D

Marks

Q1 (Start a new page)

a) Given that 1, w and w^2 are the cube roots of unity, (i.e. roots of $z^3 - 1 = 0$), simplify: 4

$$(1-w)(1-w^2)(1-w^4)(1-w^8).$$

b) Solve the equation $\dfrac{z}{3+4i} + \dfrac{z-1}{5i} = \dfrac{5}{3-4i}$ 3

c) Express $\dfrac{1+2i}{1-3i}$ in mod-arg form and hence find the value of $\left(\dfrac{1+2i}{1-3i}\right)^4$ 3

d) The complex number $w = -1 + \sqrt{3}\, i$ 5

Show that: i) $w^2 = 2\overline{w}$

ii) w is a root of the equation $x^9 - 2^9 = 0$

Q2 (Start a new page)

a) Show that the pentagon formed by the roots of $z^5 + 1 = 0$ in the Argand plane has an area of A square units, where 3

$$A = \frac{5}{2}\sin\frac{2\pi}{5}$$

b) Describe and sketch the locus of the complex number z , where 3

$1 < |z| \le 4$ and $|\arg z| \le \dfrac{3\pi}{4}$

c) i) Find the square roots of the complex number -3-4i, expressing your answers in the form a+ib. 4

ii) Hence solve the equation $z^2 - (5-2i)z + 6 - 4i = 0$ 2

d) A,B,C and D are four points in the Argand number plane representing the complex numbers Z_1, Z_2, Z_3, and Z_4 respectively. Given that Z_2 and Z_4 are purely imaginery and Z_1 and Z_3 are real; and $\arg(Z_2 - Z_3) = \arg(Z_1 - Z_4)$, determine with the aid of a diagram, the type of the quadrilateral ABCD. 3

(Start a new page) **Marks**

3 a) The polynomial equation P(x)=0 has a root α of multiplicity m. Prove that the polynomial equation P'(x) = 0 , has the root α with multiplicity m-1 . 4

Given that the polynomial equation $P(x) = x^4 + x^3 - 3x^2 - 5x - 2 = 0$ has a 3-fold root, find all the roots of P(x) = 0.

b) If α, β, γ are the roots of the cubic equation $x^3 + px + q = 0$, form the cubic equation whose roots are

i) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ 5

ii) $\beta + \gamma - 2\alpha, \quad \gamma + \alpha - 2\beta, \quad \alpha + \beta - 2\gamma$

* c) The polynomial P(x) is given by $P(x) = x^5 - 5ax + 1$ where 'a' is a real number.

i) By considering the turning points, prove that if $a < 0$, $P(x) = 0$ has just one real root, which is negative. 6

ii) Prove that P(x) = 0 has 3 distinct roots if, and only if,

$$a > \left(\frac{1}{4}\right)^{\frac{4}{5}}$$

Solutions Q1, Q2, Q3. 4 Unit ½ yearly.

Q1

(a) $(1-w)(1-w^2)(1-w^4)(1-w^8)$

$= (1-w)(1-w^2)(1-w)(1-w^2) \quad = (1-w)^2(1-w^2)^2$

$= (1-w-w^2+w^3)^2$

$= (2-w-w^2)^2 \quad$ (as $w^3=1$).

$= (2-(\omega+\omega^2))^2 \quad = (2+1)^2 = 3^2 = 9$

$= 4-2w-2w^2-2w+w^2+w^3+2w^2+w^3+w^4$

$= 4-4w-3w^2+2w^3+w^4$

$= 4-4w-3w^2+2+w$

$= 6-3w-3w^2$

$= 3(2-w-w^2)$

$= 3(3-(1+w+w^2))$

$= 3\times 3$

$= \underline{\underline{9}}$ (as $1+w+w^2=0$ sum of roots).

(4).

(b) $\dfrac{z}{3+4i} + \dfrac{z-1}{5i} = \dfrac{5}{3-4i}$

'Realizing' each denominator.

$\dfrac{z(3-4i)}{25} - \dfrac{i(z-1)}{5} = \dfrac{5(3+4i)}{25}$

$z(3-4i) - 5i(z-1) = 5(3+4i)$

$3z - 4iz - 5iz + 5i = 15 + 20i$

$3z - 9iz = 15 + 15i$

$3z(1-3i) = 15(1+i)$

$z = \dfrac{5(1+i)}{1-3i}$

$\underline{z = -1+2i}$

(3).

