C.E.M.TUITION

FINAL TRIAL HSC EXAMINATION 1999

MATHEMATICS

3/4 UNIT COMMON PAPER

Total time allowed - Two hours

(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES :

- Attempt ALL questions.
- All questions are of equal value.
- All necessary working should be shown in every question.
- Full marks may not be awarded for badly arranged work.
- Standard integrals are on the inside cover.
- Approved silent calculators may be used.
- You must hand in a blank page if a question is unanswered.

Question 1

Marks

(a) Using your standard integrals sheet, evaluate **2**

$$\int_0^2 \frac{1}{\sqrt{16+x^2}}\,dx$$

(b) Evaluate $\lim_{x\to 0}\left(\frac{\sin 3x+\tan x}{x}\right)$ **2**

(c) Differentiate $(1+x^2)\tan^{-1}x$. **2**

(d) If $P\left(5, 6\frac{1}{2}\right)$ divides the interval joining $A(-1,2)$ and $B(3,5)$ externally in the ratio $k:l$, find the value of $k:l$. **2**

(e) Using the substitution $u=3x^2-2$, find **3**

$$\int \frac{x}{\sqrt{3x^2-2}}\,dx$$

(f) Find the number of permutations of the word **1**

HALLELUJAH

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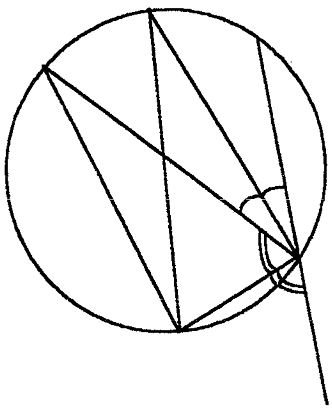
Question 2 **Marks**

(a) Using the expansion of $\tan(\alpha - \beta)$, or otherwise show that $\tan(-15^0) = \sqrt{3} - 2$. **2**

(b) (i) Show that $\sec^2 2\theta + 2\tan 2\theta = (\tan 2\theta + 1)^2$. **2**

(ii) Hence, evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{8}} (\tan 2x + 1)^2 dx$ **2**

(c)



In the above diagram, EC bisects $\angle ACB$, and FC bisects $\angle ACD$, prove the following giving reasons for your answer. **3**

(i) $\angle AFE = \angle ECB$

(ii) EF is a diameter.

(d) If $f(x) = \log_e\left(\frac{1+\cos x}{1-\cos x}\right)$, show that **3**

$f'(x) = -2\operatorname{cosec} x$.

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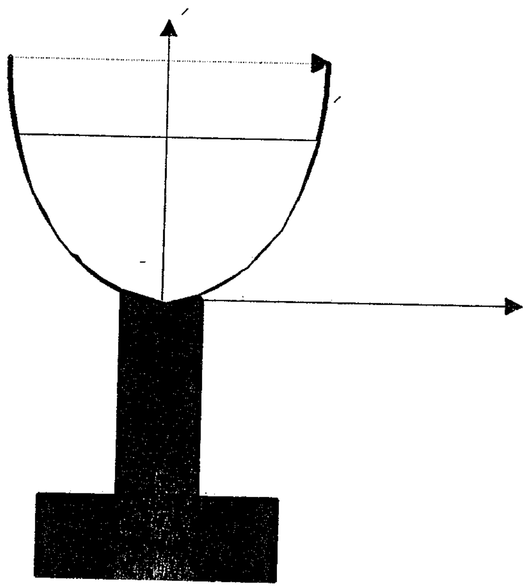
Question 3 **Marks**

(a) 6



A silver wine goblet is formed by revolving that part of the curve $y = x^4$ one complete revolution about the y–axisfrom $y = 0$ to $y = 6$.

(i) Show that the volume of wine contained in the goblet at a depth of h cm is given by $V = \frac{2\pi h^{\frac{3}{2}}}{3}$.

(ii) Find the volume of wine if the goblet is filled up. (Answer to the nearest cm^3) .

(iii) Wine is poured into an identical goblet at the rate of $10\ cm^3s^{-1}$. Find the rate at which the level h is rising when the depth is 4 cm.

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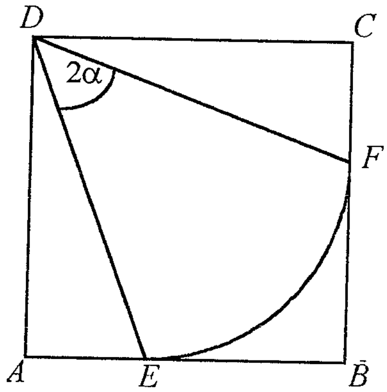
(b) **6**



In the diagram above, $ABCD$ is a square of length l .
AEF is a circular sector with $\angle FAE = 2\alpha$ radians .
The area of sector AEF is given as $\frac{l^2}{2}$ square cm.
Show that :

(i) the radius of the sector is $\dfrac{l}{\cos\left(\frac{\pi}{4}-\alpha\right)}$;

(ii) $\sin 2\alpha + 1 = 4\alpha$.

(iii) a root exist near $\alpha = 0.5$ radians for the equation

$\sin 2\alpha + 1 - 4\alpha = 0$.

Using Newton's method of approximation once, find a better approximation to α.

Continue next page ....

