C.E.M. TUITION

MINI TRIAL HSC EXAMINATION 1996

MATHEMATICS

3/4 UNIT COMMON PAPER

Total time allowed - Two hours

(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES :

- Attempt ALL questions.
- All questions are of equal value.
- All necessary working should be shown in every question.
- Full marks may not be awarded for badly arranged work.
- Standard integrals are on the inside cover.
- Approved silent calculators may be used.
- You must hand in a blank page if a question is unanswered.

Question 1 **Marks**

(a) Differentiate and simplify with respect to x, the function **2**

$$x\tan^{-1}x$$

(b) Solve $\dfrac{3x}{x-4} > 1$ **2**

(c) Solve the following equations for $0^0 \le \theta \le 360^0$ **2**

$$4\cos 2\theta - 5\sin 30^0 = 0$$

(d) Evaluate $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{1-4x^2}}$ **3**

(e) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are points on the parabola $x^2 = 4ay$. **3**

If PQ is a normal to P, show that

$$p^2 + pq + 2 = 0$$

Question 2 **Marks**

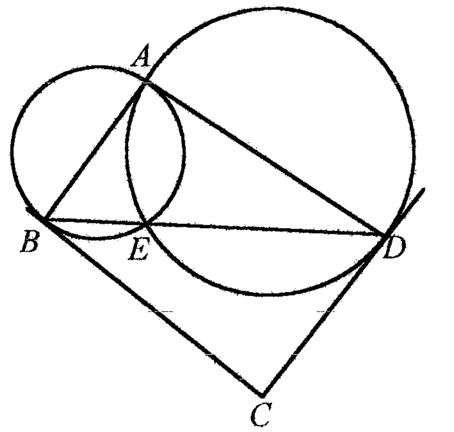
(a) Show that $\frac{d}{dx}(\sin^3 x \cos x) = 3\sin^2 x - 4\sin^4 x$. **2**

(i) Hence deduce that **1**

$$\int_0^{\frac{\pi}{2}} \sin^4 x \, dx = \frac{3}{4}\int_0^{\frac{\pi}{2}} \sin^2 x \, dx$$

(ii) And evaluate $\int_0^{\frac{\pi}{2}} \sin^4 x \, dx$ **2**

(b)



The circles intersect at A and E; BED is a straight line;

BC and CD are tangents.

(i) Copy the diagram into your booklet. **1**

(ii) Prove that $ABCD$ is a cyclic quadrilateral. **3**

(c) Sketch the graph and state the domain and range of the function **3**

$y = 2\cos^{-1} 3x$

Question 3 **Marks**

(a) Given that $f(x) = \frac{ax+2}{x-1}$ for $x \neq 1$ and that $f(2) = 8$, find **3**

(i) the value of a,

(ii) an expression for $f^{-1}(x)$ and state its domain and range.

(b) A particle moves in a straight line and its acceleration at any time t is $\cos^2 x$. **3**

If initially $v = 0$ and $x = \pi$, express v in terms of x.

(c) Find the first three terms of the expansion, in ascending powers of x, of **6**

(i) $(1-2x)^5$,

(ii) $(1+3x)^5$

(iii) Hence find the coefficient of x^6 in the expansion of $(1+x-6x^2)^5$.

Question 4 **Marks**

(a) Show by mathematical induction that if n is a positive integer then $7^n + 2$ is divisible by 3. **4**

(b) Use the substitution $u = x^2 - 1$ to evaluate **3**

$$\int_1^{\sqrt{2}} 6x(x^2 - 1)^4 \, dx$$

(c) Given that $x^3 + 4x^2 - ax + b$ is exactly divisible by $x + 2$ but leaves a remainder a^3 when divided by $x - a$, calculate values of a and of b. **3**

(d) Given that $\tan\theta = t$, find a simplified expression, in terms of t, for **2**

$\cos 2\theta$

Question 5 **Marks**

(a) Show that a root exists between 1 and 2 for the equation **4**

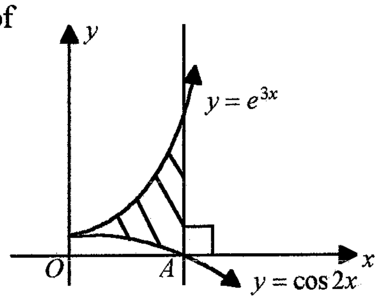
$$e^x \log_e x - 1 = 0$$

Taking $x = 1$ as the first approximation, use Newton's method to find a second approximation, correct to 3 decimal places.

