C.E.M.TUITION

Student Name :________________________________

Topic : Series Applications I

(Tutorial Exercises)

Year 12 - 2 Unit

TUTOR : PETER OOI

PHONE : 666-3331 **FAX : 316-4996**

1. An architect is employed at an initial salary of \$27 400 per annum. After each year of service he receives an increment of \$940 until he reaches the maximum salary of \$38 680.

   (a) What is his salary after seven years of service?

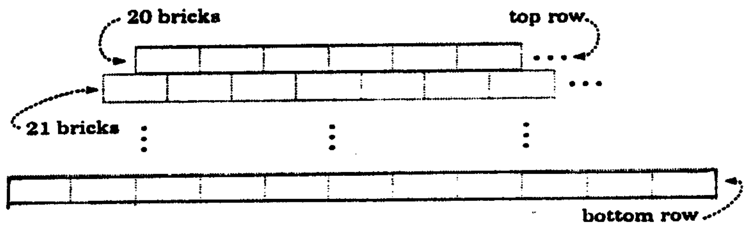
   (b) How many years does he need to work until he reaches the maximum salary?

   (c) What are his total earnings for the first 15 years?

2. **Bricks are stacked in a pile such that there are 20 bricks on the top row, 21 on the next, 22 on the next, and so on until 245 bricks in total are in the pile.**

(a) How many rows of bricks are there?

(b) How many bricks are on the bottom row?



Total number of bricks = 245.

3. A rubber ball is dropped from the top of a roof 4 metres high onto a concrete playground. The ball first rebounds from the playground and rises to a height of 3 metres and at subsequent rebounds rises to a height equal to $\frac{3}{4}$ of the previous bounce. Find the total distance travelled by the ball before coming to rest.

4. Convert the following recurring decimals to simple fractions:

(a) $0 \cdot \dot{7}$ **(b)** $0 \cdot \dot{1}\dot{5}$ **(c)** $1 \cdot 2\dot{4}$ **(d)** $0 \cdot 2\dot{1}\dot{2}$

5. The population of a town was 80 000 at the beginning of 1960 and reduces by 2% every year. In which year was the population of the town 50 000?

6. **A father offered his son a choice of two presents:**

Present *A*: $1000 plus $20 daily from the 1st of July until the end of the month;

Present *B*: 1st of July 1 cent, 2nd of July 2 cents, 3rd of July 4 cents, and so on until the end of the month.

Which present should the son accept? Why?

1. The architect's salary at the end of each year forms an arithmetic progression, with the first term $a = 27\,400$ and the common difference $d = 940$, i.e. $27\,400, 28\,340, 29\,280, \cdots$.

(a) In his seventh year of service, his salary is T_7.

$T_n = a + (n-1)d$

$a = 27\,400,\ d = 940,\ n = 7$

$\therefore\ T_8 = 27\,400 + (7-1)\times 940$
$= 27\,400 + 6\times 940$
$= 33\,040.$

Therefore, after seven years of service the architect's salary is $33 040.

(b) To find when he reaches the maximum salary of $38 680, we need to find n such that $T_n = 38\,680$

$a + (n-1)d = 38\,680$

$\boxed{a = 27\,400,\ d = 940}$

$27\,400 + (n-1)\times 940 = 38\,680$
$27\,400 + 940n - 940 = 38\,680$
$940n + 26\,460 = 38\,680$
$940n = 12\,220$
$\therefore\ n = 13.$

Therefore, the architect will reach the maximum salary after 13 years of service.

(c) For the first 15 years of service the architect will earn:

$\underbrace{27\,400 + 28\,340 + \cdots + 38\,680}_{\text{arithmetic series}}$ ↑ first 13 years

$\boxed{a = 27400,\ d = 38\,680,\ n = 13}$

$\underbrace{+\,2\times 38\,680}$ ↑ other 2 years

$= S_{13} + (2\times 38\,680)$
$= \frac{n}{2}(a + \ell) + 77\,360$
$= \frac{13}{2}(27\,400 + 38\,680) + 77\,360$
$= 429\,520 + 77\,360$
$= 506\,880.$

Therefore, the total earnings for the first 15 years of service is $506 880.