Question 4

Marks

(a) Prove by mathematical induction, or otherwise, that **5**

$$3 \times 4 + 5 \times 5 + 7 \times 6 + \ldots + (2n+1)(n+3) = \frac{n}{6}(4n^2 + 27n + 41)$$

(b) (i) Express $\sqrt{3}\cos\theta + \sin\theta$ in the form $R\cos(\theta - \alpha)$ where R and $\theta > 0$ **2**

(ii) Find, in terms of π, the general solution of the equation **3**

$$\sqrt{3}\cos\theta + \sin\theta = 1$$

(c) Two boys and four girls are executive members of a Charitable Organisation. For their annual photo shoot, the photographer asks the six executives and the President of the organisation to sit in a row with the President in the middle and the other two boys **not** to sit next to one another. How many different seating arrangements are possible ? **2**

Continue next page ....

Question 5 **Marks**

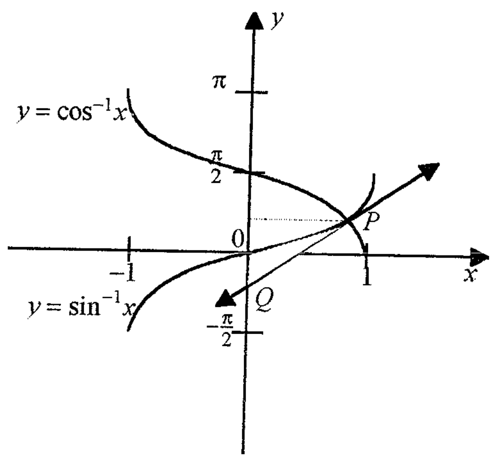
(a) The points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ move on the parabola $y^2 = 4ax$ and $p+q=2$. **6**

(i) Show that the chord PQ makes a constant angle with the x–axis.

(ii) Show that the locus of the midpoint M of PQ is part of a line which is parallel to the x–axis.

(iii) If also the point $R(ar^2, 2ar)$ moves so that $p-r=2$, find the simplest Cartesian equation of the locus of the midpoint N of PR.

(b) **6**



The diagram above shows the graph of $y = \sin^{-1}x$ and $y = \cos^{-1}x$,

(i) Show that the coordinates of P which is the point of intersection between the curves is $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$.

(ii) Show that the tangent at P makes a y– intercept at Q, where the coordinates of Q is $\left(0, \frac{\pi}{4} - 1\right)$.

(iii) Hence find the shaded area between this tangent and the curve $y = \sin^{-1}x$.

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Question 6 **Marks**

(a) Find the largest coefficient in the expansion $\left(2+\frac{2}{x}\right)^9$. **3**

(b) By substituting 0.01 for x in the binomial expansion of $(1-2x)^{10}$, find the value of $(0.98)^{10}$ correct to four decimal places. **2**

(c) Express $\dfrac{n!r^2}{(n-r)!r!}-\dfrac{2(n-1)!}{(r-1)!(n-r)!}$ as a single fraction. **2**

(d) By means of the substitution $t=\tan x$, or otherwise, find **3**

$$\int \frac{1}{1+\cos^2 x}dx$$

(e) A particle moves in a straight line and its acceleration at any time t sec is given by **2**

$$\frac{d^2x}{dt^2}=2\cos^2 x-1.$$

If the particle started from rest at $x=\frac{\pi}{12}$, find v in terms of x.

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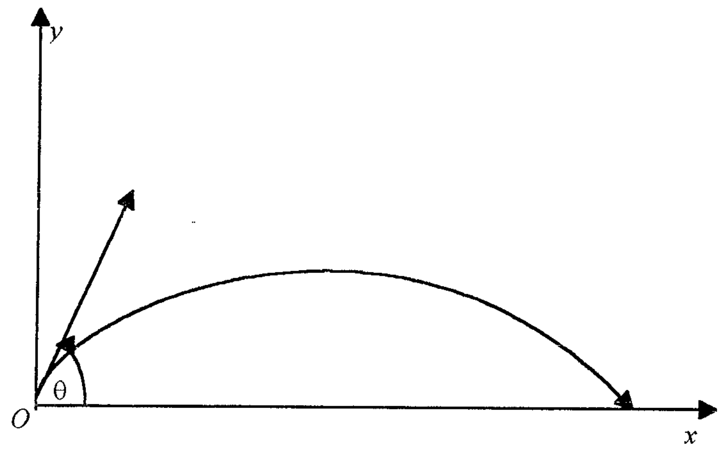
Question 7 **Marks**

(a)



A particle is projected with speed V m/s at an angle of elevation θ from a point O on a horizontal plane, and it moves freely under gravity. The horizontal and upward vertical displacements of the particle from O at any subsequent time t sec are denoted by x m and y m respectively. Assuming the following equations of motion :

$\ddot{x} = 0, \ddot{y} = -g;\ \dot{x} = V\cos\theta, \dot{y} = -gt + V\sin\alpha$

(i) Show that $y = x\tan\theta - \dfrac{gx^2}{2V^2}(1+\tan^2\theta)$ **2**

(ii) Given that $V = 50$ and $x = 200$ and taking $g = 10$ ms^{-2} , **2**

show that $y = 45 - 80\left(\tan\theta - \dfrac{5}{4}\right)^2$.