(b) The diagram shows part of the graphs of $y = e^{3x}$ and $y = \cos 2x$. Find **4**

(i) the x–coordinates of A,

(ii) the area of the shaded region. (Answer to 2 decimal places).



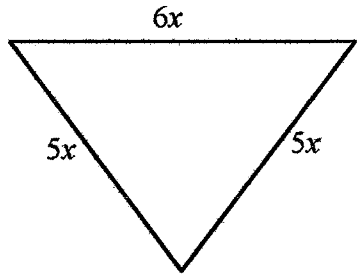
(c) The first term of a geometric progression is a, and the common ratio, r, is positive.

(i) Given that the sum of the second and third terms is $\frac{10a}{9}$, calculate the value of r. **3**

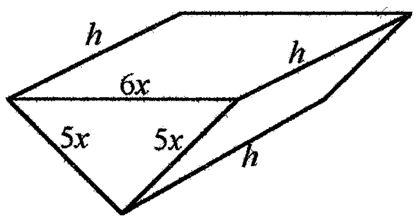
(ii) Find its limiting sum. **1**

Question 6

(a) Show that the area of the triangle below is $12x^2$ units^2. **2**



(b) A container with an open rectangular top is constructed from four pieces of cardboard sheet. The two end pieces are isosceles triangles with sides $6x$ cm, $5x$ cm and $5x$ cm as shown below.



The two side pieces are rectangles of length h cm and width $5x$ cm. The total amount of cardboard sheet used is 450 cm^2.

(i) Using the result in part (a) or otherwise, show that $h = \dfrac{45 - 2.4x^2}{x}$. **2**

(ii) Show that the volume of the container, V cm^3, is given by **2**

$$V = 540x - 28.8x^3.$$

(iii) Find the value of x for which V has a stationary value. Find this value of V and determine whether it is a maximum or a minimum. **6**

Question 7 **Marks**

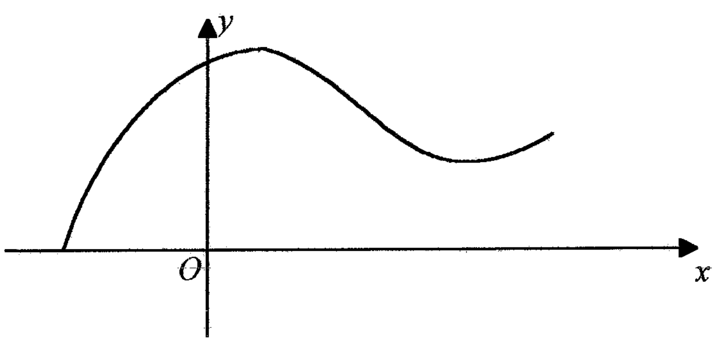
(a) A function is given by $y = 3x^4 - 4x^3 - 12x^2 + 1$.

(i) Find the coordinates of the stationary points and determine their nature. **4**

(ii) Find the point(s) of inflexion. **1**

(iii) Draw a sketch of the function showing all the important features. **2**

(b) The diagram shows the graph of a certain function $y = f(x)$.



(i) Copy the graph into your Writing Booklet. **1**

(ii) On the same set of axes, draw a sketch of the derivative $f'(x)$, of the function. **2**

(c) Given that the area of a sector is 0.6 cm^2, and its radius is 2 cm, find the value of the angle subtended at its centre in degrees and minutes. **2**

Question 8 **Marks**

(a) By considering coefficients of x^4 on both sides of **4**

$$(1+x)^{20} = (1+x)^{10}(1+x)^{10},$$

show that $\binom{20}{4} = 2\binom{10}{0}.\binom{10}{4} + 2\binom{10}{1}\binom{10}{3} + \binom{10}{2}^2$

and find the value.

