C.E.M. TUITION

Student Name : ______________________________

Review Topic : Exponential Growth & Decay

(HSC Course - Paper 2)

Year 12 - 2 Unit

1996

6. In an experiment it was found that the temperature θ of a body after t minutes is given by $\theta = 110\,e^{-0\cdot 0637\,t}$, where θ is in °C.

(a) What was the initial temperature of the body?

(b) What will the temperature of the body be after 10 minutes?

(c) After how long is the temperature of the body 75°C.

7. The amount, N, grams of a radioactive substance is given by $N = No\, e^{-kt}$, where No and k are constants and t is the time in years. In 3 years the radioactive substance decays from 30 grams to 20 grams.

(a) Show that N satisfies the equation $\frac{dN}{dt} = -kN$.

(b) After how many years does only 30% of the initial amount of the substance remain?

(c) What is the half-life of the radioactive substance?

8. On an island the population at the beginning of 1970 was 1472, and at the beginning of 1980 it was 1581. Assume that the population of the island is governed by the equation $P = Po\, e^{kt}$, where Po and k are constants and where t is the time in years.

(a) Find k, correct to two significant figures.

(b) What will the population of the island be at the beginning of 1994?

(c) In which year will the population of the island be 2000?

9. A botanist growing trees under experimental conditions discovered that for a particular species the diameter D (cm) of the tree increased according to the formula $D = Do\ e^{kt}$, where Do and k are constants and where t is the time in years.

(a) Given that the diameter D of the tree doubled every 5 years, calculate the value of k correct to 3 significant figures.

(b) If at the beginning of 1980 the diameter of the tree was 50 cm, what will the diameter be at the beginning of 1995?

(c) In how many years will the diameter of the tree be three times its initial diameter?

10. (a) The population of Comptown at the beginning of 1980 was 17 320 and at the beginning of 1990 it was 12 700. If the population is governed by the equation $P_C = Po\, e^{kt}$, find the value of Po and k.

(b) In Jonestown the population at the beginning of 1980 was 10 700 and at the beginning of 1985 it was 12 500. The population of Jonestown is governed by the equation $P_J = Po\, e^{kt}$. Find the value of Po and k.

(c) Draw a sketch of the graphs of P_C and P_J on the same set of coordinate axes.

(d) During which year will the population of Jonestown become larger than the population of Comptown?

6. $\theta = 110e^{-0\cdot 0637t}$

(a) When $t = 0$,

$$\theta = 110e^{-0\cdot 0637\times 10}$$
$$= 110e^{0} \quad \boxed{e^0 = 1}$$
$$= 110.$$

$\therefore$ The initial temperature is 110°C.

(b) When $t = 10$,

$$\theta = 110e^{-0\cdot 0637\times 10}$$
$$\approx 58$$

$\therefore$ After 10 minutes the temperature of the body will be 58°C.

(c) The question is asking us to find t when $\theta = 75$.

$$\theta = 110e^{-0\cdot 0637t}$$
$$75 = 110e^{-0\cdot 0637t}$$
$$\frac{75}{110} = e^{-0\cdot 0637t}$$
$$-0\cdot 0637t = \ln\left(\frac{75}{110}\right)$$
$$t = \frac{\ln\left(\frac{75}{110}\right)}{-0\cdot 0637}$$
$$\therefore \quad t \approx 6.$$

Therefore, the temperature of the body is 75°C after 6 minutes.

7. (a) $N = No\,e^{-kt}$ —①

$$\frac{dN}{dt} = No\,e^{-kt} \times -k$$
$$= -k \times No\,e^{-kt}$$

$\boxed{\text{But } N = No\,e^{-kt} \text{ [from ①]}}$

$$= -kN \quad \therefore \frac{dN}{dt} = -kN.$$

(b) $N = No\,e^{-kt}$

When $t = 0$, $N = 30 \Rightarrow No = 30$

$$\therefore \quad N = 30e^{-kt}$$

When $t = 3$, $N = 20$ (data to calculate the value of k).

$$\therefore \quad 20 = 30e^{-k\times 3}$$
$$\frac{20}{30} = e^{-k}$$
$$-3k = \ln\left(\frac{20}{30}\right)$$
$$k = \frac{\ln\left(\frac{20}{30}\right)}{-3}$$

$\therefore \quad k \approx 0\cdot 135$ (3 sign. figs.)