(b) The particle has to pass over a vertical wall 25 m high at a horizontal distance 200 m from O . Deduce

(i) the greatest distance by which the particle can clear the top of the wall. **3**

(ii) the possible values of $\tan\theta$ if the particle *just* clears the top of the wall. **2**

(c) Using the following natural number series : **3**

$$\sum_{r=1}^{n} r = \frac{n}{2}(n+1);\ \sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1);\ \sum_{r=1}^{n} r^3 = \frac{n^2}{4}(n+1)^2$$

Express $\sum_{r=1}^{n} r(r+1)(r+2)$ in terms of n in factored form

End of Exam

Stephanie Sim

CEM Final trial HSC 1999 oct 8th 3/4

QUESTION 1

A.) $\int_0^2 \frac{1}{\sqrt{16+x^2}}\,dx = \left(\ln(x+\sqrt{x^2+16})\right)_0^2$

$= \ln(2+\sqrt{20}) - \ln(\sqrt{16})$

$= \ln(2+2\sqrt5) - \ln 4$

$= \ln \frac{2+2\sqrt5}{4} = \underline{\ln \frac{1+\sqrt5}{2}}$ ✓

B) $\lim_{x\to 0} \frac{\sin 3x}{x} + \frac{\tan x}{x}$

$= \lim_{x\to 0} \frac{\sin 3x}{3x} \times 3 + \frac{\tan x}{x}$

$= 3 + 1 = \underline{\underline{4}}$ ✓

C) $\frac{d}{dx}(1+x^2)\tan^{-1}x = \tan^{-1}x\,(2x) + (1+x^2)\frac{1}{1+x^2}$

$= \underline{2x\tan^{-1}x + 1}$ ✓

D) $5 = \frac{k(3) - l(-1)}{k-l}$; $\frac{13}{2} = \frac{k(5) - l(2)}{k-l}$

$5k - 5l = 3k + l$

$2k - 5l = l$

$2k = 6l$

$\underline{k = 3l}$

$l = \frac{k}{3}$

$\therefore k : l$

$= k : \frac{k}{3}$

$= \underline{1 : \frac{1}{3}}$ or $\underline{3:1}$

$13k - 13l = 10k - 4l$

$3k - 13l = -4l$

$3k = 9l$

$k = 3l$

externally!!!

✓ $\therefore -3:1$

CARELESS!

0

1

E.) $\int \frac{x}{\sqrt{3x^2-2}}\,dx$ $\quad u = 3x^2-2$

$\frac{du}{dx} = 6x$

$= \frac{1}{6}\int \frac{du}{\sqrt{u}}$ $\quad du = 6x\,dx$ or $dx = \frac{du}{6x}.$

$= \frac{1}{6}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) + c$

$= \frac{1}{6} \times 2\sqrt{u} + c = \frac{\sqrt{u}}{3} + c = \frac{\sqrt{3x^2-2}}{3} + c$

F.) $\frac{10!}{2!\,3!\,2!} = \underline{151,200}$

QUESTION 2

A.) $\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}.$

$\tan(-15°) = \tan(30°-45°) = \frac{\tan 30 - \tan 45}{1+\tan 30 \tan 45}$

$= \frac{\frac{1}{\sqrt{3}} - 1}{1+\frac{1}{\sqrt{3}}} = \frac{1-\sqrt{3}}{\cancel{\sqrt{3}}} \times \frac{\cancel{\sqrt{3}}}{\sqrt{3}+1}$

$= \frac{1-\sqrt{3}}{1+\sqrt{3}} \times \frac{1-\sqrt{3}}{1-\sqrt{3}}.$

$= \frac{1+3-2\sqrt{3}}{1-3} = \frac{4-2\sqrt{3}}{-2}$

$= \frac{-\cancel{2}(\sqrt{3}-2)}{\cancel{2}} = \sqrt{3}-2$

R.T.P.

B) i.) $\sec^2 2\theta + 2\tan 2\theta = (\tan 2\theta + 1)^2$

LHS $\frac{1}{\cos^2 2\theta} + \frac{2\sin 2\theta}{\cos 2\theta} = \frac{1+2\sin 2\theta\cos 2\theta}{\cos^2 2\theta}$

$= \frac{1}{\cos^2 2\theta} + \frac{2\sin 2\theta}{\cos 2\theta}$

$= \underline{\sec^2 2\theta + 2\tan 2\theta}$

RHS $(\tan 2\theta + 1)^2 = \tan^2 2\theta + 2\tan 2\theta + 1$

$= \sec^2 2\theta + 2\tan 2\theta$ = LHS

$\therefore \sec^2 2\theta + 2\tan 2\theta = (\tan 2\theta + 1)^2$.

ii.) $\int_{\frac{\pi}{6}}^{\frac{\pi}{8}} (\tan 2x + 1)^2 dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{8}} \sec^2 2x + 2\tan 2x \, dx$.

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{8}} \sec^2 2x + \frac{2 \sin 2x}{\cos 2x} dx = \left(\frac{1}{2}\tan 2x - \ln(\cos 2x)\right)_{\frac{\pi}{6}}^{\frac{\pi}{8}}$

$= \left[\frac{1}{2}\tan\frac{\pi}{4} - \ln\left(\cos\frac{\pi}{4}\right)\right] - \left[\frac{1}{2}\tan\frac{\pi}{3} - \ln\left(\cos\frac{\pi}{3}\right.\right.$

$= \left(\frac{1}{2} - \ln\left(\frac{1}{\sqrt{2}}\right)\right) - \left(\frac{\sqrt{3}}{2} - \ln\left(\frac{1}{2}\right)\right)$

$= \frac{1}{2} - (\ln 1 - \ln\sqrt{2}) - \frac{\sqrt{3}}{2} + (\ln 1 - \ln 2)$

$= \frac{1}{2} + \frac{1}{2}\ln 2 - \frac{\sqrt{3}}{2} - \ln 2$

$= \frac{1-\sqrt{3}}{2} - \frac{1}{2}\ln 2. \quad = \frac{1}{2}\left(1 - \sqrt{3} - \ln 2\right)$ $\quad$ 7 $\quad \frac{1}{2}(1-\sqrt{3}-3\ln 2)$

c) i.) $\angle AFE = \angle ECA$ (angle is same segment extended to F and C, standing on the same arc AE)

