year. Compound interest is paid at 9% per annum on the investment. Calculate to the nearest dollar:

(a) the amount to which the 1954 investment will have grown by the beginning of 1994;

(b) the amount to which the total investment will have grown by the beginning of 1994.

9. Calculate the interest earned on an investment of $15 200 at 9% per annum, compounded monthly over 5 years.

10. A loan of $50 000 is to be repaid by equal monthly instalments. Interest at the rate of 15% per annum is calculated each month on the balance owing at the beginning of that month and added to the balance. The loan is to be paid off in equal monthly instalments of $*M* over a period of 15 years.

Let A_n be the amount owing after n months.

(a) Show that $A_1 = 50\ 000 \times 1{\cdot}0125 - M$.

(b) Show that $A_3 = 50\ 000 \times 1{\cdot}0125^3 - M\left[1 + 1{\cdot}0125 + 1{\cdot}0125^2\right]$.

(c) Find the instalment amount $*M*.

11. A company borrowed $50 000 from a building society at 15% per annum reducible quarterly. If the loan is repaid by equal quarterly instalments over a period of 10 years, find the amount of each instalment to the nearest cent.

12. Loukia borrowed $60 000 at 18% per annum, where the interest is compounded monthly on the balance owing. If she pays off the loan in equal monthly instalments over 25 years, calculate:

(a) the amount of each monthly repayment;

(b) the total amount paid (to the nearest dollar) for the loan;

(c) the interest paid;

(d) the rate of simple interest equivalent to this compound interest.

7. First \$1500 invested for 30 years amounts to

$$A_1 = P\left(1+\frac{r}{100}\right)^{30}$$
$$= 1500\left(1+\frac{10}{100}\right)^{30}$$
$$= 1500(1\cdot1)^{30}$$

> *Note* The interest factor $= 1+\frac{r}{100} = 1+\frac{10}{100} = 1\cdot1.$

The 2nd \$1500 invested will amount to $A_2 = 1500(1\cdot1)^{29}$ (only 29 years).

The 3rd \$1500 invested will amount to $A_3 = 1500(1\cdot1)^{28}$ (28 years)

⋮

The 30th (last) \$1500 will amount to $A_{30} = 1500(1\cdot1)^1$.

Total amount of superannuation

$$= A_{30} + A_{29} + \cdots + A_1$$
$$= \underbrace{1500(1\cdot1) + 1500(1\cdot1)^2 + \cdots + 1500(1\cdot1)^{30}}$$

forms a geometric series with $a = 1500\ (1{\cdot}1)$, $r = 1{\cdot}1$, and $n = 30$

$$= S_{30} = \frac{a(r^n - 1)}{r-1}$$
$$= 1500(1\cdot1)\times\frac{(1\cdot1^{30}-1)}{1\cdot1-1}$$
$$= 1500(1\cdot1)\times\frac{(1\cdot1^{30}-1)}{0\cdot1}$$
$$= 271\,415\cdot14.$$

∴ The total amount of Paul's superannuation is \$271 415·14.

8. Barry joined the superannuation scheme at the beginning of 1954 and retired at the beginning of 1994.

Therefore he contributed \$750 at the beginning of each year at 9% p.a. interest for 40 years.

(a) The 1954 investment was invested for 40 years and amounted to A_1, where

$$A_1 = 750\left(1+\frac{9}{100}\right)^{40}$$
$$= 750(1\cdot09)^{40}$$
$$= 23\,557\cdot07.$$

Therefore, the 1954 investment has grown to \$23 557·07 by the beginning of 1994.