(c) $\dfrac{1+2i}{1-3i} = \dfrac{1+2i}{1-3i} \times \dfrac{1+3i}{1+3i}$

$= -\frac{1}{2} + \frac{1}{2}i$

$= \frac{1}{\sqrt{2}}\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$

Now $\left(\dfrac{1+2i}{1-3i}\right)^4 = \left(\frac{1}{\sqrt{2}}\right)^4\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)^4$

$= \frac{1}{4}\left(\cos 3\pi + i\sin 3\pi\right).$

$= \frac{1}{4}\left(-1 + 0\right)$

$= \underline{\underline{-\frac{1}{4}}}$ (3)

(d) (i) $W = -1+\sqrt{3}\, i$

$W^2 = (-1+\sqrt{3}\, i)^2$

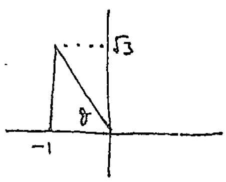
$= \underline{-2-2\sqrt{3}\, i}$ (2)

$\bar{w} = -1-\sqrt{3}\, i$

$2\bar{w} = -2-2\sqrt{3}\, i$

$\therefore\ \underline{w^2 = 2\bar{w}}$

(ii) $w = -1+\sqrt{3}\, i$



$|w| = \sqrt{1+3}$

$= \underline{2}$

$\tan\theta = \frac{\sqrt{3}}{-1}$

$\underline{\theta = \frac{2\pi}{3}.}$ (1)

$\therefore\ w = 2\,\mathrm{cis}\,\frac{2\pi}{3}.$

$w^9 = \left(2\,\mathrm{cis}\,\frac{2\pi}{3}\right)^9$

$= 2^9\,\mathrm{cis}^9\,\frac{2\pi}{3}.$

$= 2^9\left(\cos\frac{18\pi}{3} + i\sin\frac{18\pi}{3}\right).$

$= 2^9\left(\cos 6\pi + i\sin 6\pi\right)$ (3)

$= 2^9(1+0)$

$= 2^9$

$\therefore w^9 = 2^9$

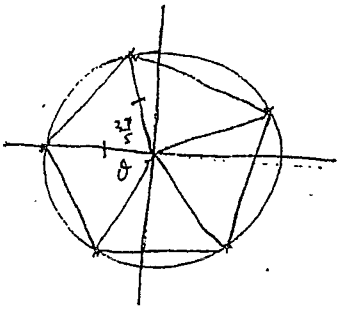
$\therefore$ <u>w is a solution of $x^9 - 2^9 = 0$.</u>

Q2

(a) $z^5 + 1 = 0$

$z^5 = -1$

$\therefore \quad z = \sqrt[5]{-1} = -1.$



$\theta = \frac{2\pi}{5}.$

Vertices will be equally spaced around circle.

$\therefore$ Pentagon is 5 congruent Δ's.

$\therefore$ Area of $1 \Delta = \frac{1}{2} ab \sin C$

$= \frac{1}{2} \times 1 \times 1 \times \sin \frac{2\pi}{5}$

$= \frac{1}{2} \sin \frac{2\pi}{5}$

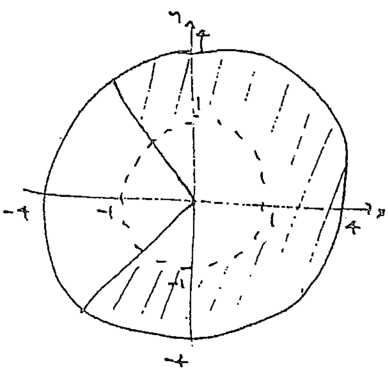
$\therefore$ Total Area $= \frac{5}{2} \sin \frac{2\pi}{5}$ units^2

(3)

(b). $1 < |z| \leq 4$

$|\arg z| \leq \frac{3\pi}{4}$

ie $\frac{-3\pi}{4} \leq \arg z \leq \frac{3\pi}{4}$



(3)

$\therefore$ annulus within circles radii 1 and 4

between the lines $y = -x$, $x < 0$

(c) (i) Let $a+ib = \sqrt{-3-4i}$

$(a+ib)^2 = -3-4i$

$\therefore a^2 - b^2 + 2abi = -3-4i$

$a^2 - b^2 = -3$ — (1)

$2ab = -4$ — (2) 1

Now $(a^2+b^2)^2 = (a^2-b^2)^2 + (2ab)^2$

$\therefore a^2+b^2 = \sqrt{9+16}$

$a^2+b^2 = 5.$ — (3) 1

(1)+(3) $\Rightarrow 2a^2 = 2$ (4)