$\angle ECA = \angle ECB$ (given b/c EC bisects $\angle ACB$)

$\therefore \angle AFE = \angle ECB$.

ii.) Let $\angle DCF = \beta$. and also let $\angle ACE$ and $\angle ECB = \underline{\alpha}$

$\angle DCF = \angle FCA$ $=\beta$ (FC bisects $\angle ACD$)

$\angle DCF + \angle FCA = 180 - 2\alpha$ (straight line $= 180°$)

$2\beta = 180 - 2\alpha$

$\underline{\beta = 90 - \alpha}$

Now, in Δ ECF, $\angle ECF = \angle ACE + \angle FCA$

$= \alpha + \beta = \alpha + 90 - \alpha = 90°$

$\therefore \angle ECF = 90°$

$\therefore$ EF is diameter ($\angle$ in semi-circle $= 90°$)

D) $f(x) = \ln\left(\frac{1+\cos x}{1-\cos x}\right)$

$= \ln(1+\cos x) - \ln(1-\cos x)$

$f'(x) = \frac{1}{1+\cos x} \times -\sin x - \left(\frac{1}{1-\cos x} \times \sin x\right)$

$= \left(\frac{-\sin x}{1+\cos x}\right) - \left(\frac{\sin x}{1-\cos x}\right)$

$= \frac{-\sin x(1-\cos x) - \sin x(1+\cos x)}{1-\cos^2 x}$

$= \frac{-\sin x + \sin x\cancel{\cos x} - \sin x - \sin x\cancel{\cos x}}{\sin^2 x}$

$= \frac{-2\cancel{\sin x}}{\sin^2 x} = \underline{-2\operatorname{cosec} x}$

QUESTION 3

A.) i.) $y = x^4$

$x^2 = \sqrt{y}$.

$v = \pi \int_0^h \sqrt{y}\,dy \quad = \pi\left(\frac{2y^{\frac{3}{2}}}{3}\right)_0^h$

$= \underline{\frac{2\pi}{3} h^{\frac{3}{2}}}$

ii.) $V = \frac{2\pi}{3} \times 6^{\frac{3}{2}} = \underline{31\text{cm}^3}$ (nearest cm^3)

iii.) $\frac{dv}{dt} = 10\text{cm}^3/\text{s}$

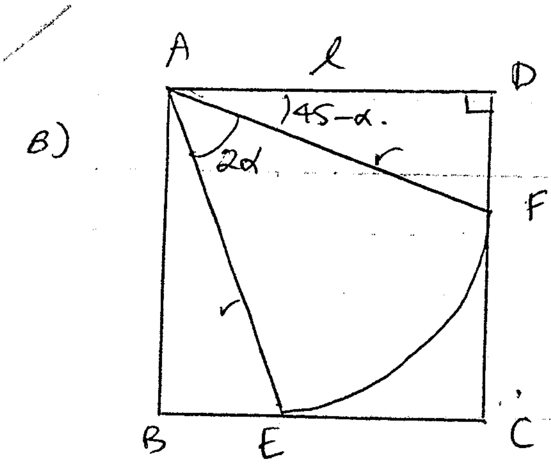
find $\frac{dh}{dt}$ when $h=4$.

$\frac{dv}{dh} = \frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{3}}{\cancel{2}}\pi h^{\frac{1}{2}} = \pi\sqrt{h}$

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{1}{\pi\sqrt{4}} \times 10 = \frac{1}{2\pi} \times 10$

$= \underline{\frac{5}{\pi}\text{cm/s}}$

B)



i.) $\angle DAF = \angle BAE$ (corresp. ∠ of ııı'Δ are equal)

$\angle DAF = \dfrac{90 - 2\alpha}{2}$ (right ∠ = 90°)

$= 45 - \alpha.$

Now, in ΔADF, $\cos(45-\alpha) = \dfrac{l}{AF}.$

$AF = \dfrac{l}{\cos(\frac{\pi}{4} - \alpha)}$

AF = radius of sector.

∴ radius $= \dfrac{l}{\cos(\frac{\pi}{4} - \alpha)}$

ii.) ~~Area of sector AEF~~ $= \frac{1}{2} \times b \times c \times \sin A$

$= \dfrac{1}{2} \times \left(\dfrac{l}{\cos(\frac{\pi}{4}-\alpha)}\right) \times \left(\dfrac{l}{\cos(\frac{\pi}{4}-\alpha)}\right) \times \sin 2\alpha$

$= \dfrac{1}{\cancel{2}}\left(\dfrac{\cancel{l^2}}{\cos^2(\frac{\pi}{4}-\alpha)}\right)\sin 2\alpha = \dfrac{\cancel{l^2}}{\cancel{2}}$

$\cos\left(\frac{\pi}{4} - \alpha\right) = \sqrt{2\alpha}$

$\cos\frac{\pi}{4}\cos\alpha + \sin\frac{\pi}{4}\sin\alpha = \sqrt{2\alpha}$

$\dfrac{\cos\alpha + \sin\alpha}{\sqrt{2}} = \sqrt{2\alpha}$

$\sin 2\alpha = \cos^2\left(\frac{\pi}{4} - \alpha\right)$

$\sin 2\alpha = \left[\cos\left(\frac{\pi}{4} - \alpha\right)\right]^2$

$\sin 2\alpha = \left(\cos\frac{\pi}{4}\cos\alpha + \sin\frac{\pi}{4}\sin\alpha\right)^2$