> *Note* The interest factor = 1·09

(b) $A_2 = 750(1\cdot09)^{39}$ (39 years)

$A_3 = 750(1\cdot09)^{38}$ (38 years)

⋮

$A_{40} = 750(1\cdot09)^1$

The total amount of the superannuation at the beginning of 1994

$$= A_{40} + A_{39} + \cdots + A_1$$
$$= \underbrace{750(1\cdot09) + 750(1\cdot09)^2 + \cdots + 750(1\cdot09)^{40}}$$

forms a geometric series with $a = 750\ (1{\cdot}09)$, $r = 1{\cdot}09$, and $n = 30$

$$S_{40} = \frac{a(r^n-1)}{r-1}$$
$$= \frac{750(1\cdot09)\times(1\cdot09^{40}-1)}{1\cdot09-1}$$
$$= \frac{750(1\cdot09)\times(1\cdot09^{40}-1)}{0\cdot09}$$
$$= 276\,218\cdot90.$$

Therefore, the amount to which the total will have grown by the beginning of 1994 is \$276 219 (to the nearest dollar).

9. $A = P\left(1+\frac{r}{100}\right)^n$

> where
> $P = 15\,200$
> $r = 9\%$ per annum
> $= \frac{9}{12} = 0\cdot75\%$ per month
> $n = 5$ years
> $= (5\times12) = 60$ months

Therefore,

$$A = 15\,200\left(1+\frac{0\cdot75}{100}\right)^{60}$$
$$= 15\,200(1\cdot0075)^{60}$$
$$= 23\,798\cdot35.$$

Interest $= A - P$

$$= 23\,798\cdot35 - 15\,200$$
$$= 8598\cdot35.$$

Therefore, the investment earns \$8598 interest.

10.

> *Note 1* If interest rate is given as $R\%$ per annum, you must convert it to $r\%$ per month if interest is reducible monthly.

Therefore, interest charged (r)

$= 15\%$ per annum

$= \frac{15}{12}\%$ per month

$= 1\cdot25\%$ per month

> *Note 2* The number of time periods is also changed, (i.e. months, not years).

> $n = 15$ years
> $= 15\times12$ months
> $= 180$ month

THEN $r = 1\cdot25\%$

$n = 180$

$P = 50\,000$

(a) After the first repayment of $\$M$ is made

> Using $A = P\left(1+\frac{r}{100}\right)^n$

$$A_1 = 50\,000\left(1+\frac{1\cdot25}{100}\right)^1 - M$$
$$= 50\,000\times(1\cdot0125) - M$$

> *Note* The interest factor $= \left(1+\frac{1\cdot 25}{100}\right) = 1\cdot 0125.$

Therefore, $A_1 = 50\,000 \times 1\cdot 0125 - M$.

(b) After the 2nd repayment of $\$M$ is made

$$A_2 = A_1(1\cdot 0125) - M$$
$$= [50\,000(1\cdot 0125) - M]1\cdot 0125 - M$$
$$= 50\,000(1\cdot 0125)^2 - 1\cdot 0125M - M.$$

After the 3rd payment of $\$M$ is made

$$A_3 = A_2(1\cdot 0125) - M$$
$$= [50\,000(1\cdot 0125)^2 - 1\cdot 0125M - M]1\cdot 0125 - M$$
$$= 50\,000(1\cdot 0125)^3 - (1\cdot 0125)^2M - 1\cdot 0125M - M$$
$$= 50\,000(1\cdot 0125)^3 - M[(1\cdot 0125)^2 + 1\cdot 0125 + 1].$$

Therefore, $A_3 = 50\,000 \times 1\cdot 0125^3 - M[1 + 1\cdot 0125 + 1\cdot 0125^2]$.

(c) Continuing the patttern in (b), we get after the nth repayment of $\$M$ is made

$$A_n = 50\,000 \times 1\cdot 0125^n - M[1 + 1\cdot 0125 + 1\cdot 0125^2 + \cdots + 1\cdot 0125^{n-1}]$$

But after 60 repayments of $\$M$ are made $A_{60} = 0$

> Amount has been repaid.

$$\therefore\ 50\,000 \times 1\cdot 0125^{60} - M[1 + 1\cdot 0125 + 1\cdot 0125^2 + \cdots + 1\cdot 0125^{59}] = 0$$

Rearranging the formula, we get

$$M = \frac{50\,000 \times 1\cdot 0125^{60}}{\underbrace{1 + 1\cdot 0125 + 1\cdot 0125^2 + \cdots + 1\cdot 0125^{59}}}$$