$a^2 = 1$

$\underline{a = \pm 1}$ 1

when $a=1$, $2ab = -4 \Rightarrow b = -2$

$a=-1$, $2ab = -4 \Rightarrow b = 2$

$\therefore$ Sqr roots of $-3-4i$ are $1-2i$ and $-1+2i$ 1

(ii) $z^2 - (5-2i)z + 6-4i = 0.$

$$z = \frac{5-2i \pm \sqrt{(5-2i)^2 - 4(6-4i)}}{2}$$

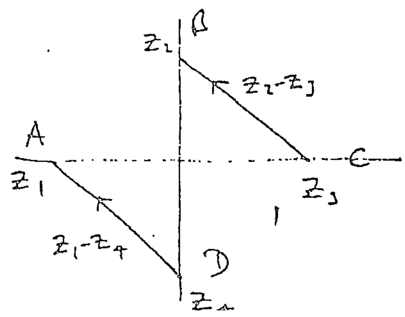
$$= \frac{5-2i \pm \sqrt{-3-4i}}{2}$$ (3)

$$= \frac{5-2i \pm (1-2i)}{2}$$

$$= \frac{6-4i}{2}, \frac{4}{2}$$

$$= \underline{3-2i \text{ or } 2}$$

(d)



B & D lie on y axis. A & C lie on x-axis

$\arg(z_2 - z_3) = \arg(z_1 - z_4)$

$\Rightarrow CB \parallel DA$ — 1 (3)

$\therefore$ ABCD is a trapezium. 1

Q3 (a) $P(x) = (x-\alpha)^m \cdot Q(x)$ where $Q(\alpha) \neq 0$.

$P'(x) = (x-\alpha)^m \cdot Q'(x) + m(x-\alpha)^{m-1} \cdot Q(x)$

$= (x-\alpha)^{m-1}\left[(x-\alpha)Q'(x) + mQ(x)\right]$

$= (x-\alpha)^{m-1} \cdot S(x)$ where $S(x)$ is a poly and $S(\alpha) \neq 0$ as $Q(\alpha) \neq 0$ 2

$\therefore$ $P'(x)$ has the root α with mult $m-1$.

$P(x) \equiv x^4 + x^3 - 3x^2 - 5x - 2 = 0$

$P'(x) = 4x^3 + 3x^2 - 6x - 5$

$P''(x) = 12x^2 + 6x - 6$

$= 6(2x-1)(x+1)$

$\therefore P''(x) = 0 \Rightarrow x = \frac{1}{2}$ or -1

$P'(-1) = P(-1) = 0$ $\therefore$ $x = -1$ is the triple root

$\therefore P(x) = (x+1)^3(x+a)$

$P(0) = 1 \times a = -2$

$\therefore a = -2$

$\therefore P(x) = (x+1)^3(x-2)$

$\therefore$ roots are $-1, -1, -1, 2$ 2

(b) (i) Let $y = \frac{1}{x}$ $\therefore$ $x = \frac{1}{y}$

$\therefore \left(\frac{1}{y}\right)^3 + p\left(\frac{1}{y}\right) + q = 0.$

$1 + py^2 + qy^3 = 0.$

ie $qy^3 + py^2 + 1 = 0.$ 2

Q3 b (iii)

Sum of roots $= \alpha+\beta+\gamma = 0.$

$\beta + \alpha - 2\alpha = \alpha+\beta+\gamma - 3\alpha$

$= -3\alpha \quad (\text{as } \alpha+\beta+\gamma = 0).$

$\gamma + \alpha - 2\beta = -3\beta$

$\alpha + \beta - 2\gamma = -3\gamma$

$\therefore$ Form an eqn with roots $-3\alpha, -3\beta, -3\gamma$

Let $y = -3x \quad \therefore \quad x = -\frac{y}{3}.$ 3

$\left(\frac{-y}{3}\right)^3 + p\left(\frac{-y}{3}\right) + q = 0$

$\frac{-y^3}{27} - \frac{py}{3} + q = 0$

$\underline{y^3 + 9py - 27q = 0.}$

(c) (i) $P(x) = x^5 - 5ax + 1$

$P'(x) = 5x^4 - 5a$

st pts $P'(x) = 0 \quad \therefore \quad x^4 = a$

if $a < 0$, then $x^4 = a$ has no solutions

and hence curve has no stat. pts. 1.

$P(x)$ is a poly of degree 5 with no st. pts

and $P'(x) = 5x^4 - 5a$

> 0 for all x (as $a < 0$)

ie $P(x)$ is always increasing

$\therefore$ Curve cuts x axis only once. 1

when $x = 0$, $y = 1 \quad \therefore \quad P(x)$ cuts y axis at 1 and is an

increasing curve, hence must cross x-axis before $x = 0$.

ie. x-intercept is negative. 1

$\therefore$ root is negative

Q 3 (c) (ii) For $\exists$ distinct real roots $P'(x) = 0$ must have two solutions. ie 2 stat pts.

