A

Question 1. *Use a separate Writing Booklet.* **Marks**

(a) Given that $z = \dfrac{2-3i}{1+i}$ **3**

find: (i) $|z|$
(ii) $\bar{z}$
(iii) $z + \bar{z}$

(b) Given that 1, ω and ω^2 are cube roots of unity (ie. roots of $z^3 - 1 = 0$), simplify $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)$ **3**

(c) If $z = -\sqrt{3} - i$ express $\dfrac{1}{z^6}$ in the for $a + ib$. **3**

(d) Sketch the locus of z such that $\arg\left(\dfrac{-z}{i}\right) = \arg\left(\dfrac{1}{z}\right)$ **3**

Question 2. *Use a separate Writing Booklet.*

(a) If $2x^3 - 4x^2 + 8x - 1 = 0$ has roots α, β and γ, **3**

find: (i) $\alpha^2 + \beta^2 + \gamma^2$
(ii) $\alpha^3 + \beta^3 + \gamma^3$

(b) Solve for complex z given that $6z^2 + (1+i)z - 1 + 3i = 0$ **4**

(c) Given that $(3-i)$ is a zero of $P(z) = z^3 - 4z^2 - 2z + m$, and that m is real, find the other roots and the value of m. **2**

(d) If α, β and γ are the roots of $x^3 + 3x - 5 = 0$ find the cubic equation whose roots are α^2, β^2 and γ^2. **3**

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(a) $z = \frac{2-3i}{1+i} \times \frac{1-i}{1-i} = \frac{2-3i-2i-3}{2}$

$= \frac{-1-5i}{2}$ ①

(i) $|z| = \sqrt{(-\frac{1}{2})^2 + (-\frac{5}{2})^2} = \sqrt{\frac{26}{4}} = \frac{\sqrt{26}}{2}$ ①

(ii) $\bar{z} = -\frac{1}{2} + \frac{5}{2}i$ ($\frac{1}{2}$)

(iii) $z + \bar{z} = -\frac{1}{2} - \frac{5}{2}i + -\frac{1}{2} + \frac{5}{2}i = -1$ ($\frac{1}{2}$) $\boxed{3}$

(b) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)$

$= (1-\omega)(1-\omega^2)(1-\omega)(1-\omega^2)$ since $\omega^4 = \omega$ & $\omega^8 = \omega^2$

$= (1-\omega-\omega^2+\omega^3)^2$

$= (2-\omega-\omega^2)^2$ since $\omega^3 = 1$

$= 4-2\omega-2\omega^2-2\omega+\omega^2+\omega^3-2\omega^2+\omega^3+\omega^4$

$= 4-4\omega-3\omega^2+2\omega^3+\omega^4$

$= 6-3\omega-3\omega^2$

$= 3(2-\omega-\omega^2)$

$= 3[3-(1+\omega+\omega^2)]$ (sum of roots)

$= 3 \times 3$ since $1+\omega+\omega^2 = 0$

$= 9$ $\boxed{3}$

) $z = -\sqrt{3} - i$

$|z| = \sqrt{3+1} = 2$ $\quad \text{Arg } z = -\pi + \tan^{-1}\frac{1}{\sqrt{3}}$

$= -\pi + \frac{\pi}{6} = \frac{-5\pi}{6}$

$\therefore z = 2 \text{ cis}(-\frac{5\pi}{6})$

$z^{-6} = 2^{-6} \text{ cis}(-\frac{5\pi}{6} \times -6)$

$= \frac{1}{64} \text{ cis } 5\pi$

$= \frac{1}{64} \cdot \text{cis } \pi = \frac{1}{64}(-1+0i) = -\frac{1}{64}$ $\boxed{3}$

$\boxed{B}$

Question 3

(a) The complex number z has modulus r and argument θ where $-\pi < \theta \le \pi$. Write down in terms of r and θ, the modulus and argument of:

(i) z^2 (ii) $\frac{1}{z}$ (iii) iz

(b) Find the square roots of $35 + 12i$.

(c) Solve $z^5 + 16z = 0$ expressing each solution in the form $z = a + bi$, where a and b are real.

