C.E.M. TUITION

FINAL TRIAL HSC EXAMINATION 1998

MATHEMATICS

3/4 UNIT COMMON PAPER

Total time allowed - TWO hours

(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES :

- Attempt ALL questions.
- All questions are of equal value.
- All necessary working should be shown in every question.
- Full marks may not be awarded for badly arranged work.
- Standard integrals are on the inside cover.
- Approved silent calculators may be used.
- You must hand in a blank page if a question is unanswered.

Question 1 **Marks**

(a) Solve the inequality $3x^2 + 5x < 2$ **2**

(b) Find the acute angle between the tangents at the point of intersection of the curves $y = x^3$ and $y = x^2 - x + 1$. **3**

(c) Using the substitution $u = x^2 - 2$, find **3**

$$\int \frac{x(x^2+1)}{x^2-2}\,dx$$

(d) If $\sec\theta = -\frac{4}{3}$ and $\frac{\pi}{2} < \theta < \pi$, evaluate $\tan 2\theta$. **3**

(e) How many ways are there of arranging all the letters of the word **1**

PARRAMATTA ?

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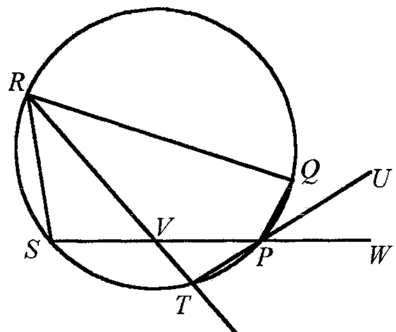
Question 2 **Marks**

(a) When the expression $x^3 + 3x^2 - 2x + c$ is divided by $x - 2$, the remainder is r. When the expression is divided by $x + 2$, the remainder is $2r$. Evaluate c. **2**

(b) (i) Show that $\frac{1-\cos 2\theta}{1+\cos 2\theta} = \sec^2\theta - 1$. **4**

(ii) Hence, evaluate $\int_0^{\frac{\pi}{4}} \frac{\sqrt{2}-\sqrt{2}\cos 2\theta}{1+\cos 2\theta}$

(c) **3**



In the above diagram, $PQRS$ is a cyclic quadrilateral. R, V, T and T, P, U are collinear points.

If PU bisects $\angle QPW$, then prove that RT also bisects $\angle SRQ$.

(d) Use Newton 's method of approximation to find $\sqrt[3]{130}$. Show that $x = 5$ may be used as a first approximation, and find a second approximation. **3**

(You may assume the function $f(x) = x^3 - 130$.)

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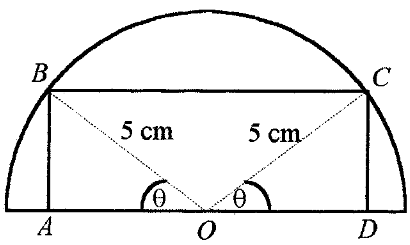
Question 3 **Marks**

(a) A sphere is increasing in volume at a rate of 20π cm^3/sec. **3**
Given that the volume of a sphere is given by $\frac{4}{3}\pi r^3$, calculate the radius of the sphere at the instant when the radius is increasing at the rate of 0.2 cm/s.

(b) **7**



The diagram shows a rectangle $ABCD$ inside a semi-circle, centre O and radius 5 cm, such that $\angle BOA = \angle COD = \theta$.

(i) Show that the perimeter, P cm, of the rectangle is given by

$$P = 20\cos\theta + 10\sin\theta.$$

(ii) Express P in the form $R\cos(\theta - \alpha)$ and hence find the value of θ for which $P = 16$.

(iii) Find the value of k for which the area of the rectangle is $k\sin 2\theta$ cm^2.

(c) Show that $\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \sin^2 2x\, dx = \frac{1}{2}\left(\frac{\pi}{8} + \frac{1}{4}\right)$ **2**

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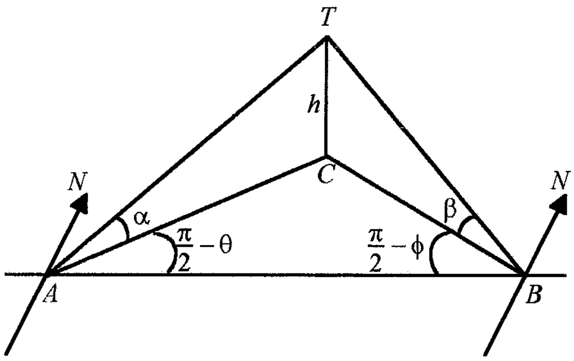
Question 4 **Marks**

(a) Prove by mathematical induction, or otherwise, that **5**

$$\sum_{r=1}^{n} r.3^r = \frac{3}{4}[1+3^n(2n-1)]$$

(b) **5**



A and B are two points on level ground and B is d metres due east of A .