(b) Find the greatest coefficient in the expansion of $(3+x)^{10}$. **5**

(c) If $(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$, show that $\sum_{r=1}^{n} r\binom{n}{r} = n2^{n-1}$. **3**

Question 1 **Marks**

(a) $\frac{dy}{dx}=x.\frac{1}{1+x^2}+\tan^{-1}x.1$ 1

$=\frac{x}{1+x^2}+\tan^{-1}x$ 1

(b) $(x-4)^2.\frac{3x}{x-4}>1.(x-4)^2$

$3x(x-4)>x^2-8x+16$

$2x^2-4x-16>0$

$(x-4)(x+2)>0$ 1

$x<-2$ or $x>4$ 1

(c) For $0^0\le\theta\le360^0$

$0^0\le2\theta\le720^0$

$4\cos2\theta=5\left(\frac{1}{2}\right)$

$\cos2\theta=\frac{5}{8}$ 1

$2\theta=51^019', 308^041', 411^019', 668^041'$

$\theta=25^040', 154^021', 205^040', 334^021'$ 1

(d) $\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{2\sqrt{\left(\frac{1}{2}\right)^2-x^2}}dx$

$=\frac{1}{2}\left[\sin^{-1}(2x)\right]_{-\frac{1}{2}}^{\frac{1}{2}}$ 1

$=\frac{1}{2}\left[\sin^{-1}1-\sin^{-1}(-1)\right]$ 1

$=\frac{1}{2}\left[2.\frac{\pi}{2}\right]$ 1

$=\frac{\pi}{2}$

(e) $y=\frac{x^2}{4a}\Rightarrow\frac{dy}{dx}=\frac{2x}{4a}=\frac{4ap}{4a}=p$

$m_{\text{tangent}}=p$ 1

$m_{PQ}=\frac{a(p^2-q^2)}{2a(p-q)}=\frac{(p+q)}{2}$ 1

Since $m_{\text{tangent}}\times m_{PQ}=-1$

$\frac{p(p+q)}{2}=-1$ 1

$p^2+pq=-2\Rightarrow p^2+pq+2=0$ as reqd.

Question 2

(a) L.H.S.$=\frac{d}{dx}\left(\sin^3x.\cos x\right)$

$=\sin^3x(-\sin x)+\cos x(3\sin^2x\cos x)$ 1

$=-\sin^4x+3\sin^2x\cos^2x$

$=-\sin^4x+3\sin^2x-3\sin^4x$ 1

$=-4\sin^4x+3\sin^2x$ =R.H.S.

(i) $\int_0^{\frac{\pi}{2}}\frac{d}{dx}\left(\sin^3x\cos x\right)dx$

$\int_0^{\frac{\pi}{2}}(3\sin^xx-4\sin^4x)dx$

$[\sin^3x\cos x]_0^{\frac{\pi}{2}}=\int_0^{\frac{\pi}{2}}3\sin^2x\,dx-\int_0^{\frac{\pi}{2}}4\sin^4x\,dx$

$[0-0]+\int_0^{\frac{\pi}{2}}4\sin^4x\,dx=3\int_0^{\frac{\pi}{2}}\sin^2x\,dx$ 1

Therefore,

$\int_0^{\frac{\pi}{2}}\sin^4x\,dx=\frac{3}{4}\int_0^{\frac{\pi}{2}}\sin^2x\,dx$ as reqd.

(ii) $\int_0^{\frac{\pi}{4}}\sin^4x\,dx$

$=\frac{3}{4}\int_0^{\frac{\pi}{4}}\left(\frac{1-\cos2x}{2}\right)dx$ 1

$=\frac{3}{8}[x-\frac{\sin2x}{2}]_0^{\frac{\pi}{2}}$

$=\frac{3}{8}[\left(\frac{\pi}{2}-0\right)-(0-0)]$ 1

$=\frac{3\pi}{16}$

(b) (ii) Construction : Join AE

Proof : Let $\angle DBC=\alpha$

$\angle BAE=\alpha$ ($\angle$ in the same segment)

Let $\angle BDC=\beta$

$\angle EAD=\beta$ ($\angle$ in the alt.segment) 1

Since $\angle BCD=180-(\alpha+\beta)$

($\angle$ sum of ΔBCD) 1

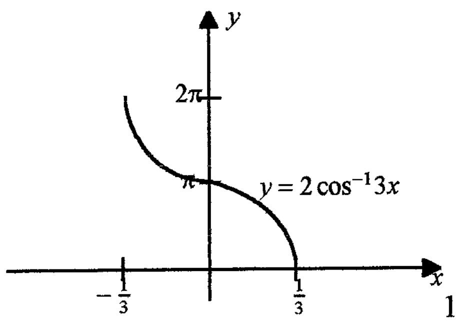
Then $\angle BCD+\angle BAD=$

$180-(\alpha+\beta)+(\alpha+\beta)=180$ 1

Therefore Quad. $ABCD$ is cyclic

(opposite angles are supplementary).

