Year 12
Mathematics Extension 1
Assessment Test 3, June 2010

Time Allowed: 60 minutes + 5 minutes reading time

Instructions:
- Start each question on a new page
- Show all necessary working
- Silent, non-programmable calculators may be used
- Pencil or liquid paper should not be used
- Answer all questions
- Marks may not be awarded for partial or incomplete answers

Question 1 (10 marks) **Marks**

a) Find: $\int \cos 4x \, dx$ 1

b) If $y = \tan^{-1}(x^2)$, find $\dfrac{d^2y}{dx^2}$ 2

c) Find the exact value of:

(i) $\displaystyle\int_0^{\frac{\pi}{12}} \sin 2x \, dx$ 2

(ii) $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2 x \, dx$

d) Find the area bounded by the curve $y = \cos x$ and the $x-axis$ between $x = 0$ and $x = \pi$ 2

Question 2 (10 marks) ***Start a new page.***

a) Use the substitution of $u = x^2 + 1$ to evaluate:

$$\int_1^7 \frac{x}{\sqrt{x^2+1}} \, dx$$ 3

b) Use the principle of mathematical induction to show that:

$4^n - 1 - 7n > 0$ for all integers $n \geq 2$ 3

c) For the function $f(x) = x^2 + 3$,

(i) Find the largest possible domain containing (1,4) for which $f(x)$ has an inverse function $f^{-1}(x)$ 1

(ii) Find this inverse function $y = f^{-1}(x)$ 1

(iii) Sketch $y = f^{-1}(x)$ 1

(iv) State the domain and range of $f^{-1}(x)$ 1

Question 3 (10 marks) ***Start a new page.***

a) Find: $\displaystyle\int \frac{5}{x^2+2} \, dx$ 1

b) Find the exact value of:

$$\tan\left\{\sin^{-1}\left(\frac{5}{13}\right) - \cos^{-1}\left(\frac{3}{5}\right)\right\}$$ 3

c) By using the substitution $x = \sin t$, evaluate:

$$\int_0^{1/2} \sqrt{1-x^2} \, dx$$ ~~3~~

d) (i) Sketch the graph: $y = 3\cos^{-1}\left(\dfrac{x}{2}\right)$ 2

(ii) State its domain and range 1

Question 4 (10 marks) ***Start a new page.***

a) Find the exact value of: $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$ 1

b) Prove that $(4^n + 14)$ is a multiple of 6 for $n \geq 1$, using Mathematical Induction. 3

c) Use the table of standard integral to show that:

$$\int_6^{15} \frac{dx}{\sqrt{x^2+64}} = \ln 2$$ 3

d) Using the method of substitution $u = e^x$, find:

$$\int \frac{e^x}{1+e^{2x}}\,dx$$ 3

Question 5 (10 marks) ***Start a new page.***

a) A particle moves in a straight line. At time t seconds, its distance x metres from a fixed point 0 on the line is given by $x = \sin(2t) + 3$

(i) Find when is the particle at rest between $0 \leq t \leq 2\pi$ 2

(ii) Sketch the graph of x as a function of t for $0 \leq t \leq 2\pi$ ~~3~~

b) A dessert, which is initially at a temperature of $25°C$ is placed in a fridge which has a constant temperature of $3°C$

The cooling rate of the dessert is proportional to the difference between the temperature of the fridge and the temperature T of the dessert. It is known that T satisfies the equation: $\dfrac{dT}{dt} = -k(T-3)$

when t is the number of minutes after the dessert is placed in the fridge.