$\sin 2\alpha = \left(\dfrac{\cos\alpha + \sin\alpha}{\sqrt{2}}\right)^2$

ii.) Area of sector AEF $= \frac{\cancel{2}\alpha}{\cancel{2}\cancel{\pi}} \times \cancel{\pi} \left(\frac{\cancel{\ell^2}}{\cos^2(\frac{\pi}{4}-\alpha)} \right) = \frac{\cancel{\ell^2}}{2}$

$$\frac{\alpha}{\cos^2(\frac{\pi}{4}-\alpha)} = \frac{1}{2} \quad ; \quad \cos^2\left(\frac{\pi}{4}-\alpha\right) = 2\alpha$$

$$1-\sin^2\left(\frac{\pi}{4}-\alpha\right) = 2\alpha$$

$$1-\left(\sin\left(\frac{\pi}{4}-\alpha\right)\right)^2 = 2\alpha$$

$$1-\left(\sin\frac{\pi}{4}\cos\alpha - \cos\frac{\pi}{4}\sin\alpha\right)^2 = 2\alpha.$$

$$1-\left(\frac{\cos\alpha-\sin\alpha}{\sqrt{2}}\right)^2 = 2\alpha$$

$$1-\left(\frac{\cos^2\alpha - 2\sin\alpha\cos\alpha+\sin^2\alpha}{2}\right) = 2\alpha$$

$$1-\left(\frac{1-2\sin\alpha\cos\alpha}{2}\right) = 2\alpha$$

$$2-1+2\sin\alpha\cos\alpha = 4\alpha.$$

$$1+2\sin\alpha\cos\alpha = 4\alpha$$

$$\therefore \underline{1+\sin 2\alpha = 4\alpha}$$

iii.) Let $P(\alpha) = \sin 2\alpha + 1 - 4\alpha = 0$

$P'(\alpha) = 2\cos 2\alpha - 4$

$\alpha = 0.5^c$

$$\alpha_1 = 0.5^c - \frac{P(0.5)}{P'(0.5)} = \underline{0.4689} \ (4\ sf)$$

QUESTION 4

A) Step 1

Let $n=1$.

LHS $(2+1)(1+3) = 3\times4 = 12$

RHS $\frac{1}{6}(4+27+41) = \frac{1}{6}\times 72 = 12 = LHS$

$\therefore$ true for $n=1$.

Step 2

Assume true for $n=k$.

$3\times4 + 5\times5 + 7\times6 + \ldots (2k+1)(k+3) = \frac{k}{6}(4k^2+27k+41)$

R.T.P. also true for $n=k+1$

$3\times4 + 5\times5 + \ldots (2k+3)(k+4) = \frac{k+1}{6}(4(k+1)^2+27(k+1)+41)$

$= \frac{k+1}{6}(4k^2+8k+4+27k+27+41)$

$= \frac{k+1}{6}(4k^2+35k+72)$

$\therefore$ R.T.P. $3\times4 \ldots (2k+3)(k+4) = \frac{k+1}{6}(4k^2+35k+72)$

LHS

$3\times4+5\times5+\ldots(2k+1)(k+3)+(2k+3)(k+4) = \frac{k}{6}(4k^2+27k+41)+(2k+3)(k+4)$

$= \frac{k(4k^2+27k+41)+6(2k+3)(k+4)}{6}$

$= \frac{4k^3+27k^2+41k+6(2k^2+11k+12)}{6} = \frac{4k^3+39k^2+107k+72}{6}$

$\cancel{= \frac{k(4k^2+39k+107)+72}{6}}$ $= \frac{(k+1)}{6}(4k^2+35k+72)$

$= RHS.$

Step 3

if true for $n=k$ and $n=k+1$ AND also true for $n=1$, then it is true for $n=1+1=2$, $n=2+1=3\ldots$ and so on. $\therefore$ by the POMI it is true for all positive integers, n.

Stephanie Lim

b) i.) $\sqrt{3}\cos\theta + \sin\theta = R\cos(\theta - \alpha)$

$= R(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$

$\sqrt{3} = R\cos\alpha$ — ②

$1 = R\sin\alpha$ — ①

$\frac{①}{②} = \tan\alpha = \frac{1}{\sqrt{3}}$

$\alpha = \frac{\pi}{6}$ $\qquad R = \sqrt{3+1} = 2$

$\therefore \sqrt{3}\cos\theta + \sin\theta = 2\cos(\theta - \frac{\pi}{6})$

ii.) $2\cos(\theta - \frac{\pi}{6}) = 1$

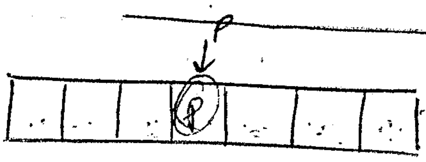
$\cos(\theta - \frac{\pi}{6}) = \frac{1}{2}$ $\qquad$ S | A ✓ / T | C ✓

$\theta - \frac{\pi}{6} = \cos^{-1}(\frac{1}{2})$ $\qquad$ ($\theta > 0$ $\therefore \cos^{-1}(\frac{1}{2}) \neq -\frac{\pi}{3}$)

