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# C.E.M. TUITION

## YEAR 12 – MATHS EXT.1

## REVIEW TOPIC: POLYNOMIALS I

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**EXERCISES:**

(1) Given $f(x) = 2x^3 + 3x^2 - 11x - 6$

(a) show that $f(-3) = 0$. [1]

(b)Hence factorise $f(x)$ completely. [3]

(c) Solve the equation $f(x) = 0$ completely [2]

(2) The polynomial $x^3 - x^2 + ax + b$ has $x - 2$ as a factor. When the polynomial is divided by $x + 5$ there is a remainder of $-56$.

(a) By obtaining two simultaneous equations, find the values of $a$ and $b$. [4]

(b) Find the other factors of the polynomial. [3]

(3) The cubic polynomial $x^3 + Ax - 12$ is exactly divisible by $(x + 3)$.

Find the constant $A$, and solve the equation $x^3 + Ax - 12 = 0$ for this value of $A$. [10]

(4) (a) Show that $(x-2)$ is a factor of the polynomial $x^3+x^2-x-10$. [1]

(b) Find the other factor. Hence show that there is only one solution of the equation $x^3+x^2-x-10=0$. [6]

(5) When the polynomial $2x^3 + ax^2 + bx - 6$ is divided by $x - 2$ there is a remainder of 12. When the polynomial is divided by $x + 3$ there is a remainder of $-18$. By obtaining two simultaneous equations, find the values of $a$ and $b$. [8]

(6) Given the polynomial $x^3 - 4x^2 - 17x + 60$

(a) show that $x - 3$ is a factor. [1]

(b) By dividing, find the other factor and hence factorise the polynomial completely. [3]

(c) Hence solve the equation $x^3 - 4x^2 - 17x + 60 = 0$. [2]

(7) Given the polynomial $f(x) = x^3 - x^2 - 12x - 12$,

(a) show $f(-2) = 0$. [1]

(b) Hence factorise the polynomial as a product of two factors. [3]

(c) Find, to 2 decimal places, the other values of $x$ for which $f(x) = 0$. [2]

(8) When the polynomial $2x^3 - 5x^2 + ax + b$ is divided by $(x + 1)$, the remainder is 20. When the polynomial is divided by $(x - 2)$ the remainder is $-4$.

(a) By obtaining two simultaneous equations, find the values of $a$ and $b$. [8]

(b) Factorise the polynomial completely. [4]

(9) The cubic function $f$ is given by $f(x) = x^3 + ax^2 - 28x + b$ where $a$ and $b$ are constants. $(x + 2)$ is a factor of $f(x)$ and, when $f(x)$ is divided by $(x - 1)$, a remainder of $-84$ is obtained.

Find the values of $a$ and $b$. [8]

(10) A quadratic function is exactly divisible by $(x-2)$ and leaves a remainder of $-18$ when divided by $(x+1)$. (Assume $a=1$)

(a) Find the quadratic function. [4]

(b) Factorise it completely. [1]

(11)

Of the three roots of the cubic equation $x^3 - 15x + 4 = 0$, two are reciprocals.

(i) Find the other root.

(ii) Find all the roots and verify that two of them are reciprocals.

(12)

The cubic polynomial equation $x^3 = ax^2 + bx + c$ has three real roots, two of which are opposites. Prove that

(i) one of the roots is $a$

(ii) the other roots are $\sqrt{b}$ and $-\sqrt{b}$

(iii) $ab + c = 0$.

**SOLUTIONS:**

(1)

(a)

$$f(-3) = 2(-3)^3 + 3(-3)^2 - 11(-3) - 6$$
$$= -54 + 27 + 33 - 6$$
$$f(-3) = 0$$

(b) If $f(-3) = 0$ then $(x+3)$ is a factor

To find the other factor, divide $f(x)$ by $(x+3)$

$$\begin{array}{r} 2x^2 - 3x - 2 \\ (x+3)\overline{)\,2x^3 + 3x^2 - 11x - 6} \\ \underline{2x^3 + 6x^2} \qquad\qquad \\ -3x^2 - 11x \qquad \\ \underline{-3x^2 - 9x} \qquad \\ -2x - 6 \\ \underline{-2x - 6} \\ - \quad - \end{array}$$

$\Rightarrow$

$$f(x) = (x+3)(2x^2 - 3x - 2)$$
$$f(x) = (x+3)(2x+1)(x-2)$$

(c)

$$f(x) = 0$$
$$\Rightarrow \quad (x+3)(2x+1)(x-2) = 0$$
$$\Rightarrow \quad x+3 = 0 \quad \text{or} \quad 2x+1 = 0 \quad \text{or} \quad x-2 = 0$$
$$x = -3, \qquad x = -\frac{1}{2}, \qquad x = 2$$