$\therefore \quad N = No\,e^{-0\cdot 135t}$.

The question is asking us to find t when $N = 30\%$ of No

$$= \frac{30}{100} \times No = \frac{3}{10}No.$$

When $N = \frac{3}{10}No$,

$$\frac{3}{10}No = No\,e^{-0\cdot 135t}$$

(Divide both sides by No)

$$\Rightarrow \quad \frac{3}{10} = e^{-0\cdot 135t}$$
$$-0\cdot 135t = \ln\left(\frac{3}{10}\right)$$
$$t = \frac{\ln\left(\frac{3}{10}\right)}{-0\cdot 135}$$
$$\therefore \quad t \approx 8\cdot 92.$$

Therefore, after 8·92 years, only 30% of the initial amount of the substance remains.

(c) Half-life of the substance $\Rightarrow$ find t when $N = \frac{1}{2}No$.

When $N = \frac{1}{2}No$,

$$\tfrac{1}{2}No = No\,e^{-0\cdot 135t}$$
$$\tfrac{1}{2} = e^{-0\cdot 135t}$$
$$-0\cdot 135t = \ln\left(\tfrac{1}{2}\right)$$
$$t = \frac{\ln\left(\frac{1}{2}\right)}{-0\cdot 135}$$
$$\therefore \quad t \approx 5\cdot 13.$$

Therefore, the half-life of the substance is 5·13 years.

8. (a) Let the beginning of 1970 be $t = 0$.

Therefore, when $t = 0$,

$P = 1472 \Rightarrow Po = 1472$

$$\therefore \quad P = 1472\,e^{kt}.$$

Beginning of 1980,

$P = 1581 \Rightarrow$ when $t = 10$,

$P = 1581$ [Data to calculate k.]

$$1581 = 1472\,e^{k\times 10}$$
$$\frac{1581}{1472} = e^{10k}$$
$$10k = \ln\left(\frac{1581}{1472}\right)$$
$$t = \frac{\ln\left(\frac{1581}{1472}\right)}{10}$$
$$\therefore \quad k \approx 7\cdot 2\times 10^{-3}$$
$$\therefore \quad P = 1472\,e^{7\cdot 2\times 10^{-3}t}.$$

(b) Beginning of 1994 $\Rightarrow t = 24$

When $t = 24$,

$$P = 1472\,e^{7\cdot 2\times 10^{-3}\times 24}$$
$$\approx 1747$$

Therefore, at the beginning of 1994, the population of the island will be 1750.

(c) The question is asking us to find t when $P = 2000$.

$$2000 = 1472\,e^{7\cdot 2\times 10^{-3}t}$$
$$\frac{2000}{1472} = e^{7\cdot 2\times 10^{-3}t}$$
$$\therefore \; 7\cdot 2\times 10^{-3}t = \ln\left(\frac{2000}{1472}\right)$$
$$t = \frac{\ln\left(\frac{2000}{1472}\right)}{7\cdot 2\times 10^{-3}}$$
$$\approx 42\cdot 6.$$

$\boxed{\textit{Note} \text{ In the year 1970, } t = 0}$

Therefore the population of the island will reach 2000 in the year 2017.

$\boxed{\text{1970 + 42·6 years is the year 2013}}$

9. (a) $D = Do\,e^{kt}$

When $t = 5$, $D = 2Do$

$$\Rightarrow \quad 2Do = Do\,e^{k\times 5}$$
$$2 = e^{5k}$$

(Divide both sides by Do)

$$5k = \ln 2$$
$$k = \frac{\ln 2}{5}$$
$$\approx 0\cdot 139.$$
$$\therefore \quad D = Do\,e^{0\cdot 139t}.$$

(b) Let the beginning of 1980 be $t = 0$, $\therefore$ when $t = 0$,

$$D = 50 \Rightarrow Do = 50$$
$$\therefore \quad D = 50\,e^{0\cdot 139t}.$$

The beginning of 1995 $\Rightarrow t = 15$.