(i) Show that $T = 3 + Ae^{-kt}$ satisfies the equation. 1

(ii) The temperature of the dessert is $11°C$ after 10 minutes. Find the temperature of the dessert after 15 minutes. 3

(iii) When will the dessert reach $5°C$? 1

End of Examination

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Q1 (a) $\int \cos 4x\,dx = \frac{1}{4}\sin 4x + C$

(b) $y = \tan^{-1}(x^2)$

$\frac{dy}{dx} = \frac{2x}{1+x^4}$

$\frac{d^2y}{dx^2} = \frac{(1+x^4)(2) - 2x(4x^3)}{(1+x^4)^2}$

$= \frac{2(1+x^4-4x^4)}{(1+x^4)^2}$

$= \frac{2(1-3x^4)}{(1+x^4)^2}$

(c)(i) $\int_0^{\pi/12} \sin 2x\,dx$

$= \left[-\frac{1}{2}\cos 2x\right]_0^{\pi/12}$

$= -\frac{1}{2}\left(\cos\frac{\pi}{6} - \cos 0\right)$

$= -\frac{1}{2}\left(\frac{\sqrt{3}}{2} - 1\right)$

$= -\frac{\sqrt{3}}{4} + \frac{1}{2}$

$= \frac{2-\sqrt{3}}{4}$

(ii) $\int_0^{\pi/4} \cos^2 x\,dx$

$= \int_0^{\frac{\pi}{4}} \frac{1}{2}(1+\cos 2x)\,dx$

$= \frac{1}{2}\int_0^{\pi/4} (1+\cos 2x)\,dx$

$= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\pi/4}$

$= \frac{1}{2}\left[\frac{\pi}{4} + \frac{1}{2}\sin\frac{\pi}{2} - 0\right]$

$= \frac{1}{2}\left[\frac{\pi}{4} + \frac{1}{2}(1)\right]$

$= \frac{\pi}{8} + \frac{1}{4}$

$= \frac{\pi+2}{8}$

(d) $A = \int_0^{\pi/2} \cos x\,dx + \left|\int_{\frac{\pi}{2}}^{\pi} \cos x\,dx\right|$

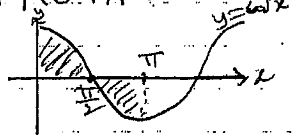
$= \left[\sin x\right]_0^{\pi/2} + \left|\left[\sin x\right]_{\frac{\pi}{2}}^{\pi}\right|$

$= (\sin\frac{\pi}{2} - \sin 0) + \left|(\sin\pi - \sin\pi/2)\right|$

$= (1-0) + |(0-1)|$

$= 1+1$

$= 2u^2$



Q2 (a) $\int_1^7 \frac{x}{\sqrt{x^2+1}}\,dx$, $u = x^2+1$

$\int_1^7 \frac{x}{\sqrt{x^2+1}}\,dx$

$u = x^2+1$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

when $x=7$, $u=50$

when $x=1$, $u=2$

$= \frac{1}{2}\int_1^7 \frac{2x}{\sqrt{x^2+1}}\,dx$

$= \frac{1}{2}\int_2^{50} \frac{du}{\sqrt{u}}$

$= \frac{1}{2}\int_2^{50} u^{-1/2}\,du$

$= \frac{1}{2}\left[2u^{1/2}\right]_2^{50}$

$= \left[\sqrt{u}\right]_2^{50}$

$= \sqrt{50} - \sqrt{2}$

$= 5\sqrt{2} - \sqrt{2}$

$= 4\sqrt{2}$

(b) $4^n - 1 - 7n > 0$, $n \geq 2$.

Let $n=2$.

LHS: $4^2 - 1 - 7(2) = 16 - 1 - 14$

$= 1$

> 0

$\therefore n=2$ is true.

Assume true for $n=k$

$4^k - 1 - 7k > 0$.

$4^k > 7k+1$

Prove true for $n=k+1$.

LHS

$4^{(k+1)} - 1 - 7(k+1)$

$= 4(4^k) - 1 - 7k - 7$

$= 4(7k+1) - 7k - 8$

$= 28k + 1 - 7k - 8$

$= 21k - 4$

> 0.

Since true for $n=2$, the result is true for $n=k$ and is also true for $n=k+1$. Since true for $n=2$, the result is also true for $n=3, n=4, \ldots$ Hence the result is true for all integers $n \geq 2$.