A tower of height h, standing on the same level ground, is in a direction $\theta^0 T$ of A and $(2\pi-\phi)$ of B. The angle of elevation of the top of the tower from A is α and from B is β.

Prove that :

(i) $h\sin(\theta+\phi) = d\cos\phi\tan\alpha$

(ii) $\phi = \cos^{-1}\left(\frac{\cos\theta\tan\beta}{\tan\alpha}\right)$

(iii) $h = 80$ m if $\alpha = \frac{\pi}{6}, \beta = \frac{\pi}{4}, \theta = \frac{\pi}{3}$ and $d = 160$ m.

(c) If $\frac{dv}{dt} = \frac{1}{x^2+4}$, find an expression for v in terms of x **2**

if the $v = 0$ at $x = 2$.

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Question 5 **Marks**

(a) A class of twenty pupils consists of 12 girls and 8 boys. **5**
For a discussion session four "officers" are to be chosen at random as "Chairman", "Recorder", "Proposer" and "Opposer".
Find, giving your answers correct to three significant figures.

(i) the probability that all four officers are girls

(ii) the probability that two officers are girls and two are boys.

(iii) the probability that the Proposer and Opposer are both girls

(iv) the probability that the Proposer and Opposer are of opposite sex given that the Chairman and Recorder are both girls.

(b) If $y = \sqrt{1-x^2} \, . \sin^{-1}x$, find $\frac{dy}{dx}$ **3**

Show that $\left(\frac{1}{x} - x\right)\left(1 - \frac{dy}{dx}\right) = y$

(c) Sketch the graph of $y = 2\sin^{-1}3x$, showing its domain and range. **4**

Hence, find the area under the curve $y = 2\sin^{-1}3x$ between the ordinates at $x = 0$ and $x = \frac{1}{3}$.

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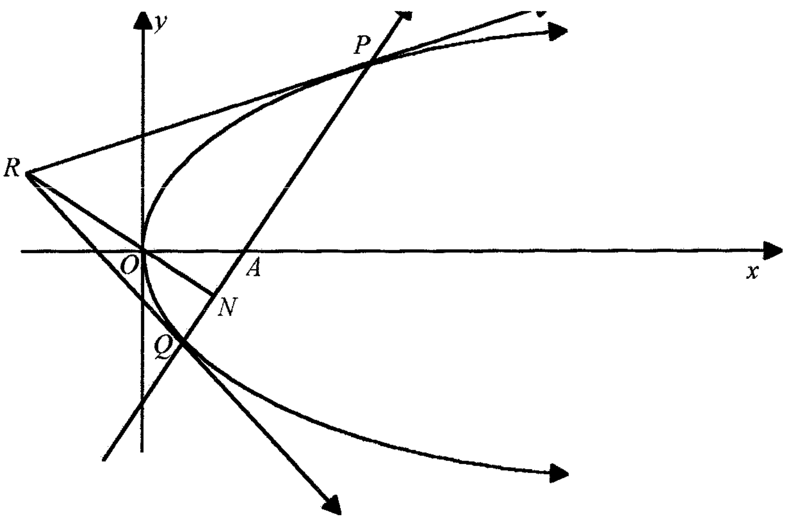
Question 6 **Marks**

(a)



A parabola is defined by the parametric equations

$$x = t^2,\ y = 2t$$

(i) Find the equation of the tangents at the points $P(p^2, 2p)$ and $Q(q^2, 2q)$.

(ii) Show that these tangents intersect at the point $R(pq, p+q)$. **2**

(iii) Show that the equation of the line PQ is $(p+q)y = 2x + 2pq$. **2**

The points P and Q move on the parabola in such a way that $pq = -2$.