(2) (a)

$$f(x) = x^3 - x^2 + ax + b$$

$(x-2)$ is a factor of $f(x)$

$$\Rightarrow \quad f(2) = 0$$
$$(2)^3 - (2)^2 + a(2) + b = 0$$
$$2a + b = -4 \qquad - \quad \{1\}$$

When $f(x)$ is divided by $(x+5)$ there is a remainder of $-56$

$$\Rightarrow \quad f(-5) = -56 \text{ using the remainder theorem}$$
$$(-5)^3 - (5)^2 + a(-5) + b = -56$$
$$-125 - 25 - 5a + b = -56$$
$$5a - b = -94 \qquad - \quad \{2\}$$

$\{1\} + \{2\}$

$$7a = -98$$
$$a = -14$$



Substitute $a = -14$ in $\{1\}$

$$2(-14) + b = -4$$
$$b = 24$$

$$\Rightarrow \quad f(x) = x^3 - x^2 - 14x + 24$$

(b) To find the other factors divide $f(x)$ by $x - 2$

$$\begin{array}{r} x^2 + x - 12 \\ (x-2)\overline{)\,x^3 - x^2 - 14x + 24} \\ \underline{x^3 - 2x^2} \qquad\qquad \\ x^2 - 14x \qquad \\ \underline{x^2 - 2x} \qquad \\ -12x + 24 \\ \underline{-12x + 24} \\ - \quad - \end{array}$$

$$f(x) = (x-2)(x^2 + x - 12)$$
$$f(x) = (x-2)(x+4)(x-3)$$

(3)

$$f(x) = x^3 + Ax - 12$$

$f(x)$ is exactly divisible by $(x+3)$

$$\Rightarrow \quad f(-3) = 0$$
$$(-3)^3 + A(-3) - 12 = 0$$
$$-27 - 3A - 12 = 0$$
$$A = -13$$
$$f(x) = x^3 - 13x - 12$$

Divide $f(x)$ by $(x+3)$ to find the other factor

$$\begin{array}{r} x^2 - 3x - 4 \\ x+3\overline{)\,x^3 + 0x^2 - 13x - 12} \\ \underline{x^3 + 3x^2} \qquad\qquad \\ -3x^2 - 13x \qquad \\ \underline{-3x^2 - 9x} \qquad \\ -4x - 12 \\ \underline{-4x - 12} \\ - \quad - \end{array}$$

$$f(x) = (x+3)(x^2 - 3x - 4)$$
$$= (x+3)(x+1)(x-4)$$

To solve

$$x^3 - 13x - 12 = 0$$
$$(x+3)(x+1)(x-4) = 0$$
$$\Rightarrow \quad x+3 = 0 \qquad x+1 = 0 \qquad x-4 = 0$$
$$x = -3, -1, 4$$

(4)

(a)

$$\begin{aligned} f(x) &= x^3 + x^2 - x - 10 \\ f(2) &= (2)^3 + (2)^2 - (2) - 10 \\ &= 0 \end{aligned}$$

$\Rightarrow$ $(x-2)$ is a factor of the polynomial

(b) To find the other factor divide $f(x)$ by $(x-2)$

$$\begin{array}{r} x^2 + 3x + 5 \\ (x-2)\overline{)x^3 + x^2 - x - 10} \\ \underline{x^3 - 2x^2} \qquad\qquad \\ 3x^2 - x \qquad \\ \underline{3x^2 - 6x} \qquad \\ 5x - 10 \\ \underline{5x - 10} \\ - \quad - \end{array}$$

$$f(x) = (x-2)(x^2 + 3x + 5)$$

$$x^3 + x^2 - x - 10 = 0$$

$\Rightarrow$ $(x-2)(x^2 + 3x + 5) = 0$

$x - 2 = 0$ or $x^2 + 3x + 5 = 0$

$x = 2$ $\quad a = 1,\ b = 3,\ c = 5$

$b^2 - 4ac = 3^2 - 4 \times 1 \times 5$

$= -11 < 0$

$\Rightarrow$ The quadratic has no real solution

$\Rightarrow$ The equation has one real solution only

$x = 2$

(5) $f(x) = 2x^3 + ax^2 + bx - 6$

$f(x)$ has a remainder of 12 when divided by $(x-2)$

$\Rightarrow$ $f(2) = 2(2)^3 + a(2)^2 + b(2) - 6 = 12$

$4a + 2b = 2$ $\quad\{1\}$

$f(x)$ has a remainder of $-18$ when divided by $(x+3)$

$\Rightarrow$ $f(-3) = 2(-3)^3 + a(-3)^2 + b(-3) - 6 = -18$ using the remainder theorem