$\theta - \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{3}$

$\theta = 2n\pi \pm \frac{\pi}{3} + \frac{\pi}{6}$

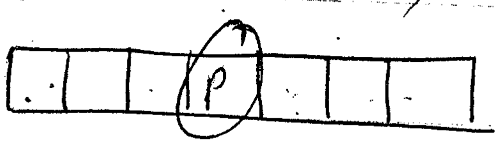
c) ~~prob.~~



~~no. of ways 2 boys sit next to each other = 4 × 2 = 8~~

~~no. of dif. seating arrangements that are possible = 6! − 8 = 712~~

$(4+3+4+4+3+4) \times 4! = 528$



$(4+3+4+4+3+4) \times 4!$

$6! - 5! = 600$ ways

A8K

QUESTION 5

A.) $y^2 = 4ax$

$x = \frac{y^2}{4a}$

$\frac{dx}{dy} = \frac{2y}{4a} = \frac{y}{2a}$; $\underline{\frac{dy}{dx} = \frac{2a}{y}}$.

i.) Grad of $P(ap^2, 2ap)$, $Q(aq^2, 2aq)$ $= \frac{2aq - 2ap}{aq^2 - ap^2}$

$= \frac{2\cancel{a}\cancel{(q-p)}}{\cancel{a}\cancel{(q-p)}(q+p)} = \underline{\frac{2}{p+q}}$

$(p+q = 2)$

$\therefore$ grad. of $PQ = \frac{2}{2} = \underline{1}$

Let θ be angle PQ makes with x-axis

$\therefore \tan\theta = 1$

$\theta = \frac{\pi}{4}$. $\therefore$ PQ makes a CONSTANT angle with x-axis

ii.) $M = \left(\frac{ap^2 + aq^2}{2}, \frac{2ap + 2aq}{2}\right)$

$= \left(\frac{a}{2}(p^2+q^2), a(p+q)\right)$

~~$x = \frac{a}{2}(p^2+q^2)$ | $y = a(p+q)$~~

~~now, sub in (1)~~ ~~$y^2 = a^2(p^2+2pq+q^2)$~~

~~$p^2+2pq+q^2 = \frac{y^2}{a^2}$~~

~~$p^2+q^2 = \frac{y^2}{a^2} - 2pq$~~ — (1)

~~$x = \frac{a}{2}\left(\frac{y^2}{a^2} - 2pq\right)$~~

~~we know that $p+q = 2$~~

~~$p = 2 - q$~~

$x = \frac{a}{2}(p^2+q^2)$, $y = a(p+q)$ $(p+q = 2)$

$\underline{y = 2a}$

a is a constant.

$\therefore$ locus of M is $y = 2a$, a line ∥ to x-axis

iii.) $N = \left(\frac{ap^2+ar^2}{2}, \frac{2ap+2ar}{2}\right)$

$N = \left(\frac{a}{2}(p^2+r^2), a(p+r)\right)$

$x = \frac{a}{2}(p^2+r^2)$, $y = a(p+r)$

$\boxed{p^2+r^2 = \frac{2x}{a}}$ — ①

$y^2 = a^2(p^2+2pr+r^2)$

GIVEN

$p-r=2$

$p^2-2pr+r^2=4$

$2pr = p^2+r^2-4$

$y^2 = a^2(p^2+r^2+p^2+r^2-4)$

$y^2 = a^2(2p^2+2r^2-4)$

$y^2 = 2a^2(p^2+r^2-2)$

NOW sub in ①

$y^2 = 2a^2\left(\frac{2x}{a} - 2\right)$

$y^2 = \frac{4a^2x}{a} - 4a^2 = 4ax - 4a^2$

$= 4a(x-a)$

$\therefore$ $y^2 = 4a(x-a)$

B) i.) $y = \sin^{-1} x$ — ①

$y = \cos^{-1} x$ — ②

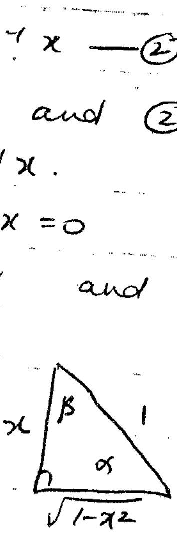
solve ① and ② simultaneously.

$\sin^{-1} x = \cos^{-1} x$.

$\sin^{-1} x - \cos^{-1} x = 0$

Let $\sin^{-1} x = \alpha$ and $\cos^{-1} x = \beta$

$\sin \alpha = x$ $\cos\beta = x$



$\therefore \underline{\alpha - \beta = 0}$

$\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$

$= x.x - \sqrt{1-x^2}\sqrt{1-x^2}$

$= x^2 - 1 + x^2 = 2x^2 - 1$

$2x^2 - 1 = 0$

$2x^2 = 1$

$x^2 = \frac{1}{2}$; $x = \pm \frac{1}{\sqrt{2}}$ ($x > 0$ b/c x-value of $P > 0$)

$\therefore x = \frac{1}{\sqrt{2}}$. $y = \sin^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$

$\therefore P = \underline{\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)}$

ii.) tgt at P is the tangent to $y = \sin^{-1} x$.

$y' = \frac{1}{\sqrt{1-x^2}}$.

At P, $y' = \frac{1}{\sqrt{1-\frac{1}{2}}} = \frac{1}{\sqrt{\frac{1}{2}}} = \underline{\sqrt{2}}$.

Eqt of tgt at P is $(y - \frac{\pi}{4}) = \sqrt{2}\left(x - \frac{1}{\sqrt{2}}\right)$

At Q, $x = 0$.