Prove that :

(iv) the line PQ always passes through the point $A(2, 0)$. **1**

(v) the line through R and the origin O is always perpendicular to PQ. **2**

(vi) the locus of N is a circle, stating its centre and radius. **2**

(b) Find the term independent of x in the expansion of $\left(x + \frac{1}{x^2}\right)^9$ **3**

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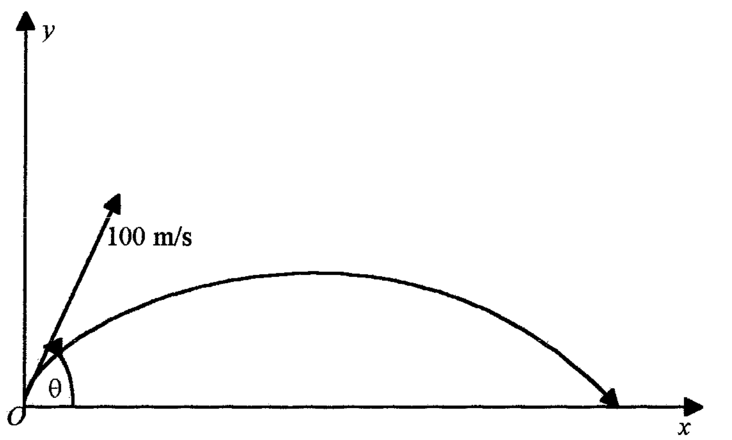
Question 7 **Marks**

(a)



A particle is projected with speed 100 m/s at an angle of elevation θ from a point O on a horizontal plane, and it moves freely under gravity. The horizontal and upward vertical displacements of the particle from O at any subsequent time t sec are denoted by x m and y m respectively.

(i) Express x and y in terms of θ and t, and hence **2**

(ii) show that $y = x\tan\theta - \dfrac{x^2}{2\,000}(1+\tan^2\theta)$ **1**

(iii) Given that the particle passes through the point (800, 160), find the two possible values of $\tan\theta$. **2**

(iv) Given instead that the particle passes through the point $(a, 160)$, show that $(a\tan\theta - 1\,000)^2 = 680\,000 - a^2$ and **2**

(v) deduce the greatest possible value of a as θ varies.- **2**

(b) Using the expansion of $(1+x)^{3n} = (1+x)^{2n}(1+x)^n$, or otherwise **3**

prove that $\binom{3n}{3} = \binom{n}{3} + \binom{2n}{1}\binom{n}{2} + \binom{2n}{2}\binom{n}{1} + \binom{2n}{3}$

End of Exam

(1)(a) $-2 < x < \frac{1}{3}$

(b) $\theta = 26^0 34'$

(c) $\frac{1}{2}[x^2 - 2 + 3\ln(x^2 - 3)] + c$

(d) $3\sqrt{7}$

(e) 37 800

(2) (a) $c = -24$

(b)(i) Proof (ii) $\sqrt{2}\left(1 - \frac{\pi}{4}\right)$

(c) Proof

(d) $f(4) < 0$ and $f(6) > 0$, therefore 5 is close to the root

$x_2 = 5.07$ (to 2 d.p.)

(3)(a) $r = 5$

(b) (i) Proof

(ii) $P = 10\sqrt{5}\cos(\theta - 26^0 34')$

$\theta = 70^0 53'$

(c) Proof

(4)(a), (b) Proofs

(c) $v = \pm\sqrt{\tan^{-1}\left(\frac{x}{2}\right) - \frac{\pi}{4}}$

(5)(a)(i) $\frac{^{12}C_4}{^{20}C_4} = 0.102$

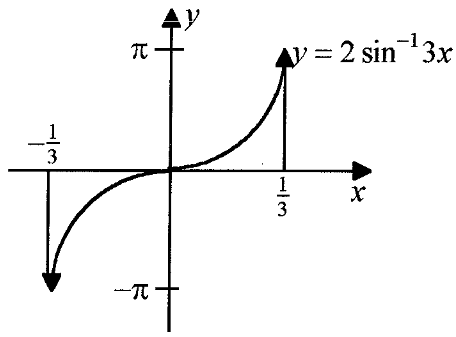
(ii) $\frac{^{12}C_2 \times {}^8C_2}{^{20}C_4} = 0.381$

(iii) $\frac{^{12}C_2}{^{20}C_2} \times 1 = \frac{12}{20} \times \frac{11}{19} \times 1 = 0.347$

(iv) $\frac{^{12}C_2 \times {}^{10}C_1 \times {}^8C_1}{^{12}C_2 \times {}^{18}C_2} = \frac{5280}{10098} = 0.523$

(b) Proof

(c)



$D: -\frac{1}{3} \le x \le \frac{1}{3}$
$R: -\pi \le x \le \pi$

Area = $\frac{\pi - 2}{3}$ units^2

(6)(a)(i) $x - py + p^2 = 0$ and $x - qy + q^2 = 0$.

(ii) to (v) Proofs

(vi) $(x-1)^2 + y^2 = 1$

(b) $T_4 = 84$

(7)(a)(i), (ii) Proofs

(iii) $\tan\theta = \frac{3}{2}$ or 1

(iv) Proof

(v) $a \le 200\sqrt{17}$

(b) Proof