2. Row 1: 20 bricks T_1 (top row)
Row 2: 21 bricks T_2
Row 3: 22 bricks T_3
⋮
Row n: T_n bricks T_n (bottom row)

The rows of bricks form an arithmetic series

$20 + 21 + 22 + \cdots + T_n = 245$

where T_n is the number of bricks in the bottom row.

(a) To find the number of rows of bricks there are, we need to find n such that $S_n = 245$, where $a = 20$, $d = 1$.

$\frac{n}{2}\left[2a + (n-1)d\right] = 245$

$\frac{n}{2}\left[40 + (n-1)\times 1\right] = 245$

(multiply both sides by 2)

$n(40 + n - 1) = 490$
$n(39 + n) = 490$
$39n + n^2 = 490$
$n^2 + 39n - 490 = 0$
(quadratic equation)
$(n + 49)(n - 10) = 0$
$\therefore\ n = -49$ or $n = 10.$

But since n has to be a positive integer, then $n = 10$,
$\therefore$ there are 10 rows of bricks.

(b) The bottom row has T_n bricks, where

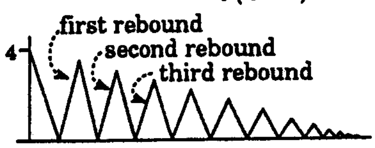
$T_n = a + (n-1)d$

$\boxed{a = 20,\ d = 1,\ n = 10}$

$T_{10} = 20 + (10-1)\times 1$
$= 20 + 9$
$= 29.$

Therefore, the bottom row has 29 bricks.

3. First rebound = 3 m
Second rebound = $\frac{3}{4}\times 3$ m
Third rebound = $\frac{3}{4}\left(\frac{3}{4}\times 3\right)$ m



Therefore, the rebounds form an infinite geometric series in which $a = 3$ and $r = \frac{3}{4}$, i.e.

$3 + \frac{3}{4}\times 3 + \left(\frac{3}{4}\right)^2\times 3 + \left(\frac{3}{4}\right)^3\times 3 + \cdots$

> *Note* Theoretically, the rebounds continue indefinitely.

Therefore, distance rebounded

$= S_\infty = \frac{a}{1-r}$
$= \frac{3}{1-\frac{3}{4}} = \frac{3}{\frac{1}{4}}$
$= 12$ metres.

Also, the ball falls through a distance equal to the sum of rebounds and it originally fell through a height of 4 metres.

$\therefore$ total distance $= 12 + 12 + 4$
$= 28$ metres.

Therefore, the ball travels 28 metres before coming to rest.

4. (a) $0{\cdot}\dot{7} = 0{\cdot}777\,77$

$= \frac{7}{10} + \frac{7}{10^2} + \frac{7}{10^3} + \cdots$

(an infinite series in which $a = \frac{7}{10}$, $r = \frac{1}{10}$)

$= S_\infty = \frac{a}{1-r}$
$= \frac{\frac{7}{10}}{1-\frac{1}{10}} = \frac{\frac{7}{10}}{\frac{9}{10}}$
$= \frac{7}{9}.$

(b) $0{\cdot}\dot{1}\dot{5} = 0{\cdot}151\,515\cdots$

$= \frac{15}{10^2} + \frac{15}{10^4} + \frac{15}{10^6} + \cdots$

(an infinite geometric series in which $a = \frac{15}{100}$, $r = \frac{1}{100}$)

$= S_\infty = \frac{a}{1-r}$
$= \frac{\frac{15}{100}}{1-\frac{1}{100}} = \frac{\frac{15}{100}}{\frac{99}{100}}$
$= \frac{15}{99} = \frac{5}{33}.$

Therefore, $0{\cdot}\dot{1}\dot{5} = \frac{5}{33}.$

(c) $1{\cdot}2\dot{4} = 1{\cdot}2444\cdots$

$= 1{\cdot}2 + \frac{4}{10^2} + \frac{4}{10^3} + \frac{4}{10^4} + \cdots$

> $\frac{4}{10^2} + \frac{4}{10^3} + \frac{4}{10^4} + \cdots$
> an infinite geometric series in which $a = \frac{4}{100}$, $r = \frac{1}{10}$

