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# C.E.M. TUITION

## YEAR 12 – MATHS EXT.1

## REVIEW TOPIC: PARAMETRIC EQUATIONS OF THE PARABOLA

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**EXERCISES:**

(1) The parabola $C$ is given by $y^2 = 4ax$ where $a$ is a constant. $P$, with co-ordinates $(ap^2, 2ap)$, is a point on the curve.

(a) Prove that the point $P$ lies on $C$, [1]

(b) Find the equation of the tangent to $C$ at $P$. Also show that the equation of the normal at $P$ is

$$y + px - 2ap - ap^3 = 0.$$ [5]

(c) The tangent at $P$ meets the $y$ axis at $R$, and the normal at $P$ meets the $y$ axis at $S$. If $RS = 6a$, show that

$$p^3 + p - 6 = 0.$$ [4]

(2) A curve $C$ is given by the parametric equations

$$x = 3t^2, \qquad y = \frac{3}{t} \qquad\qquad t \neq 0$$

(a) Find the cartesian equation of $C$. [2]

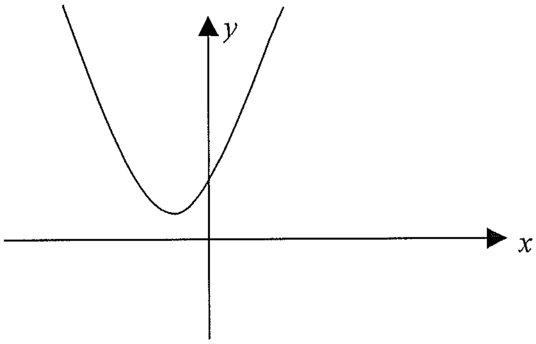
(b) Find the value of the parameter at the point $P(27, -1)$
Find the equation of the tangent at this point. [5]

(3) The curve $C$ is given by $y^2 = 4ax$ where $a$ is a constant. $P(ap^2, 2ap)$, and $Q(aq^2, 2aq)$ lie on the curve.

(a) Find the equations of the tangents at $P$ and $Q$ to the curve. Show that if $pq = -1$ the tangents intersect at right angles. [6]

(b) Show also that the point $F(a, 0)$ lies on the line $PQ$ if the tangents intersect at right angles. [4]

(4) The equation of a curve is given parametrically by the equations $x = 2t - 1$, $y = t^2 + 1$.




The diagram of the curve is as follows:

(a) Show that the point $P(3, 5)$ lies on the curve.
What is the value of $t$ corresponding to this point? [2]

(b) Find the equation of the tangent at $P\ (3, 5)$. [5]

(c) Find the equation of the tangent to the curve which is perpendicular to the tangent at $P$. If this tangent touches the curve at $Q$, find the co-ordinates of $Q$. [4]

(5) The parabola $C$ is given by $y^2 = 4ax$ where $a$ is a constant. $P$, with co-ordinates $(ap^2, 2ap)$, is a point on the curve.

(a) Prove that $P$ lies on the curve. [1]

(b) Find the equation of the tangent to $C$ at $P$. [4]

(c) Show that the tangents at the points $P$ and $Q$ $(aq^2, 2aq)$ meet at the point $L$ with co-ordinates $(apq, a(p+q))$. [3]

(d) If $L$ lies on the line $x + a = 0$, show that $pq = -1$ , and also show that the line through $P$ and $Q$ passes through the point with co-ordinates $(a, 0)$ . [5]

(6) (a)

(i) Show that the points defined by $x = 2\cos t,\ y = \cos 2t$ lie on a parabolic arc.

(ii) Sketch the arc, showing its end points and the focus and directrix of the parabola.

(7)

$P\left(2at,\ at^2\right)$ is a variable point on the parabola $x^2 = 4ay$, whose focus is $S$.

$Q(x,\ y)$ divides the interval from $P$ to $S$ in the ratio $t^2 : 1$.

(i) Find $x$ and $y$ in terms of $a$ and $t$.

(ii) Verify that $\dfrac{y}{x} = t$.

(iii) Prove that, as $P$ moves on the parabola, $Q$ moves on a circle, and state its centre and radius.