When $t = 15$,

$$D = 50\,e^{0\cdot 139\times 15}$$
$$\approx 402\cdot 2.$$

Therefore, the diameter of the tree at the beginning of 1995 will be 402·2 cm.

(c) The question is asking us to find t when $D = 3Do$.

Using $D = Do\,e^{0\cdot 139t}$

then $3Do = Do\,e^{0\cdot 139t}$

$$3 = e^{0\cdot 139t}$$

(Divide both sides by Do)

$$0\cdot 139t = \ln 3$$
$$t = \frac{\ln 3}{0\cdot 139}$$
$$\approx 7\cdot 9.$$

Therefore, it will take 7·9 years for the diameter of the tree to be three times its initial diameter.

10. **(a)** When $t = 0$ (beginning of 1980), $P = 17\,320$

$\Rightarrow P_0 = 17\,320$

$\therefore\ P_C = 17320\, e^{kt}$

When $t = 10$ (beginning of 1990), $P = 12\,700$

$\therefore\ 12700 = 17320\, e^{k \times 10}$

$\frac{12700}{17320} = e^{10k}$.

$10k = \ln\left(\frac{12700}{17320}\right)$

$k = \frac{\ln\left(\frac{12700}{17320}\right)}{10}$

$\therefore\ k \approx -0{\cdot}0310$ (3 sign. figs.)

$\therefore\ \boxed{P_C = 17320\, e^{-0\cdot0310\,t}}$

(b) When $t = 0$ (beginning of 1980), $P = 10\,700$

$\Rightarrow P_0 = 10\,700$

$\therefore\ P_J = 10700\, e^{kt}$

When $t = 5$ (beginning of 1985), $P = 12\,500$

$\therefore\ 12500 = 10700\, e^{k \times 5}$

$\frac{12500}{10700} = e^{5k}$

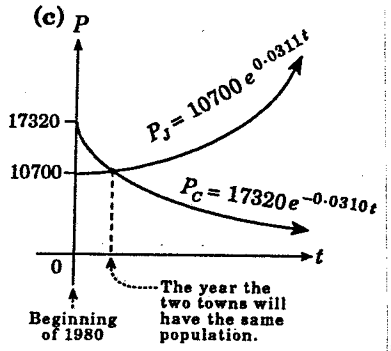
$\therefore\ 5k = \ln\left(\frac{12500}{10700}\right)$

$k = \frac{\ln\left(\frac{12500}{10700}\right)}{5}$

$k \approx 0{\cdot}0311$ (3 sign. figs.)

$\therefore\ \boxed{P_J = 10700\, e^{0\cdot0311t}}$

(c)



(d) The population of Jonestown becomes larger than the population of Comptown when $P_J > P_C$,

i.e. $10700\, e^{0\cdot0311t} > 17320\, e^{-0\cdot031t}$

$\frac{e^{0\cdot0311t}}{e^{-0\cdot031t}} > \frac{17320}{10700}$ (Dividing both sides by $e^{-0\cdot031t}$ and 10700.)

$\boxed{\frac{e^a}{e^b} = e^{a-b}}$

$e^{0\cdot0621t} > \frac{17320}{10700}$

i.e. $0{\cdot}0621t > \ln\left(\frac{17320}{10700}\right)$

$e^x = a \Rightarrow x = \ln a$

$t > \frac{\ln\left(\frac{17320}{10700}\right)}{0\cdot062}$

$\approx 7{\cdot}8$ (1dp).

Therefore, the population of Jonestown becomes bigger than the population of Comptown 7·8 years on from the start of 1980, which is during the year 1987.