The denominator forms a geometric series with $a = 1$, $r = 1\cdot 0125$, $n = 60$

$$= \frac{50\,000 \times 1\cdot 0125^{60}}{S_{60}}$$

> where $S_{60} = \frac{a(r^n - 1)}{r - 1} = \frac{1[1\cdot 0125^{60} - 1]}{1\cdot 0125 - 1} = \frac{1\cdot 0125^{60} - 1}{0\cdot 0125}$

$$\therefore\ M = 50\,000 \times 1\cdot 0125^{60} \times \frac{0\cdot 0125}{1\cdot 0125^{60} - 1}$$
$$= 1189\cdot 50.$$

Therefore, the monthly repayment is $1189·50.

11. > # Always work these out first.

#1 Interest charged = 15% per annum

$= \frac{15}{4}\%$ per quarter = 3·75% per quarter

#2 The quarterly interest factor is $\left(1+\frac{3\cdot 75}{100}\right) = 1\cdot 0375$.

#3 $n = 10$ years
$= 10 \times 4$
$= 40$ quarters

> Then $r = 3\cdot 75$, $n = 40$, $P = 50\,000$
> $\left(1+\frac{r}{100}\right) = \left(1+\frac{3\cdot 75}{100}\right) = 1\cdot 0375$

Let $\$M$ be the quarterly instalment and A_n be the amount owing after n repayments.

Using $A = P\left(1+\frac{r}{100}\right)^n$

After the first repayment of $\$M$ ismade,

$A_1 = 50\,000(1{\cdot}0375) - M$.

After the second repayment of $\$M$ is made,

$$\begin{aligned} A_2 &= A_1(1{\cdot}0375) - M \\ &= [50\,000(1{\cdot}0375) - M]1{\cdot}0375 - M \\ &= 50\,000(1{\cdot}0375)^2 - 1{\cdot}0375M - M. \end{aligned}$$

After the third repayment of $\$M$ is made,

$$\begin{aligned} A_3 &= A_2(1{\cdot}0375) - M \\ &= [50\,000(1{\cdot}0375)^2 - 1{\cdot}0375M - M]1{\cdot}0375 - M \\ &= 50\,000(1{\cdot}0375)^3 - 1{\cdot}0375^2M - 1{\cdot}0375M - M \\ &= 50\,000(1{\cdot}0375)^3 - M[1{\cdot}0375^2 + 1{\cdot}0375 + 1] \\ &= 50\,000(1{\cdot}0375)^3 - M[1 + 1{\cdot}0375 + 1{\cdot}0375^2]. \end{aligned}$$

Continuing this pattern, we get after n repayments are made:

$$A_n = 50\,000(1{\cdot}0375)^n - M[1 + 1{\cdot}0375 + 1{\cdot}0375^2 + \dots + 1{\cdot}0375^{n-1}]$$

But after 40 repayments, the loan is paid off.

Therefore $A_{40} = 0 \quad (n = 40).$,

i.e. $50\,000(1{\cdot}0375)^{40} - M[1 + 1{\cdot}0375 + 1{\cdot}0375^2 + \dots + 1{\cdot}0375^{39}] = 0$

Rearranging the formula we get

$$M = \frac{50\,000 \times 1{\cdot}0375^{40}}{\underbrace{1 + 1{\cdot}0375 + 1{\cdot}0375^2 + \cdots}_{\text{geometric series with } a = 1,\ r = 1{\cdot}0375,\ n = 40}}$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{40} = \frac{1[1{\cdot}0375^{40} - 1]}{1{\cdot}0375 - 1} = \frac{1{\cdot}0375^{40} - 1}{0{\cdot}0375}$$

$$\begin{aligned} \therefore\ M &= \frac{50\,000(1{\cdot}0375)^{40}}{S_{40}} \\ &= 50\,000(1{\cdot}0375)^{40} \times \frac{0{\cdot}0375}{1{\cdot}0375^{40} - 1} \\ &= 2432{\cdot}97. \end{aligned}$$

Therefore, each quarterly instalment is $2432·97.