**SOLUTIONS:**

(1) (a) $P$ has co-ordinates $(ap^2, 2ap)$

$\Rightarrow$ At $P$ $x = ap^2$ , $y = 2ap$

$4ax = 4a \times ap^2$ , $y^2 = (2ap)^2$

$\Rightarrow \quad = 4a^2p^2$ , $= 4a^2p^2$

$\therefore$ At $P$ $y^2 = 4ax$

$\Rightarrow$ $P$ lies on the curve $C$

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(b) $x = ap^2$ , $y = 2ap$

$\dfrac{dx}{dp} = 2ap$ , $\dfrac{dy}{dp} = 2a$

$$\frac{dy}{dx} = \frac{dy}{dp} \times \frac{dp}{dx}$$
$$= 2a \times \frac{1}{2ap}$$
$$= \frac{1}{p}$$

At $P$ the gradient of the tangent is $\dfrac{1}{p}$

Equation of tangent is $(y - y_1) = m(x - x_1)$

Equation of tangent at $P$

$$y - 2ap = \frac{1}{p}\left(x - ap^2\right)$$

$\times p$ $\quad py - 2ap^2 = x - ap^2$

$$py = x + ap^2$$

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The normal is perpendicular to the tangent at $P$

$\Rightarrow$ Gradient of normal $= -p$ (Using $m_1m_2 = -1$)

Equation of normal at $P$

$$y - 2ap = -p(x - ap^2)$$

$\Rightarrow$ $\quad y - 2ap = -px + ap^3$

$\Rightarrow$ $\quad y + px - 2ap - ap^3 = 0$

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(c) When a curve cuts the $y$ axis $x = 0$

The point at which the tangent intersects with the $y$ axis is given by

$$py = ap^2$$

$\div p$ $\quad y = ap$

$\Rightarrow$ Tangent intersects with the $y$ axis at $R(0, ap)$

The point at which the normal intersects with the $y$ axis is given by

$$y - 2ap - ap^3 = 0$$
$$y = 2ap + ap^3$$

$\Rightarrow$ Normal intersects with the $y$ axis at $S(0, 2ap + ap^3)$

$$RS = OS - OR$$

$$= 2ap + ap^3 - ap$$
$$= ap^3 + ap$$

But $RS = 6a$

$\Rightarrow \quad ap^3 + ap = 6a$

$\div a \quad p^3 + p - 6 = 0$

(2) (a) $x = 3t^2 \qquad y = \dfrac{3}{t}$

By eliminating the parameter $t$, we can obtain the cartesian equation of $C$.

$$t = \frac{3}{y}$$

Substituting $x = 3\left(\dfrac{3}{y}\right)^2$

$$xy^2 = 27$$

(b) At the point $P(27, -1)$

$3t^2 = 27$ and $\dfrac{3}{t} = -1$

$\Rightarrow \quad t = -3$ at $P$

$x = 3t^2 \quad , \quad y = 3t^{-1}$

$\dfrac{dx}{dt} = 6t \qquad \dfrac{dy}{dt} = -\dfrac{3}{t^2}$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$
$$= -\frac{3}{t^2} \times \frac{1}{6t}$$
$$= -\frac{1}{2t^3}$$

Gradient of tangent at the point $P(27, -1)$ where $t = -3$

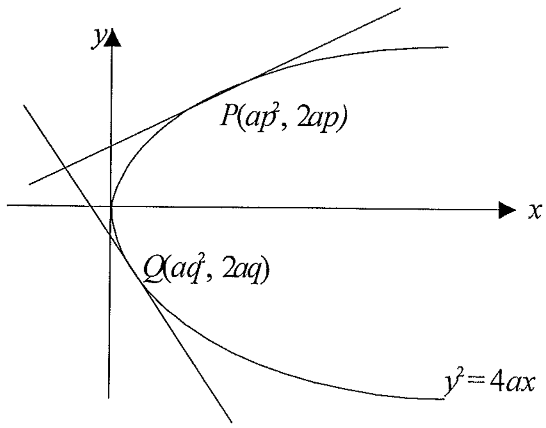
$$= -\frac{1}{2(-3)^3}$$
$$= \frac{1}{54}$$

Equation of tangent at $P(27, -1)$

$$(y - y_1) = m(x - x_1)$$
$$y - (-1) = \frac{1}{54}(x - 27)$$

$\times 54 \quad 54y + 54 = x - 27$

$$x - 54y = 81$$




(3) (a)

The parametric equations of the curve $C$ are

$$x = at^2 \qquad\qquad y = 2at$$

$$\frac{dx}{dt} = 2at \qquad\qquad \frac{dy}{dt} = 2a$$

$$\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx}$$

$$= 2a\times\frac{1}{2at}$$

$$= \frac{1}{t}$$

At the point $P(ap^2, 2ap)$ where $t = p$

Gradient of tangent $= \dfrac{1}{p}$

Equation of tangent

$$y - y_1 = m(x - x_1)$$

$$y - 2ap = \frac{1}{p}\left(x - ap^2\right)$$

$\times p$ $\quad py - 2ap^2 = x - ap^2$

$\Rightarrow$ $\quad py = x + ap^2$

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At the point $Q(aq^2, 2aq)$ $\quad t = q$