$y - \frac{\pi}{4} = \sqrt{2}\left(-\frac{1}{\sqrt{2}}\right)$

$y = -1 + \frac{\pi}{4} = \underline{\frac{\pi}{4} - 1}$

$\therefore Q = \underline{\underline{\left(0, \frac{\pi}{4} - 1\right)}}$

iii) $y = \sin^{-1}x$; $x = \sin y$

$$\text{Area} = \left[\frac{1}{2} \times \left(\frac{\pi}{4} - \left(\frac{\pi}{4} - 1\right)\right) \times \frac{1}{\sqrt{2}}\right] - \int_0^{\frac{\pi}{4}} \sin y \, dy$$

$$= \frac{1}{2\sqrt{2}} - \left(-\cos y\right)_0^{\frac{\pi}{4}}$$

$$= \frac{\sqrt{2}}{4} - \left(-\cos\frac{\pi}{4} + \cos 0\right) = \frac{\sqrt{2}}{4} + \frac{1}{\sqrt{2}} - 1$$

$$= \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{2} - 1$$

$$= \frac{\sqrt{2} + 2\sqrt{2} - 4}{4}$$

$$= \frac{3\sqrt{2} - 4}{4} \; u^2$$

QUESTION 6

A.) $\left(2 + \frac{2}{x}\right)^9$. $\quad T_{r+1} = {}^9C_r (2)^{9-r} (2)^r (x)^{-r}$

$T_r = {}^9C_{r-1} (2)^{10-r} (2)^{r-1} (x)^{1-r}$

For largest coeff,

c of $T_{r+1} \geq T_r$ ie $\frac{T_{r+1}}{T_r} \geq 1$

$$\frac{{}^9C_r \cancel{(2)^{9-r}} (\cancel{2^{r}})}{{}^9C_{r-1} \cancel{(2)^{10-r}} \cancel{(2)^{r-1}}} \geq 1$$

$$\cancel{2} \times {}^9C_r \geq \cancel{2} \times {}^9C_{r-1}$$

$$\frac{\cancel{9!}}{(9-r)!\, r!} \geq \frac{\cancel{9!}}{(10-r)!\, (r-1)!}$$

$$(9-r)!\, r! \leq (10-r)!\, (r-1)!$$

$$\cancel{(9-r)!}\, r \cancel{(r-1)!} \leq (10-r) \cancel{(9-r)!} \cancel{(r-1)!}$$

$$r \leq 10 - r$$

$r - 10 + r \leq 0$; $\quad 2r - 10 \leq 10$

$2r \leq 10$

$r \leq 5$.

$\therefore r = 5, 4, 3, 2, 1$

$\therefore T_6 \geq T_5 \geq T_4 \geq T_3 \geq T_2 \geq T_1$

$\therefore T_6$ ~~is~~ has greatest coeff, when $r = 5$

Greatest coeff. $= {}^9C_4 (2)^5 (2)^4 = \boxed{64,5}$

B) $(1-2x)^{10} = {}^{10}C_0 + {}^{10}C_1(-2x) + {}^{10}C_2(-2x)^2 \ldots {}^{10}C_{10}(-2x)^{10}$.

Let $x = 0.01$

$(1-0.02)^{10} = 0.98^{10} = \underline{0.8171}$ (4dp)

c) $\dfrac{n!\,r^2}{(n-r)!\,r!} - \dfrac{2(n-1)!}{(r-1)!\,(n-r)!} = \dfrac{n(n-1)!\,r^2}{(n-r)!\,r(r-1)!} - \dfrac{2(n-1)!}{(r-1)!\,(n-r)!}$

$= \dfrac{n(n-1)!\,r^2 - 2r(n-1)!}{(n-r)!\,r!} = \dfrac{r(n-1)!\,[nr-2]}{(n-r)!\,r!}$

$= \dfrac{(n-1)!\,(nr-2)}{(n-r)!\,(r-1)!}$.

D)

$\displaystyle\int \frac{1}{1+\cos^2 x}\,dx$

$t = \tan x$ $\qquad$ $t^2 = \tan^2 x$

$\dfrac{dt}{dx} = \sec^2 x$. $\qquad$ $t^2+1 = \sec^2 x$.

$\cos^2 x = \dfrac{1}{t^2+1}$

$dx = dt\cos^2 x$

$dx = \underline{\dfrac{dt}{t^2+1}}$

$= \displaystyle\int \frac{1}{1+\frac{1}{t^2+1}} \times \frac{dt}{t^2+1}$

$= \displaystyle\int \frac{\cancel{t^2+1}}{t^2+2} \times \frac{1}{\cancel{t^2+1}}\,dt$

$= \displaystyle\int \frac{1}{t^2+2}\,dt = \frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}} + c$

$= \underline{\dfrac{1}{\sqrt{2}}\tan^{-1}\left(\dfrac{\tan x}{\sqrt{2}}\right) + c}$

e.) $\frac{d^2x}{dt^2} = 2\cos^2 x - 1$

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\frac{1}{2}v^2 = \int 2\cos^2 x - 1 \, dx.$

$\frac{1}{2}v^2 = \int 2\left(\frac{1+\cos 2x}{2}\right) - 1 \, dx \quad ; \quad \frac{1}{2}v^2 = \int \cos 2x \, dx$

$= \frac{1}{2}\sin 2x + c$

$v^2 = \sin 2x + c$

when $x = \frac{\pi}{12}$, $\dot{v} = 0$.