$9a - 3b = 42$ $\quad\{2\}$

$\{1\} \div 2$ $\quad 2a + b = 1$ $\quad\{3\}$

$\{2\} \div 3$ $\quad 3a - b = 14$ $\quad\{4\}$

$\{3\} + \{4\}$ $\quad 5a = 15$

$\Rightarrow$ $\quad a = 3$



Substitute $a = 3$ in $\{3\}$

$$\begin{aligned} 2(3) + b &= 1 \\ b &= -5 \end{aligned}$$

$$f(x) = 2x^3 + 3x^2 - 5x - 6$$

(6) (a)

$$\begin{aligned} f(x) &= x^3 - 4x^2 - 17x + 60 \\ f(3) &= (3)^3 - 4(3)^2 - 17(3) + 60 \\ &= 27 - 36 - 51 + 60 \\ &= 0 \end{aligned}$$

$\Rightarrow$ $(x-3)$ is a factor of $f(x)$

(b)

$$\begin{array}{r} x^2 - \quad x - 20 \\ (x-3)\overline{)x^3 - 4x^2 - 17x + 60} \\ \underline{x^3 - 3x^2} \qquad\qquad\quad \\ -x^2 - 17x \qquad \\ \underline{-x^2 + 3x} \qquad \\ -20x + 60 \\ \underline{-20x + 60} \\ - \quad - \end{array}$$

$$\begin{aligned} f(x) &= (x-3)(x^2 - x - 20) \\ f(x) &= (x-3)(x+4)(x-5) \end{aligned}$$

(c)

$f(x) = 0$

$\Rightarrow$ $(x-3)(x+4)(x-5) = 0$

$\Rightarrow$ $x - 3 = 0$ $\quad x + 4 = 0$ $\quad x - 5 = 0$

$x = 3, -4, 5$

(7) (a)

$$\begin{aligned} f(-2) &= (-2)^3 - (-2)^2 - 12(-2) - 12 \\ &= -8 - 4 + 24 - 12 \\ f(-2) &= 0 \end{aligned}$$

(b) $\Rightarrow$ $(x+2)$ is a factor

To find the other factor divide $f(x)$ by $(x+2)$

$$\begin{array}{r} x^2 \quad -3x \quad -6 \\ (x+2)\overline{)x^3 - x^2 - 12x - 12} \\ \underline{x^3 + 2x^2} \qquad\qquad \\ -3x^2 - 12x \qquad \\ \underline{-3x^2 - 6x} \qquad \\ -6x - 12 \\ \underline{-6x - 12} \\ \end{array}$$

$$f(x) = (x+2)(x^2 - 3x - 6)$$

(c) The other values of $x$ for which $f(x) = 0$ are given by

$$x^2 - 3x - 6 = 0$$

This does not factorise

$a = 1 \qquad b = -3 \qquad c = -6$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-3) \pm \sqrt{(-3)^2 - 4\times1\times(-6)}}{2\times1}$$

$$x = \frac{3 \pm \sqrt{33}}{2}$$

$x = -1.37$ or $x = 4.37$ to 2 decimal places

(8) (a)

$$f(x) = 2x^3 - 5x^2 + ax + b$$

$f(x)$ divided by $(x + 1)$ gives a remainder of 20.

$\Rightarrow \quad f(-1) = 20$ using the remainder theorem

$2(-1)^3 - 5(-1)^2 + a(-1) + b = 20$

$\Rightarrow \quad a - b = -27 \qquad \{1\}$

$f(x)$ divided by $(x - 2)$ gives a remainder of $-4$

$\Rightarrow \quad f(2) = -4$ using the remainder theorem

$2(2)^3 - 5(2)^2 + a(2) + b = -4$

$2a + b = 0 \qquad \{2\}$

$\{1\} + \{2\} \qquad 3a = -27 \qquad a = -9$

Substitute $a = -9$ in $\{2\}$

$2(-9) + b = 0$

$b = 18$

$\Rightarrow \quad f(x) = 2x^3 - 5x^2 - 9x + 18$



(b)

$$\begin{aligned} f(1) &= 2(1)^3 - 5(1)^2 - 9(1) + 18 \\ &= -6 \end{aligned}$$

$\Rightarrow$When $f(x)$ is divided by $(x - 1)$ a remainder of $-6$ is obtained. Hence $(x - 1)$ cannot be a factor

$$\begin{aligned} f(-1) &= 2(-1)^3 - 5(-1)^2 - 9(-1) + 18 \\ &= 20 \end{aligned}$$