(c)



1

$D: -1 \le 3x \le 1 \Rightarrow -\frac{1}{3} \le x \le \frac{1}{3}$ 1

$R: 0 \le y \le 2\pi$ 1

Question 3

(a) (i) $f(2) = \dfrac{2a-2}{2-1} = 8$

$a = 3$ 1

(ii) $y = \dfrac{3x-2}{x-1}$

$xy - y = 3x + 2$

$x(y-3) = y + 2$

$x = \dfrac{y+2}{y-3}$ 1

$f^{-1}(x) = \dfrac{x+2}{x-3}$

$\{D: x \ne 3\}, \{R: y \ne 1\}$ 1

(b) If $\frac{d}{dx}(\frac{1}{2}v^2) = \cos^2 x$, then

$\frac{1}{2}v^2 = \int \cos^2 x\, dx$

Using the identity : $\cos 2x = 2\cos^2 x - 1$

$\frac{1}{2}v^2 = \int \frac{1}{2}(\cos 2x + 1)\, dx$ 1

$v^2 = \dfrac{\sin 2x}{2} + x + c$

when $v = 0, x = \pi \Rightarrow c = -\pi$ 1

$v = \sqrt{\frac{\sin 2x}{2} + x - \pi}$ 1

(c) (i) $1 - 10x + 40x^2$ 1

(ii) $1 + 15x + 90x^2$ 1

(iii) $[(1-2x)(1+3x)]^5$ 1

$= (1-2x)^5(1+3x)^5$ 1

$= (1 - 10x + 40x^2 + ..)(1 + 15x + 90x^2 + ...)$

Consider coeff. of the term in x^6

$-10.243 + 40.405 - 80.270 + 80.90 - 32.15$

$= -1110$ 2

Question 4

(a) Step 1: Prove true for $n = 1$
i.e. $7^1 + 2 = 9$ which is divisible by 9 1

Step 2 : Assume true for $n = k$
i.e. $7^k + 2 = 3M$ where M is an integer
To prove true for $n = k+1$
i.e. $7^{k+1} + 2 = 3N$ where N is another integer
Proof : L.H.S. $= 7^K.7^1 + 2$
$= 7(7^K + 2) - 14 + 2$
$= 7(3M) - 12$
$= 3(7M - 4) = 3N$ 2

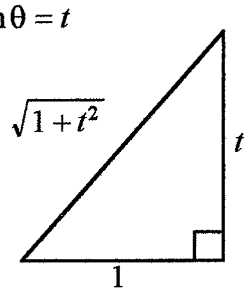
Step 3: If it is true for $n = 1$, and also true for $n = 1 + 1 = 2$, then it is true for all positive integer value n.

(b) $\int_1^{\sqrt{2}} 6x(x^2-1)^4\, dx$
Let $u = x^2 - 1 \Rightarrow \dfrac{du}{dx} = 2x$
if $x = 1, u = 0; x = \sqrt{2}, u = 1$ 1
$\int_0^1 3u^4\, du = 3\left[\dfrac{u^5}{5}\right]_0^1$ 1
$= 3\left(\frac{1}{5}\right) = \frac{3}{5}$

(c) Let $P(x) = x^3 + 4x^2 - ax + b$ 1
$P(-2) = -8 + 4(-2)^2 + 2a + b = 0$

$2a+b=-8$ ..... (i)
Also, $P(a)=a^3+4a^2-a^2+b=a^3$
$3a^2+b=0$ ... (ii)
Sub. (ii) into (i)
$3a^2-8-2a=0$
$(3a+4)(a-2)=0$
$a=-\frac{4}{3}$ or 2
$b=\frac{-16}{3}$ or -12

(d) If $\tan\theta=t$



$\cos 2\theta=\cos^2\theta-\sin^2\theta$ 1
$=\frac{1}{1+t^2}-\frac{t^2}{1+t^2}$

$=\frac{1-t^2}{1+t^2}$ 1

Question 5

(a) Let $f(x)=e^x\ln x-1$
$f(1)=e\ln 1-1<0$
$f(2)=e^2\ln 2-1>0$ 1
Therefore a root exists between 1 and 2.