(d) Sketch the locus of all complex numbers z if: arg O + arg O

(i) $|z - 2| = \text{Re}(z)$ (ii) $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$

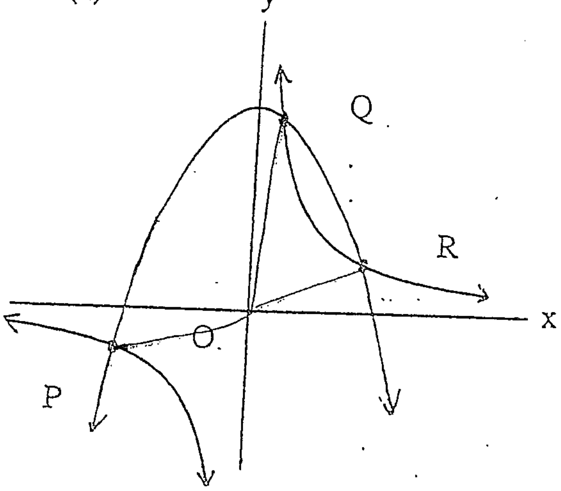
(iii) $|\arg(z-1)| \le \frac{\pi}{4}$ $-\frac{\pi}{4} \le \arg(z-1) \le \frac{\pi}{4}$

Question 4

(a) The equation $x^3 - 4x^2 + mx + 4 = 0$ has one root equal to the sum of the other two roots. Solve the equation and find the value of m.

(b) Given $z = 2 - i$ is a zero of $P(z) = z^4 - 4z^3 + 6z^2 - 4z + 5$, factorise $P(z)$ fully over the complex field.

(c)



The curves $y = \frac{1}{x}$ and $y = k - x^2$, for some real number k, intersect at P, Q and R which have x-values α, β and γ.

(i) Show that the monic cubic equation with coefficients in terms of k with roots α^2, β^2 and γ^2 is given by $x^3 - 2kx^2 + k^2x - 1 = 0$.

(ii) Find the monic cubic equation with coefficients in terms of k with roots $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$ and $\frac{1}{\gamma^2}$.

(iii) Hence show that for O the origin: $OP^2 + OQ^2 + OR^2 = k^2 + 2k$

3)(a) $|z| = r$, $\arg z = \theta$

$|z^2| = \underline{r^2}$, $\arg(z^2) = \underline{2\theta}$

$\left|\frac{1}{z}\right| = \underline{\frac{1}{r}}$, $\arg\left(\frac{1}{z}\right) = \underline{-\theta}$

$|iz| = \underline{r}$, $\arg(iz) = \underline{\theta + \frac{\pi}{2}}$

(b) Let $a + bi = \sqrt{35 + 12i}$

$a^2 - b^2 + 2abi = 35 + 12i$

$\Rightarrow a^2 - b^2 = 35$ and $2ab = 12 \Rightarrow b = \frac{6}{a}$

$\Rightarrow a^2 - \left(\frac{6}{a}\right)^2 = 35 \Rightarrow a^4 - 35a^2 - 36 = 0$

$\Rightarrow (a^2 - 36)(a^2 + 1) = 0 \Rightarrow a = \pm 6$ (a real)

$\Rightarrow b = \pm 1$ $\therefore \underline{\sqrt{35 + 12i} = \pm(6 + i)}$

(c) $z^5 + 16z = 0 \Rightarrow z(z^4 + 16) = 0$

$\Rightarrow z = 0$ or $z^4 = -16$

$= 16\,\text{cis}(\pi + 2k\pi)$

$\Rightarrow \underline{z = 2\,\text{cis}\left(\frac{\pi}{4} + \frac{k}{2}\pi\right)}$

$k = -2 \Rightarrow z = 2\,\text{cis}\left(-\frac{3\pi}{4}\right) = -\sqrt{2} - \sqrt{2}i$

$k = -1 \Rightarrow z = 2\,\text{cis}\left(-\frac{\pi}{4}\right) = \sqrt{2} - \sqrt{2}i$

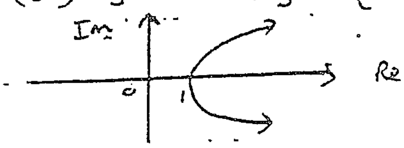
$k = 0 \Rightarrow z = 2\,\text{cis}\left(\frac{\pi}{4}\right) = \sqrt{2} + \sqrt{2}i$