Equation of tangent

$$qy = x + aq^2$$

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Gradient of tangent at $P$ $= \dfrac{1}{p}$

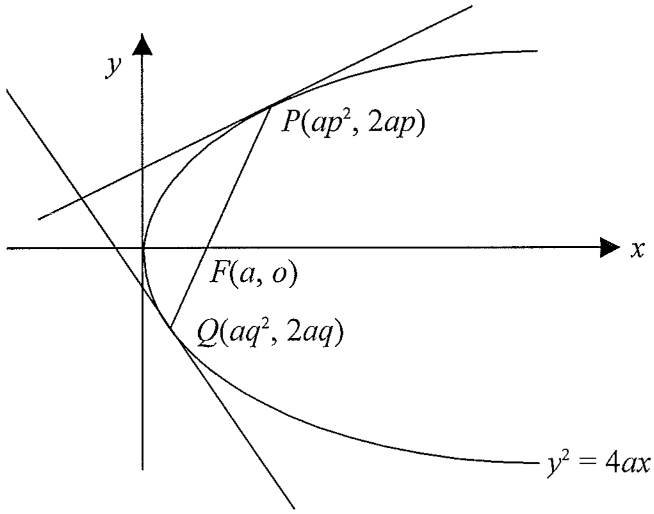
Gradient of tangent at $Q$ $= \dfrac{1}{q}$

Condition for perpendicular lines is, $m_1m_2 = -1$

$$\Rightarrow \quad \frac{1}{p}\times\frac{1}{q} = -1$$

$$\Rightarrow \quad pq = -1$$

(b)




To show $F$ lies on the line $PQ$ we can show

Gradient of line $PF$ = Gradient of line $FQ$

$$\text{Gradient of line } PF = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2ap-0}{ap^2-a} = \frac{2ap}{a\left(p^2-1\right)} = \frac{2p}{\left(p^2-1\right)}$$

$$\text{Gradient of line } FQ = \frac{0-2aq}{a-aq^2} = \frac{-2q}{\left(1-q^2\right)}$$

However, as the tangents intersect at right angles, $pq = -1$

$$\Rightarrow \quad q = -\frac{1}{p}$$

$$\therefore \text{Gradient of line } FQ = \frac{-2\left(-\frac{1}{p}\right)}{1-\left(-\frac{1}{p}\right)^2}$$

$$= \frac{\frac{2}{p}}{\left(1-\frac{1}{p^2}\right)}$$

Multiply top and bottom by $p^2$

Gradient of line $FQ$ $= \dfrac{\frac{2}{p}\times p^2}{\left(1-\frac{1}{p^2}\right)p^2}$

$$= \frac{2p}{p^2-1}$$

$\therefore$ Gradient of line $PF$ = Gradient of line $FQ$

$\Rightarrow$ $F$ lies on the line $PQ$

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(4)

(a) If the point $P(3, 5)$ lies on the curve then

$$3 = 2t - 1$$

$\Rightarrow$ $t = 2$

when $t = 2$ $\quad y = (2)^2 + 1$

$$= 5$$

$\Rightarrow P(3, 5)$ lies on the curve.

The value of the parameter corresponding to this point is $t = 2$

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(b)

$x = 2t - 1$ $\qquad$ $y = t^2 + 1$

$\dfrac{dx}{dt} = 2$ $\qquad$ $\dfrac{dy}{dt} = 2t$

$$\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx}$$

$$= 2t\times\frac{1}{2}$$

$$= t$$

At the point $P(3, 5)$ $\quad t = 2$

$\Rightarrow$ Gradient of tangent at $P$ = 2

Equation of tangent is $y - y_1 = m(x - x_1)$

$$y - 5 = 2(x - 3)$$

Equation of tangent at $P$ is $y = 2x - 1$

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(c) Gradient of tangent at $P$ = 2

Gradient of perpendicular tangent $= -\dfrac{1}{2}$

Let $t'$ be the value of the parameter at the point where the perpendicular tangent touches the curve

$\Rightarrow \quad t' = -\frac{1}{2}$ because $\frac{dy}{dx} = t'$ using result from (b)

Co-ordinates of point of contact $Q\left(2\left(-\frac{1}{2}\right)-1,\ \left(-\frac{1}{2}\right)^2+1\right)$

$Q\left(-2,\ 1\frac{1}{4}\right)$

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Equation of tangent at $Q$

$$y-\frac{5}{4} = -\frac{1}{2}\left(x-(-2)\right)$$

$\times 4 \quad 4y-5 = -2x-4$

$\Rightarrow \quad 2x+4y = 1$

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(5)