(c)(i) $x \geq 0$

(ii) $y = x^2+3$

Interchange:

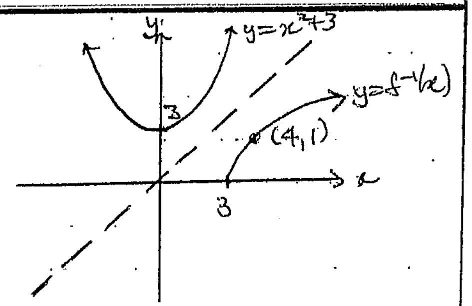
$x = y^2+3$

$y^2 = x-3$

$y = \pm\sqrt{x-3}$

$\therefore y = \sqrt{x-3}$ $\quad \therefore f^{-1}(x) = \sqrt{x-3}$

(iii)



(iv) Domain: $x \geq 3$

Range: $y \geq 0$.

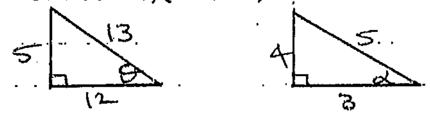
Q3 (a) $\int \frac{5}{x^2+2}\,dx$.

$= 5\int \frac{1}{x^2+2}\,dx$.

$= \frac{5}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$

(b) $\tan\left\{\sin^{-1}\left(\frac{5}{13}\right) - \cos^{-1}\left(\frac{3}{5}\right)\right\}$

Let $\theta = \sin^{-1}(5/13)$ & $\alpha = \cos^{-1}(3/5)$



$\tan(\theta - \alpha) = \frac{\tan\theta - \tan\alpha}{1 + \tan\theta\tan\alpha}$

$= \frac{5/12 - 4/3}{1 + (5/12)(4/3)}$

$= \frac{-11/12}{14/9}$

$= -\frac{33}{56}$

Q3(c) $\int_0^{1/2} \sqrt{1-x^2}\,dx$, $x=\sin t$, $\frac{dx}{dt}=\cos t$

$dx=\cos t\,dt$

when $x=0, t=0$

" $x=\frac{1}{2}, t=\frac{\pi}{6}$

Now, $\int_0^{\frac{1}{2}} \sqrt{1-x^2}\,dx$

$=\int_0^{\pi/6} (\sqrt{1-\sin^2 t})\cos t\,dt$

$=\int_0^{\pi/6} \sqrt{\cos^2 t}\,\cos t\,dt$

$=\int_0^{\pi/6} \cos^2 t\,dt$

$=\int_0^{\pi/6} \frac{1}{2}(1+\cos 2t)\,dt$

$=\frac{1}{2}\int_0^{\pi/6} (1+\cos 2t)\,dt$

$=\frac{1}{2}\left[x+\frac{1}{2}\sin 2t\right]_0^{\pi/6}$

$=\frac{1}{2}\left[\frac{\pi}{6}+\frac{1}{2}\sin\frac{\pi}{3}-0\right]$

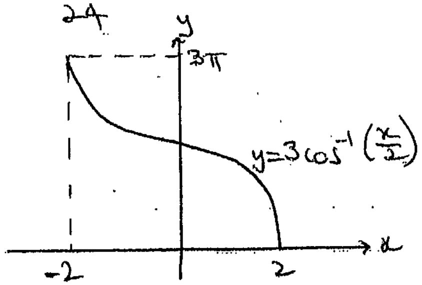
$=\frac{1}{2}\left[\frac{\pi}{6}-\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\right]$

$=\frac{1}{2}\left[\frac{\pi}{6}+\frac{\sqrt{3}}{4}\right]$

$=\frac{1}{2}\left[\frac{2\pi+3\sqrt{3}}{12}\right]$

$=\frac{2\pi+3\sqrt{3}}{24}$

(d)(i)



(ii) Domain: $-2\le x\le 2$

Range: $0\le y\le 3\pi$

Q4 (a) $\int \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\left(\frac{x}{2}\right)+c$.