$k = 1 \Rightarrow z = 2\,\text{cis}\left(\frac{3\pi}{4}\right) = -\sqrt{2} + \sqrt{2}i$

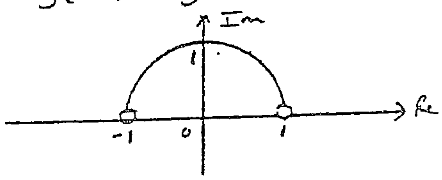
$\therefore$ Solns $\underline{z = 0,\ \pm\sqrt{2} \pm \sqrt{2}i}$

d)(i) $|z - 2| = \text{Re}(z)$

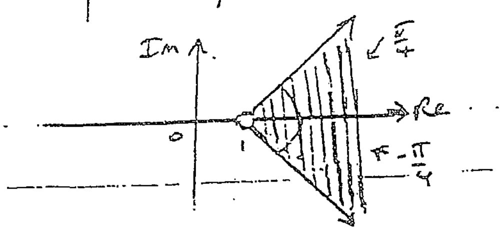
$\Rightarrow (x-2)^2 + y^2 = x^2 \Rightarrow y^2 = 4(x-1)$



(ii) $\arg(z-1) - \arg(z+1) = \frac{\pi}{2}$



(iii) $|\arg(z-1)| \le \frac{\pi}{4}$



4)(a) $x^3 - 4x^2 + mx + 4 = 0$

Let roots be $\alpha, \beta, \alpha + \beta$

$\therefore \alpha + \beta + (\alpha + \beta) = 4 \Rightarrow \alpha + \beta = 2$

and $\alpha\beta(\alpha + \beta) = -4 \Rightarrow \alpha\beta = -2$

$\therefore \alpha, \beta$ roots to $x^2 - 2x - 2 = 0$

$\therefore x = \frac{2 \pm \sqrt{4 + 8}}{2}$

$= \underline{1 \pm \sqrt{3}}$

$\therefore$ Roots are $\underline{1 \pm \sqrt{3},\ 2}$

Now $2(1+\sqrt{3}) + 2(1-\sqrt{3}) + (1+\sqrt{3})(1-\sqrt{3}) = m$

$2 + \cancel{2\sqrt{3}} + 2 - \cancel{2\sqrt{3}} + 1 - 3 = m$

$\underline{2 = m}$

(b) $P(z) = z^4 - 4z^3 + 6z^2 - 4z + 5$

$z = 2 - i$ a zero $\Rightarrow$ $\underline{z = 2 + i}$ is too

$\therefore (z - (2-i))(z - (2+i))$ is a factor

ie $z^2 - 4z + 5$

$$\begin{array}{r l}
 & z^2 \qquad\quad + 1 \\
z^2 - 4z + 5 \,\big) & z^4 - 4z^3 + 6z^2 - 4z + 5 \\
 & z^4 - 4z^3 + 5z^2 \\
\hline
 & z^2 - 4z + 5
\end{array}$$

$\therefore P(z) = (z^2 - 4z + 5)(z^2 + 1)$

$= \underline{(z - (2-i))(z - (2+i))(z + i)(z - i)}$

(c) P, Q, R are pts of int of $y = k - x^2$ and $y = \frac{1}{x}$. So α, β, γ are roots of $k - x^2 = \frac{1}{x}$

ie $0 = x^3 - kx + 1$

(i) Let $y = x^2 \Rightarrow x = \sqrt{y}$

$0 = \sqrt{y}^3 - k\sqrt{y} + 1$

$(-1)^2 = (\sqrt{y}(y - k))^2$

$1 = y(y^2 - 2ky + k^2)$

$\therefore$ Eqn with roots $\alpha^2, \beta^2, \gamma^2$

is $\underline{0 = x^3 - 2kx^2 + k^2x - 1}$ (A)

(ii) Let $z = \frac{1}{x}$ ie $x = \frac{1}{z}$ in (A)

$0 = \left(\frac{1}{z}\right)^3 - 2k\left(\frac{1}{z}\right)^2 + k^2\left(\frac{1}{z}\right) - 1$