$0 = \sin\frac{\pi}{6} + c \quad ; \quad 0 = \frac{1}{2} + c \; ; \; c = -\frac{1}{2}$

$\therefore \quad v^2 = \sin 2x - \frac{1}{2}$

$v = \pm\sqrt{\sin 2x - \frac{1}{2}}$

$\therefore$ greatest distance by which it can clear the wall

$= 45 - 25 = \underline{\underline{20\,m}}$ ✓

$$25 = 45 - 80\left(\tan\theta - \frac{5}{4}\right)^2$$

$$80\left(\tan\theta - \frac{5}{4}\right)^2 = 20$$

$$\left(\tan\theta - \frac{5}{4}\right)^2 = 0.25$$

$$\tan^2\theta + \frac{25}{16} - \frac{5}{2}\tan\theta - \frac{1}{4} = 0$$

$$\tan^2\theta - \frac{5}{2}\tan\theta + \frac{21}{16} = 0$$

$$\tan\theta = \frac{\frac{5}{2} \pm \sqrt{\frac{25}{4} - 4\left(\frac{21}{16}\right)}}{2}$$

$$= \frac{\frac{5}{2} \pm \sqrt{\frac{25}{4} - \frac{21}{4}}}{2} = \frac{\frac{5}{2} \pm \sqrt{1}}{2}$$

$$= \frac{\frac{5}{2} \pm 1}{2}.$$

$$\tan\theta = \frac{\frac{5}{2} + 1}{2} \quad \text{or} \quad \frac{\frac{5}{2} - 1}{2}$$

$$\boxed{= \frac{7}{4}} \quad \boxed{\text{OR} \quad \frac{3}{4}}$$ ✓

QUESTION 7

A) GIVEN = $\ddot{x} = 0$ | $\ddot{y} = -g$

$\dot{x} = V\cos\theta$ | $\dot{y} = -gt + V\sin\theta$

$x = Vt\cos\theta$ | $y = -\frac{gt^2}{2} + Vt\sin\theta$

i.) $x = Vt\cos\theta$

$t = \frac{x}{V\cos\theta}$ (sub into y.)

$$y = -\frac{g}{2}\left(\frac{x^2}{V^2\cos^2\theta}\right) + V\sin\theta\left(\frac{x}{V\cos\theta}\right)$$

$$y = -\frac{gx^2}{2V^2}(1+\tan^2\theta) + x\tan\theta$$

ii.) if $V = 50$, $x = 200$ and $g = 10$,

$$y = \frac{-10(40{,}000)}{2(2500)}(1+\tan^2\theta) + 200\tan\theta$$

$$y = -80(1+\tan^2\theta) + 200\tan\theta$$

$$y = -80 - 80\tan^2\theta + 200\tan\theta.$$

$$y = 45 - 125 - 80\tan^2\theta + 200\tan\theta$$

$$y = 45 - 80\left(\tan^2\theta - \frac{5}{2}\tan\theta + \frac{25}{16}\right)$$

$$\therefore y = 45 - 80\left(\tan\theta - \frac{5}{4}\right)^2$$

B)

i.) when $x = 200$, find the greatest y value.

$$y = 45 - 80\left(\tan\theta - \frac{5}{4}\right)^2$$

$$= 45 - 80\left(\tan^2\theta + \frac{25}{16} - \frac{5}{2}\tan\theta\right)$$

$$\frac{dy}{d\theta} = -80\left(2\tan\theta\sec^2\theta - \frac{5}{2}\sec^2\theta\right).$$

At max/min y, $\frac{dy}{d\theta} = 0$

$$\frac{dy}{d\theta} = -80\left(2\tan\theta\sec^2\theta - \frac{5}{2}\sec^2\theta\right).$$

At max/min y, $\frac{dy}{d\theta} = 0$

$$-80\left(\frac{2\sin\theta}{\cos\theta} \times \frac{1}{\cos^2\theta} - \frac{5}{2\cos^2\theta}\right) = 0$$

$$\frac{2\sin\theta}{\cos^3\theta} - \frac{5}{2\cos^2\theta} = 0$$

$$\frac{4\sin\theta - 5\cos\theta}{2\cos^3\theta} = 0$$

$$4\sin\theta - 5\cos\theta = 0$$

$$\frac{4\sin\theta}{\cos\theta} = \frac{5\cos\theta}{\cos\theta}.$$

$$4\tan\theta = 5$$

$$\tan\theta = \frac{5}{4} \qquad (\theta \text{ is acute})$$

$$\theta = 51^\circ 20'$$

Determine nature.

$$\cancel{\frac{d^2y}{d\theta^2} = 2\cos^3\theta(4\cos\theta + 5\sin\theta) - (4\sin\theta - 5\cos\theta)6\cos^2\theta \times -\sin\theta}$$

θ	$51^\circ 20'^-$	$51^\circ 20'$	$51^\circ 20'^+$
$\frac{d^2y}{d\theta^2}$	>0	0	<0

$\therefore$ y is max when $\theta = 51^\circ 20'$

$$y = 45 - 80\left(\tan 51^\circ 20' - \frac{5}{4}\right)^2 = 44.99$$
$$= 45\text{m}$$

$$\sum_{r=1}^{n} r(r+1)(r+2) = \sum_{r=1}^{n} (r^2+r)(r+2) = \sum_{r=1}^{n} r^3+3r^2+2r$$

$$= \frac{n^2}{4}(n+1)^2 + \frac{3n}{6}(n+1)(2n+1) + n(n+1)$$

$$= \frac{n^2}{4}(n+1)^2 + \frac{n}{2}(n+1)(2n+1) + n(n+1)$$

$$= [n(n+1)]\left(\frac{n}{4}(n+1) + \frac{(2n+1)}{2} + 1\right)$$

$$= n(n+1)\left(\frac{n^2+n+4n+2+4}{4}\right)$$

$$= \frac{n(n+1)}{4}\left(n^2+5n+6\right) = \frac{n(n+1)(n+3)(n+2)}{4}$$