(b) (4^n+14) is a mult. of 6, $n\ge 1$

Let $n=1$,

$4^1+14=18$

$=6\times 3$

$\therefore n=1$ is true.

Assume true for $n=k$,

$4^k+14=6M$, M is an integer

$4^k=6M-14$

Prove true for $n=k+1$

LHS. $4^{k+1}+14=4(4^k)+14$

$=4(6M-14)+14$

$=24M-56+14$

$=24M-42$

$=6(4M-7)$.

Since true for $n=1$, the result is true for $n=k$ and is also true for $n=k+1$. Since true for $n=2$, the result is also true for $n=3, n=4, \ldots$. Hence the result is true for all integers $n\ge 1$.

Q4(c) $\int_6^{15} \frac{dx}{\sqrt{x^2+64}} = \ln 2$.

LHS: $\int_6^{15} \frac{dx}{\sqrt{x^2+64}}$

$=\left[\ln\left(x+\sqrt{x^2+64}\right)\right]_6^{15}$

$=\ln\left(15+\sqrt{15^2+64}\right)-\ln\left(6+\sqrt{6^2+64}\right)$

$=\ln(15+17)-\ln(6+10)$

$=\ln 32-\ln 6$

$=\frac{\ln 32}{\ln 6}$

$=\ln 2$

$=RHS$

(d) $\int \frac{e^x}{1+e^{2x}}\,dx$, $u=e^x$

$\frac{du}{dx}=e^x$

$du=e^x\,dx$.

Now, $\int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{1}{1+u^2}\,du$

$=\tan^{-1}(u)+c$

$=\tan^{-1}(e^x)+c$.

Q5(a)(i) $x=\sin(2t)+3$.

$\frac{dx}{dt}=2\cos 2t$

$\frac{dx}{dt}=0$, $v=$ rest.

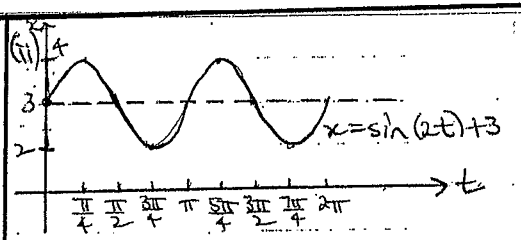
$2\cos 2t=0$.

$\cos 2t=0$

$2t=\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$t=\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

(ii)



(b)(i) $T=3+Ae^{-kt}$

$\frac{dT}{dt}=-kAe^{-kt}$

$=-k(Ae^{-kt})$

$=-k(T-3)$, since $Ae^{-kt}=T-3$

(ii) when $t=0$, $T=25$

$T=3+Ae^{-kt}$

$25=3+Ae^{-k(0)}$

$22=Ae^0$

$\therefore A=22$

$\therefore T=3+22e^{-kt}$

when $t=10$, $T=11$

$11=3+22e^{-k(10)}$

$8=22e^{-10k}$

$\frac{4}{11}=e^{-10k}$

$\ln\left(\frac{4}{11}\right)=\ln e^{-10k}$

$-10k=\ln\left(\frac{4}{11}\right)$

$k=\frac{\ln\left(\frac{4}{11}\right)}{-10}$

$=0.101160091$

$\therefore T=3+22e^{-0.10116t}$

when $t=15$,

$T=3+22e^{-0.10116(15)}$

$=7.8°C$.

Q5 (iii) when $T=5$,

$$5 = 3 + 22e^{-0.10116t}$$

$$2 = 22e^{-0.10116t}$$

$$\frac{1}{11} = e^{-0.10116t}$$

$$\ln(1/11) = \ln e^{-0.10116t}$$

$$-0.10116t = \ln(1/11)$$

$$t = \frac{\ln(1/11)}{-0.10116}$$

$$= 23.7 \text{ min.}$$