12. † Always work these out first, for every individual problem.

†1 $P = 60\,000$

†2 Interest charged $= 18\%$ per annum
$= \frac{18}{12}\%$ per month
$= 1{\cdot}5\%$ per month

†3 The monthly interest factor $= \left(1 + \frac{1{\cdot}5}{100}\right) = 1{\cdot}015$

†4 $n = 25$ years
$= 25 \times 12$
$= 300$ months

Then $P = 60\,000,\quad r = 1{\cdot}5,$
$1 + \frac{r}{100} = 1{\cdot}015,\quad n = 300$

(a) Let $\$M$ be the monthly instalment and A_n be the amount owing after n repayments.

After the first repayment is made $A_1 = 60\,000\,(1{\cdot}015) - M$.

After the second repayment is made

$$\begin{aligned} A_2 &= A_1(1{\cdot}015) - M \\ &= \left[60\,000\,(1{\cdot}015) - M\right]1{\cdot}015 - M \\ &= 60\,000\,(1{\cdot}015)^2 - 1{\cdot}015M - M \end{aligned}$$

After the third repayment is made

$$\begin{aligned} A_3 &= A_2(1{\cdot}015) - M \\ &= \left[60\,000\,(1{\cdot}015)^2 - 1{\cdot}015M - M\right]1{\cdot}015 - M \\ &= 60\,000\,(1{\cdot}015)^3 - 1{\cdot}015^2M - 1{\cdot}015M - M \\ &= 60\,000\,(1{\cdot}015)^3 - M\left[1{\cdot}015^2 + 1{\cdot}015 + 1\right] \\ &= 60\,000\,(1{\cdot}015)^3 - M\left[1 + 1{\cdot}015 + 1{\cdot}015^2\right]. \end{aligned}$$

Continuing this pattern, we get after n repayments

> *Note* Always work out up to A_3 before predicting the formula for A_n.

$$A_n = 60\,000\,(1{\cdot}015)^n - M\left[1 + 1{\cdot}015 + 1{\cdot}015^2 + \dots + 1{\cdot}015^{n-1}\right]$$

But after 300 repayments the loan is paid off, therefore $A_{300} = 0$ (and $n = 300$),

i.e. $60\,000\,(1{\cdot}015)^{300} - M\left[1 + 1{\cdot}015 + 1{\cdot}015^2 + \dots + 1{\cdot}015^{299}\right] = 0$.

Rearranging the formula we get

$$M = \frac{60\,000 \times 1{\cdot}015^{300}}{\underbrace{1 + 1{\cdot}015 + 1{\cdot}015^2 + \cdots}_{\substack{\text{geometric series with} \\ a = 1,\ r = 1{\cdot}015,\ n = 300}}}$$

$$\therefore\ M = \frac{60\,000(1{\cdot}015)^{300}}{S_{300}}$$

$$= 60\,000(1{\cdot}015)^{300} \times \frac{0{\cdot}015}{1{\cdot}015^{300} - 1} = 910{\cdot}46.$$

> $$S_n = \frac{a(r^n - 1)}{r - 1}$$
> $$S_{300} = \frac{1\left[1{\cdot}015^{300} - 1\right]}{1{\cdot}015 - 1} = \frac{1{\cdot}015^{300} - 1}{0{\cdot}015}$$

Therefore, the amount of each monthly repayment is \$910.

(b) The total amount repaid
= amount of each repayment × number of repayments
= \$910 × 300 = \$273 000.

(c) Interest paid = total amount repaid − amount borrowed
= \$273 000 − \$60 000 = \$213 000.

(d) $I = \dfrac{PRn}{100}$ [Simple Interest formula]

$R = \dfrac{100I}{Pn}$ [making R the subject of the formula]

$= \dfrac{100 \times 213\,000}{60\,000 \times 25} = 14{\cdot}2\%$ $\boxed{I = 213\,000,\ P = 60\,000,\ n = 25}$

Therefore, the rate of simple interest equivalent to this compound interest is 14·2% per annum (used $n = 25$ years).