NAME :

Centre of Excellence in Mathematics
S201 / 414 GARDENERS RD. ROSEBERY 2018
www.cemtuition.com.au

MOBILE 0412 880 475

PHONE 9666 3331

YEAR 12 – MATHS EXT.2
TUTORIAL EXERCISES – 19A
TOPIC : HARDER 3U – PERMS & COMBS & INDUCTION

Received on			**Check corrections on pages:**
Completed on			
Checked by			
	Tutor's Initials	**Dated on**	

PAST HSC QUESTIONS:

HSC 08

(7)

(a) An urn contains n red balls, n white balls and n blue balls. Three balls are drawn at random from the urn, one at a time, without replacement.

(i) What is the probability, p_s, that the three balls are all the same colour? **2**

$$\boxed{\frac{(n-1)(n-2)}{(3n-1)(3n-2)}}$$

(ii) What is the probability, p_d, that the three balls are all of different colours? **1**

$$\boxed{\frac{2n^2}{(3n-1)(3n-2)}}$$

(iii) What is the probability, p_m, that two balls are of one colour and the third is of a different colour? **1**

$$\boxed{\frac{6n(n-1)}{(3n-1)(3n-2)}}$$

(iv) If n is large, what is the approximate ratio $p_s : p_d : p_m$? **1**

$\boxed{1:2:6}$

HSC 07

(5)

(a) A bag contains 12 red marbles and 12 yellow marbles. Six marbles are selected at random without replacement.

(i) Calculate the probability that exactly three of the selected marbles are red. Give your answer correct to two decimal places. **1**

$\boxed{0.36 \text{ (to 2 d.p)}}$

(ii) Hence, or otherwise, calculate the probability that more than three of the selected marbles are red. Give your answer correct to two decimal places. **2**

$\boxed{0.32 \text{ (to 2 d.p)}}$

(6) **1**

(a) (i) Use the binomial theorem: $(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + b^n$

to show that, for $n \geq 2$, $2^n > \binom{n}{2}$.

(ii) Hence show that, for $n \geq 2$, **2**

$$\frac{n+2}{2^{n-1}} < \frac{4n+8}{n(n-1)}.$$

(iii) Prove by induction that, for integers $n \geq 1$, **3**

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \cdots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}.$$

(iv) Hence determine the limiting sum of the series **1**

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \cdots .$$

4

<u>HSC 06</u>

(7)

(c) The sequence $\{x_n\}$ is given by

$$x_1 = 1 \text{ and } x_{n+1} = \frac{4 + x_n}{1 + x_n} \text{ for } n \geq 1.$$

(i) Prove by induction that for $n \geq 1$ **4**

$$x_n = 2\left(\frac{1 + \alpha^n}{1 - \alpha^n}\right), \qquad \text{where } \alpha = -\tfrac{1}{3}$$

(ii) Hence find the limiting value of x_n as $n \to \infty$. **1**

$x_n \to 2$

HSC 05

(b) Mary and Ferdinand are competing against each other in a competition in which the winner is the first to score five goals. The outcome is recorded by listing, in order, the initial of the person who scores each goal. For example, one possible outcome would be recorded as *MFFMMFMM*.

(i) Explain why there are five different ways in which the outcome could be recorded if Ferdinand scores only one goal in the competition. **1**

(ii) In how many different ways could the outcome of this competition be recorded? **2**

252 ways

HSC 04

(5)

(b) (i) In how many ways can n students be placed in two distinct rooms so that neither room is empty? **1**

$\boxed{2^n - 2}$

(ii) In how many ways can five students be placed in three distinct rooms so that no room is empty? **2**

$\boxed{150}$

HSC 03

(4)

(c) A hall has n doors. Suppose that n people each choose any door at random to enter the hall.

(i) In how many ways can this be done? **1**

$\boxed{n^n}$

(ii) What is the probability that at least one door will not be chosen by any of the people? **2**

$$\boxed{1 - \frac{n!}{n^n} = \frac{n^{n-1} - (n-1)!}{n^{n-1}}}$$

(6)

(b) A sequence s_n is defined by $s_1 = 1$, $s_2 = 2$ and, for $n > 2$,

$$s_n = s_{n-1} + (n-1)s_{n-2}\,.$$

(i) Find s_3 and s_4. **1**

$\boxed{s_3 = 4;\ s_4 = 10}$

(ii) Prove that $\sqrt{x} + x \geq \sqrt{x(x+1)}$ for all real numbers $x \geq 0$. **2**

(iii) Prove by induction that $s_n \geq \sqrt{n!}$ for all integers $n \geq 1$. **3**

HSC 02

(4)

(c) From a pack of nine cards numbered $1, 2, 3, \ldots, 9$, three cards are drawn at random and laid on a table from left to right.

(i) What is the probability that the number formed exceeds 400? **1**

$\boxed{\frac{2}{3}}$

(ii) What is the probability that the digits are drawn in descending order? **2**

$\boxed{\frac{1}{6}}$

HSC 01

(5) (c) A class of 22 students is to be divided into four groups consisting of 4, 5, 6 and 7 students.

(i) In how many ways can this be done? Leave your answer in unsimplified form. **2**

$\boxed{{}^{22}C_4 . {}^{18}C_5 . {}^{13}C_6 . {}^{7}C_7}$

(ii) Suppose that the four groups have been chosen.

In how many ways can the 22 students be arranged around a circular table if the students in each group are to be seated together? Leave your answer in unsimplified form.

$\boxed{3!4!5!6!7!}$

(6)

(a) (i) Let x be a fixed, non-zero number satisfying $x > -1$. Use the method of mathematical induction to prove that **4**

$$(1+x)^n > 1+nx$$

for $n = 2, 3, \ldots$.

(ii) Deduce that $\left(1-\frac{1}{2n}\right)^n > \frac{1}{2}$ for $n = 2, 3, \ldots$.