$\therefore$ eqn with roots $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$ is

$\underline{x^3 - k^2x^2 + 2kx - 1 = 0}$

(iii) Now P, Q, R all lie on $y = \frac{1}{x}$

$\therefore P$ is $\left(\alpha, \frac{1}{\alpha}\right)$, Q is $\left(\beta, \frac{1}{\beta}\right)$, R is $\left(\gamma, \frac{1}{\gamma}\right)$

$\therefore OP^2 + OQ^2 + OR^2 = \alpha^2 + \left(\frac{1}{\alpha}\right)^2 + \beta^2 + \left(\frac{1}{\beta}\right)^2 + \gamma^2 + \left(\frac{1}{\gamma}\right)^2$

$= \alpha^2 + \beta^2 + \gamma^2 + \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}\right)$

$= \underline{2k + k^2}$

C

Question 2. Start a new page.

MARKS

a) Let $z=-\sqrt{3}+i$ and $u=3-3i$ 4

i) Express z in modulus-argument form.

ii) Hence, or otherwise, express $\frac{z}{u}$ in modulus argument form.

b) Find the equation in Cartesian form of the locus of the point z if 4

$$\text{Re}\left(\frac{z-4}{z}\right)=0$$

c) z is a point on the circle $|z-1|=1$ and $\arg z=\theta$ 4

i) Find $\arg(z-1)$ in terms of θ

ii) Hence, or otherwise find $\arg(z^2-3z+2)$ in terms of θ

d) On the Argand diagram, the points P, Q, R represent the complex numbers p, q, and r respectively. 3

If $p-q+iq-ir=0$ what type of triangle is ΔPQR? Give reasons for your answer.

Question 6. Start a new page.

MARKS

a) i) Show that $2+i$ is a root of $2z^3-5z^2-2z+15=0$ 4

ii) Find the other roots.

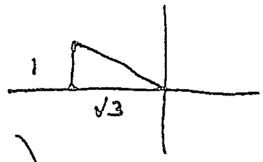
b) The roots of the equation $x^3-px^2+qx-r=0$ are α, β, and γ 4

i) Find the equation with roots $\alpha+\beta$, $\beta+\gamma$, $\gamma+\alpha$

ii) Hence, or otherwise, find the value of $(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)$

Question 2

a) (i) $z = -\sqrt{3} + i$



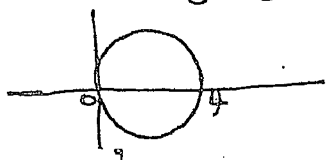
$$= 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

$$\frac{z}{u} = \frac{2\,\text{cis}\,\frac{5\pi}{6}}{3\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)}$$

$$= \frac{\sqrt{2}}{3}\left(\cos\left(-\frac{11\pi}{12}\right) + i\sin\left(-\frac{11\pi}{12}\right)\right)$$

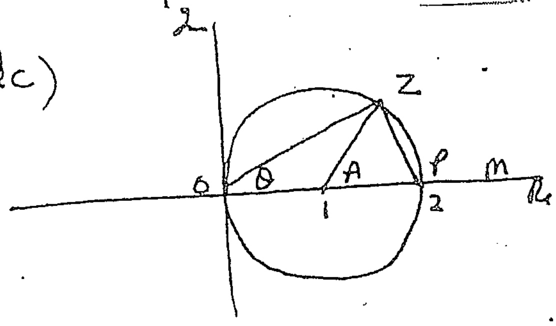
b) $\text{Re}\left(\frac{z-4}{z}\right) = 0$

$\therefore \arg(z-4) - \arg z = \pm\pi/2$



$(x-2)^2 + y^2 = 4$ exclude $(0,0)$

c)



AZ represents $z-1$

$\arg(z-1) = \angle ZAP$

$AO = AZ$ (radii)

$\therefore \angle ZAP = 2\theta$ (ext. $\angle$ = sum int opp $\angle$s)

$\therefore \arg(z-1) = 2\theta$.