$\Rightarrow(x + 1)$ cannot be a factor.

$$\begin{aligned} f(2) &= 2(2)^3 - 5(2)^2 - 9(2) + 18 \\ &= -4 \end{aligned}$$

$\Rightarrow(x - 2)$ cannot be a factor

$$\begin{aligned} f(-2) &= 2(-2)^3 - 5(-2)^2 - 9(-2) + 18 \\ &= 0 \end{aligned}$$

$(x + 2)$ is a factor

Divide $f(x)$ by $(x + 2)$ to find the other factor

$$\begin{array}{r} 2x^2 - 9x \;+9 \\ (x+2)\overline{)2x^3 - 5x^2 - 9x + 18} \\ \underline{2x^3 + 4x^2} \qquad\qquad \\ -9x^2 - 9x \qquad \\ \underline{-9x^2 - 18x} \qquad \\ 9x + 18 \\ \underline{9x + 18} \\ - \quad - \end{array}$$

$$f(x) = (x + 2)(2x^2 - 9x + 9)$$
$$f(x) = (x + 2)(2x - 3)(x - 3)$$

(9)

$$f(x) = x^3 + ax^2 - 28x + b$$

$(x + 2)$ is a factor

$\Rightarrow \quad f(-2) = 0$

$\Rightarrow \quad (-2)^3 + a(-2)^2 - 28(-2) + b = 0$

$4a + b = -48 \qquad \{1\}$

$f(x)$ divided by $(x - 1)$ gives a remainder of $-84$

$\Rightarrow \quad f(1) = -84$ using the remainder theorem

$(1)^3 + a(1)^2 - 28(1) + b = -84$

$a + b = -57 \qquad \{2\}$

$\{1\} - \{2\} \qquad 3a = 9$

$\Rightarrow \quad a = 3$

Substitute $a = 3$ in {2}

$$\begin{aligned} 3 + b &= -57 \\ b &= -60 \\ f(x) &= x^3 + 3x^2 - 28x - 60 \end{aligned}$$

(10) (a) $f(x) = x^2 + bx + c$

Where $b$ and $c$ are constants ($a = 1$)

$f(x)$ is exactly divisible by $(x - 2)$

$$\begin{aligned} \Rightarrow \quad f(2) &= 0 \\ (2)^2 + b(2) + c &= 0 \\ 2b + c &= -4 \qquad \{1\} \end{aligned}$$

$f(x)$ leaves a remainder of $-18$ when dividing by $(x + 1)$

$$\begin{aligned} \Rightarrow \quad f(-1) &= -18 \text{ using the remainder theorem} \\ (-1)^2 + b(-1) + c &= -18 \\ b - c &= 19 \qquad \{2\} \end{aligned}$$

{1} + {2}

$$\begin{aligned} 3b &= 15 \\ b &= 5 \end{aligned}$$

Substitute $b = 5$ in {2}

$$\begin{aligned} 5 - c &= 19 \\ c &= -14 \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad f(x) &= x^2 + 5x - 14 \\ f(x) &= (x + 7)(x - 2) \end{aligned}$$

(11)

Let the roots be $\alpha$, $\frac{1}{\alpha}$, $\beta$.

(i) $(\alpha)\left(\frac{1}{\alpha}\right)(\beta) = -4$

$\beta = -4$

(ii)

$$\begin{aligned} x^3 - 15x + 4 &= 0 \\ (x+4)(x^2 - 4x + 1) &= 0 \\ \therefore \quad x &= -4 \\ \text{or} \quad x^2 - 4x + 4 &= 3 \\ (x-2)^2 &= 3 \end{aligned}$$

$\therefore\ x = 2 + \sqrt{3}$ or $x = 2 - \sqrt{3}$

and $(2 + \sqrt{3})(2 - \sqrt{3}) = 1$



(12)

Let the roots be $\alpha$, $-\alpha$, and $\beta$.

$x^3 - ax^2 - bx - c = 0$

(i)

$$\begin{aligned} \alpha + (-\alpha) + \beta &= a \\ \beta &= a \end{aligned}$$

(ii)

$$\begin{aligned} (\alpha)(-\alpha) + (\alpha)(a) + (-\alpha)(a) &= -b \\ \alpha^2 &= b \end{aligned}$$

$\therefore\ \alpha = \sqrt{b}$ or $\alpha = -\sqrt{b}$

(iii)

$$\begin{aligned} (a)(\sqrt{b})(-\sqrt{b}) &= c \\ -ab &= c \\ ab + c &= 0 \end{aligned}$$