$= 1{\cdot}2 + S_\infty \quad = 1{\cdot}2 + \frac{a}{1-r}$

$= 1{\cdot}2 + \frac{\frac{4}{100}}{1-\frac{1}{10}} \quad = 1{\cdot}2 + \frac{\frac{4}{100}}{\frac{9}{10}}$

$= 1{\cdot}2 + \frac{4}{90}$ $\boxed{1\frac{2}{10} + \frac{4}{90}}$

$= 1\frac{11}{45}$.

Therefore, $1{\cdot}2\dot{4} = 1\frac{11}{45}$.

(d) $0{\cdot}2\dot{1}\dot{2} = 0{\cdot}212\,121\,2\cdots$

$= \frac{2}{10} + \frac{12}{10^3} + \frac{12}{10^5} + \frac{12}{10^7} + \cdots$

$\frac{12}{10^3} + \frac{12}{10^5} + \frac{12}{10^7} + \cdots$ an infinite geometric series in which $a = \frac{12}{1000}$, $r = \frac{1}{100}$

$= \frac{2}{10} + S_\infty \quad = \frac{2}{10} + \frac{a}{1-r}$

$= \frac{2}{10} + \frac{\frac{12}{1000}}{1-\frac{1}{100}} = \frac{2}{10} + \frac{\frac{12}{1000}}{\frac{99}{100}}$

$= \frac{2}{10} + \frac{12}{990} \quad = \frac{210}{990}$

$= \frac{7}{33}$.

Therefore, $0{\cdot}2\dot{1}\dot{2} = \frac{7}{33}$.

5. At the beginning of 1960, the population was 80 000. At the beginning of 1961 the population was 80 000 (0·98). At the beginning of 1962 the population was 80 000 (0·98) × 0·98

$= 80\,000\,(0{\cdot}98)^2$

and so on.

The population of the town at the beginning of each year (from 1960) forms a geometric sequence, with first term $a = 80\,000$ and $r = 0{\cdot}98$, i.e. 80 000, 80 000 (0·98), $80\,000\,(0{\cdot}98)^2, \cdots$.

To find the year in which the population is 50 000, we need to find n such that $T_n = 50\,000$.

$T_n = ar^{n-1}$

$\therefore\ 80\,000 \times (0{\cdot}98)^{n-1} = 50\,000$

$(0{\cdot}98)^{n-1} = \frac{50\,000}{80\,000} = \frac{5}{8}$

(take logs of both sides)

$\therefore\quad \log_e (0{\cdot}98)^{n-1} = \log_e\left(\frac{5}{8}\right)$

then $(n-1)\log 0{\cdot}98 = \log\left(\frac{5}{8}\right)$

$\boxed{\log_e a^b = b\log_e a}$

$\therefore\quad n - 1 = \frac{\log\left(\frac{5}{8}\right)}{\log 0{\cdot}98}$

$\therefore\quad n = \frac{\log\left(\frac{5}{8}\right)}{\log 0{\cdot}98} + 1$

$\approx 24{\cdot}3$.

Therefore the population of the town will be 50 000, 24·3 years after the beginning of 1960. That is, in the year 1984.

6. The son should accept the present which is worth more money. Calculate how much each present is worth, then choose the greater.

Present A:

Total = 1000 + 20 × 31

(*Note* July has 31 days)

= 1628.

Therefore, present A is worth $1628.

Present B:

(There are 31 terms in the series, since July has 31 days.)

$\underbrace{1 + 2 + 4 + 8 + \cdots}$ which is a geometric series, with $a = 1$, $r = 2$, and $n = 31$.

$\text{Total} = S_n = \frac{a(r^n - 1)}{r-1}$

$= \frac{1(2^{31}-1)}{2-1}$

$= 2^{31} - 1$

$= 2\,147\,483\,646$ (cents).

Therefore, present B is worth $21 474 836·46.

The most sensible choice of presents is present B because it is worth quite a bit more than present A.