(ii) $\arg(z^2 - 3z + 2) = \arg(z-1)(z-2)$

$= \arg(z-1) + \arg(z-2)$

$= 2\theta + \theta + \frac{\pi}{2}$

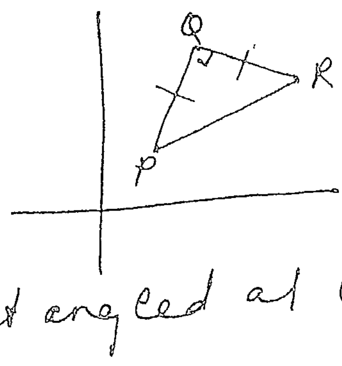
$= 3\theta + \frac{\pi}{2}$

($\arg(z-2) = \angle ZPM = \angle POZ + \angle OZP$ (ext $\angle$ = sum int opp $\angle$s)

$\angle OZP = \frac{\pi}{2}$ (angle in semi circle)

$\therefore \arg(z-2) = \theta + \frac{\pi}{2}$.

d) $p - q + iq - ir = 0$



$\therefore p - q = i(r-q)$

$|p-q| = |r-q|$

ie. $PQ = QR$.

and $PQ \perp RQ$

$\therefore \triangle PQR$ is isosceles & rt angled at Q.

Question 6

a) $P(z) = 2z^3 - 5z^2 - 2z + 15$

$P(2+i) = 2(2+i)^3 - 5(2+i)^2 - 2(2+i) + 15$

$= 2(8+12i-6-i) - 5(4+4i-1) - 2(2+i)+15$

$= 0$

$\therefore$ $2+i$ is a root of $P(z)=0$.

Since coeffs of $P(z)$ are integers
$2-i$ is also a root of $P(z)=0$.

$(z-2-i)(z-2+i) = z^2 - 4z + 5$

$P(z) = (z^2-4z+5)(2z+3)$

$\therefore$ Roots are $2+i$, $2-i$, $-3/2$.

b) $x^3 - px^2 + qx - r = 0$ has roots α, β, γ

$\therefore \alpha+\beta+\gamma = p$

Let $y = p - x$ $\therefore x = p - y$

Eqn with roots $\alpha+\beta$, $\beta+\gamma$, $\gamma+\alpha$ is

$(p-y)^3 - p(p-y)^2 + q(p-y) - r = 0$

$p^3 - 3p^2y + 3py^2 - y^3 - p(p^2 - 2py + y^2) + qp - qy - r = 0$

$-p^2y + 2py^2 - y^3 + qp - qy - r = 0$

$y^3 - 2py^2 + (p^2+q)y - qp + r = 0$

$\therefore (\alpha+\beta)(\beta+\gamma)(\gamma+\alpha) = qp - r$

1. (d) $\arg\left(-\frac{z}{i}\right) = \arg\left(\frac{1}{z}\right)$

$\therefore \arg(-z) - \arg i = \arg 1 - \arg z$

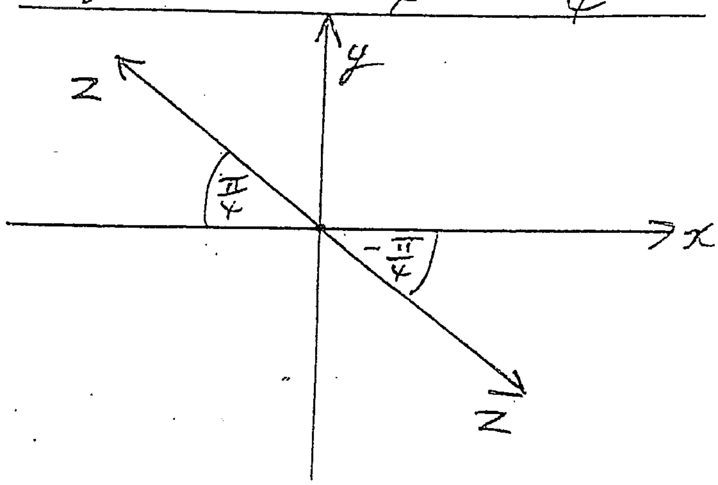
$\arg(-1.z) - \frac{\pi}{2} = 0 - \arg z$