$f'(x)=e^x.\ln x+e^x.\frac{1}{x}$
$=e^x\left(\frac{1}{x}+\ln x\right)$ 1
$f(1)=-1; f'(1)=e(1+\ln 1)=e$

Using Newton's method of approximation,
$x_2=x_1-\frac{f(x)}{f'(x)}$ 1

$=1-\frac{-1}{e}=1+\frac{1}{e}$

$=1.368$ (to 3 d.p) 1

(b) (i) $y=0,\ \cos 2x=0$
$2x=\frac{\pi}{2},\ x=\frac{\pi}{4}$
$A(\frac{\pi}{4},0)$ 1

(ii) Area $=\int_0^{\frac{\pi}{4}} e^{3x}-\cos 2x\,dx$ 1

$=\left[\frac{e^{3x}}{3}-\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}}$ 1

$=\left[\frac{e^{\frac{3\pi}{4}}}{3}-\frac{1}{2}\right]-\frac{1}{3}$

$=2.68$(to 2 d.p) 1

(c) (i) $\frac{10a}{9}=T_2+T_3=ar^1+ar^2$
$10=9r+9r^2$ 1
$(3r+5)(3r-2)=0$ 1
$r=-\frac{5}{3}$ or $\frac{2}{3}$
Therefore $r=\frac{2}{3}$ for positive r. 1

(ii) $S_\infty=\frac{a}{1-\frac{2}{3}}=3a$ 1

Question 7

(a) $y=3x^4-4x^3-12x^2+1$
$y'=12x^3-12x^2-24x$
$y''=36x^2-24x-24$ 1

For Stationary points, $y'=0$
$12x(x-2)(x+1)=0$
$x=0, 2$ or -1 1

Testing nature of: 2

$x=0, y=1, f''(0)<0$
$(0,1)$ is a local max.

$x=2, y=-31, f''(2)>0$
$(2,-31)$ is a local min.

$x=-1, y=-4, f''(-1)>0$
$(-1,-4)$ is a local min.

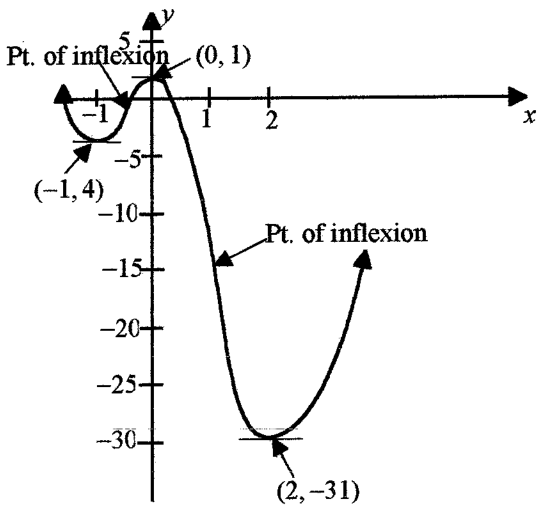
(ii) For pts of inflexion, $y''=0$ 1
$3x^2-2x-2=0$

$$x = \frac{2 \pm 2\sqrt{7}}{6} = \frac{1 \pm \sqrt{7}}{3}$$

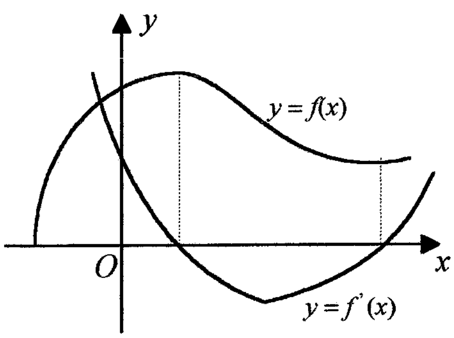
Test for concavity,
$f''(\alpha^-) < 0$ and $f''(\alpha^+) > 0$ then
α is an inflexion point.