(a) $P$ has co-ordinates $(ap^2, 2ap)$

$\Rightarrow$at $P$ $\quad x = ap^2$, $\quad y = 2ap$

$4ax = 4a.ap^2 = 4a^2p^2$ $\qquad y^2 = (2ap)^2 = 4a^2p^2$

$\Rightarrow$at $P$ $\quad y^2 = 4ax$

$\Rightarrow$ $P$ lies on the curve $C$

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(b)

$x = ap^2 \qquad y = 2ap$

$\frac{dx}{dp} = 2ap \qquad \frac{dy}{dp} = 2a$

$$\frac{dy}{dx} = \frac{dy}{dp}\times\frac{dp}{dx} = 2a\times\frac{1}{2ap} = \frac{1}{p}$$

Equation of tangent at $P$

$$(y-y_1) = m(x-x_1)$$

$$y-2ap = \frac{1}{p}\left(x-ap^2\right)$$

$\times p \quad py = x+ap^2$

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(c) The equation of the tangent at $Q(aq^2,\ 2aq)$ is

$$qy = x + aq^2$$

Solve the equations of the tangents simultaneously in order to find the point of intersection.

$$py = x + ap^2 \quad -\{1\}$$

$$qy = x + aq^2 \quad -\{2\}$$

$\{1\} - \{2\}$

$$(p-q)y = a(p^2 - q^2)$$

$$y = \frac{a(p+q)(p-q)}{(p-q)}$$

$$y = a(p+q)$$

Substitute $y = a(p+q)$ in equation $\{1\}$

$$ap(p+q) = x + ap^2$$

$\Rightarrow$ $x = apq$

$\Rightarrow$ Point of intersection $L$ has co-ordinates $(apq, a(p+q))$

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(d) Let $A$ be point $(a, o)$.
Then the line through $P$ and $Q$ passes through $A$ if gradient of $PA$ is equal to the gradient of $AQ$

$$\text{gradient of } PA = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2ap - 0}{ap^2 - a} = \frac{2ap}{a(p^2-1)} = \frac{2p}{(p^2-1)}$$

and

$$\text{gradient of } AQ = \frac{0 - 2aq}{a - aq^2} = \frac{-2aq}{a(1-q^2)} = \frac{-2q}{1-q^2}$$

$$= \frac{-2\left(-\frac{1}{p}\right)}{\left(1 - \frac{1}{p^2}\right)} \quad \text{using } pq = -1$$

$$= \frac{2p}{(p^2-1)}$$

$=$ gradient of $PA$ as required

$\therefore$ $(a, o)$ lies on the line $PQ$

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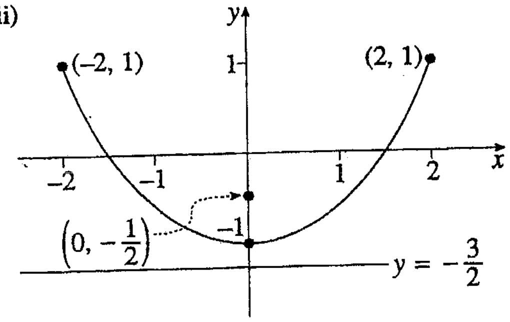
(6) (a) (i)

$$y = 2\cos^2 t - 1$$
$$= 2\left(\frac{x}{2}\right)^2 - 1$$
$$x^2 = 2(y+1)$$

Since $-1 \le \cos t \le 1$, $-2 \le x \le 2$.

(ii)




(7)

$P(2at, at^2)$, $S(0, a)$

(i) $x = \dfrac{2at+0}{t^2+1}$, $\quad y = \dfrac{at^2+at^2}{t^2+1}$

$= \dfrac{2at}{t^2+1} \qquad = \dfrac{2at^2}{t^2+1}$

(ii) $\dfrac{y}{x} = \dfrac{2at^2}{t^2+1} \cdot \dfrac{t^2+1}{2at} = t$

(iii) $x = \dfrac{2at}{t^2+1}$

$$= \frac{2a\left(\frac{y}{x}\right)}{\left(\frac{y}{x}\right)^2 + 1}$$

$$= \frac{2axy}{y^2+x^2}$$

$$\therefore \quad x\left(x^2+y^2\right) = 2axy$$
$$x^2+y^2 = 2ay$$
$$x^2+y^2-2ay+a^2 = a^2$$
$$x^2+(y-a)^2 = a^2,$$

which is the circle, centre $(0, a)$, radius $a$.