$\arg z + \arg(-1) + \arg z = \frac{\pi}{2}$

$2\arg z \pm \pi = \frac{\pi}{2}$

$2\arg z = \frac{\pi}{2} \mp \pi = -\frac{\pi}{2}$ or $\frac{3\pi}{2}$

$\therefore \arg z = -\frac{\pi}{4}$ or $\frac{3\pi}{4}$ 2



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$\boxed{3}$

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2. (a) $2x^3 - 4x^2 + 8x - 1 = 0$

$\Sigma\alpha = -\frac{b}{a} = \frac{4}{2} = 2$ $\quad \Sigma\alpha\beta = \frac{c}{a} = \frac{8}{2} = 4$ $\Big\}$ $\frac{1}{2}$

$\alpha\beta\gamma = -\frac{d}{a} = \frac{1}{2}$

i) $\alpha^2 + \beta^2 + \gamma^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$

$= 2^2 - 2.4$

$= 4 - 8 = \underline{-4}$ 1

ii) $2x^3 = 4x^2 - 8x + 1$

$\therefore 2\alpha^3 = 4\alpha^2 - 8\alpha + 1$

$2\beta^3 = 4\beta^2 - 8\beta + 1$

$2\gamma^3 = 4\gamma^2 - 8\gamma + 1$

$2(\alpha^3+\beta^3+\gamma^3) = 4(\Sigma\alpha^2) - 8(\Sigma\alpha) + 3$

$2\Sigma\alpha^3 = 4.-4 - 8.2 + 3$

$= -16 - 16 + 3$

$= -29$

$\therefore \alpha^3+\beta^3+\gamma^3 = -\frac{29}{2}$

$= \underline{-14\frac{1}{2}}$ $\frac{1}{2}$

$\boxed{3}$

(b) $6z^2 + (1+i)z - 1 + 3i = 0$

$$z = \frac{-1-i \pm \sqrt{(1+i)^2 - 4.6(-1+3i)}}{2.6}$$

$$= \frac{-1-i \pm \sqrt{1+2i-1+24-72i}}{12}$$

$$= \frac{-1-i \pm \sqrt{24-70i}}{12} \quad 1.$$

Let $\sqrt{24-70i} = a + ib$

$\therefore 24 - 70i = a^2 - b^2 + 2abi$

$\therefore a^2 - b^2 = 24$ & $2ab = -70$

$ab = -35$

$a = \frac{-35}{b}$

$\therefore \left(\frac{-35}{b}\right)^2 - b^2 = 24$

$1225 - b^4 = 24b^2$

$b^4 + 24b^2 - 1225 = 0$

$(b^2+49)(b^2-25) = 0 \qquad b = \pm 5, \; a = \mp 7 \qquad 2.$

$$\therefore z = \frac{-1-i(-7+5i)}{12}, \; \frac{-1-i(+7-5i)}{12}$$

$$= \frac{-8+4i}{12}, \; \frac{6-6i}{12}$$

$$z = \frac{-2+i}{3}, \; \frac{1-i}{2} \quad 1.$$

$\boxed{4}$

) $P(z) = z^3 - 4z^2 - 2z + m$

Since $(3-i)$ is a root, so is $(3+i)$ as all co-eff. are real

Sum of roots: $(3-i) + (3+i) + \alpha = \frac{4}{1}$ $\qquad (z^2 - 6z + 10)$

$\frac{1}{2} \qquad \therefore \alpha = -2$

$\therefore$ Roots are $\boxed{3+i}$, $3-i$, $\boxed{-2}$ $\quad 1\frac{1}{2}$

Prod. of roots: $(3+i)(3-i) \cdot -2 = -m$

$\therefore m = 20 \quad \frac{1}{2}$

$\boxed{2}$

Q (d) $x^3 + 3x - 5 = 0$ Eqn with roots $\alpha^2, \beta^2, \gamma^2$

Let $y = x^2$ i.e. $x = \sqrt{y}$

$$y^{\frac{3}{2}} + 3y^{\frac{1}{2}} - 5 = 0$$

$$y^{\frac{3}{2}} + 3y^{\frac{1}{2}} = 5$$

Squaring: $y^3 + 6y^2 + 9y = 25$

or i.e. $\underline{x^3 + 6x^2 + 9x - 25 = 0}$ $\boxed{3}$