$f''(\beta^-) < 0$ and $f''(\beta^+) > 0$ then
β is also an inflexion point.

(iii)



(b)



(c) $A = \frac{1}{2} r^2 \theta$

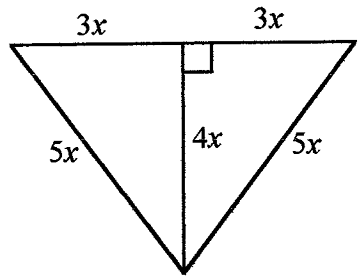
$0.6 = \frac{1}{2} \times 4\theta$

$\theta = 0.3$ radians 1

$\theta = 0.3 \times \frac{180^0}{\pi} = 17^0 11'$ 1

Question 6

(b) (i)



By Pythagoras' Theorem ,

Height of $\Delta = 4x$ 1

Area $= \frac{1}{2} \times 6x \times 4x = 12x^2$ 1

(b) (i) $S.A. = 2(12x^2 + 5xh)$

$450 = 24x^2 + 10xh$ 1

Therefore, $h = \frac{450 - 24x^2}{10x}$

$= \frac{10(45 - 2.4x^2)}{10x}$ 1

$=$ R.H.S

(ii) $V = 12x^2 h = 12x^2 \left(\frac{45 - 2.4x^2}{x} \right)$ 1

$= \frac{540x^2 - 28.8x^4}{x}$ 1

$= 540x - 28.8x^3$

(iii) $\frac{dV}{dx} = 540 - 86.4x^2$ 1

$\frac{d^2V}{dx^2} = -172.8x$ 1

For stationary values, $V' = 0$
therefore,
$x = \pm 2.5$, for $x = 2.5$ $V'' < 0$ 2

Therefore, Volume is a maximum
$V = 540(2.5) - 28.8(2.5)^3$
$= 900$ units^3 2

Question 8 :

(a) Coefficient of x^4 in $(1+x)^{20}$ is $\binom{20}{4}$

Terms in x^4 in $(1+x)^{10}(1+x)^{10}$ are given by

$\binom{10}{0}\binom{10}{4}x^4+\binom{10}{1}x\binom{10}{3}x^3+\binom{10}{2}x^2\binom{10}{2}x^2$

$+\binom{10}{3}x^3\binom{10}{1}x+\binom{10}{4}x^4\binom{10}{0}$ 2

Coefficient in x^4 is $2.\binom{10}{0}\binom{10}{4}+2.\binom{10}{1}\binom{10}{3}$

$+\binom{10}{2}^2=420+2400+2045=4845$ 2

(b) $\dfrac{T_{k+1}}{T_k}=\dfrac{\binom{10}{k}3^{10-k}x^k}{\binom{10}{k-1}3^{11-k}x^{k-1}}$

$=\dfrac{10!}{(10-k)!}\times\dfrac{[10-(k-1)]!(k-1)!x}{3\times 10!}$

$=\dfrac{10-k+1}{3k}\times x$

$=\dfrac{11-k}{3k}\times x$ 2

For the coefficient of $T_{k+1}>T_k$, the coefficient of $\dfrac{T_{k+1}}{T_k}>1$

i.e. $\dfrac{11-k}{3k}>1$ 1

$11-k>3k\Rightarrow k<2.75$ 1

So for $k=1,2,$ the coefficient of $T_{k+1}>T_k$,

for $k=3,4,5,\ldots$ the coefficient of $T_{k+1}<T_k$,

therefore the term with the greatest

coefficient is $k=2$, i.e.

$\binom{10}{2}3^{10-2}=45\times 3^8=295\ 245.$ 1

(c) $(1+x)^n=\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\ldots+\binom{n}{n}x^n$

Differentiating both sides w.r.t. x gives

$n(1+x)^{n-1}=\binom{n}{1}+2\binom{n}{2}x+\ldots+n\binom{n}{n}x^{n-1}$ 1

Let $x=1$, therefore 1

$n(2)^{n-1}=\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+\ldots+n\binom{n}{n}$ 1

$n.2^{n-1}=\sum_{r=1}^{n} r\binom{n}{r}$ as required.