$\therefore$ $5x^4 - 5a = 0$ has 2 distinct solutions

$x^4 = a$

$x = \pm a^{1/4}$ $(a > 0)$

and $P(-a^{1/4})$ and $P(a^{1/4})$ must be opp in sign

$P(-a^{1/4}) = -a^{5/4} + 5a^{5/4} + 1$

$= 4a^{5/4} + 1$

$> 0.$

$P(a^{1/4}) = a^{5/4} - 5a^{5/4} + 1$

$= 1 - 4a^{5/4}$

$\therefore$ For $\exists$ distinct real roots

$P(a^{+1/4}) = 1 - 4a^{5/4}.$

$< 0.$ (as $P(a^{1/4})$ is opp in sign to $P(a^{-1/4})$).

$\therefore 1 - 4a^{5/4} < 0$

$1 < 4a^{5/4}$

$\frac{1}{4} < a^{5/4}.$

$\therefore$ $\underline{a > \left(\frac{1}{4}\right)^{4/5}}.$

E

QUESTION 2

Marks

8

a) Let $z = 2(\cos \frac{2\pi}{9} + i \sin \frac{2\pi}{9})$

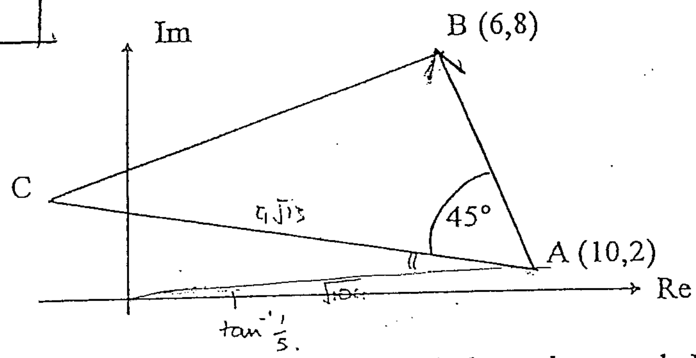
i) Write down the modulus and argument of the complex numbers

z, $\bar{z}$, z^2, and $\frac{1}{z}$

ii) Hence, or otherwise, clearly plot the points on an Argand diagram corresponding to z, $\bar{z}$, z^2 and $\frac{1}{z}$ labelling them A, B, C and D respectively.

b) Find the two square roots of $-3 + 4i$ expressing each root in the form $a + ib$ where a, b are real. 4

c) 3



(NOT TO SCALE)

Δ ABC is drawn on the Argand plane where angle BAC = 45°, A represents the complex number $10 + 2i$ and B represents $6 + 8i$.

If the length of side AC is twice the length of AB then find the complex number that point C represents.

2

(a) (i)

$z = 2 \operatorname{cis} \frac{2\pi}{9}$

$|z| = 2$

$\arg z = \frac{2\pi}{9}$

$\bar{z} = 2 \operatorname{cis}\left(-\frac{2\pi}{9}\right)$

$\therefore |\bar{z}| = 2$

$z^2 = \left(2 \operatorname{cis} \frac{2\pi}{9}\right)^2$

$= 4 \operatorname{cis} \frac{4\pi}{9}$

$|z^2| = 4$

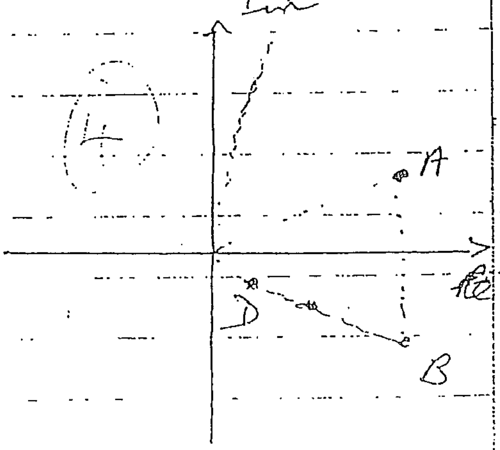
$\arg z^2 = \frac{4\pi}{9}$

$\frac{1}{z} = \frac{1}{2 \operatorname{cis} \frac{2\pi}{9}}$

$= \frac{1}{2} \operatorname{cis}\left(-\frac{2\pi}{9}\right)$

$\left|\frac{1}{z}\right| = \frac{1}{2}$

$\arg\left(\frac{1}{z}\right) = -\frac{2\pi}{9}$



(b)

Let $(a+ib)^2 = -3+4i$ $\quad a, b \in R$

$a^2 - b^2 + 2iab = -3+4i$

$\therefore a^2 - b^2 = -3$

$ab = 2$

$\Longrightarrow$

$a^2 - \left(\frac{2}{a}\right)^2 = -3$

$a^4 + 3a^2 - 4 = 0$

$(a^2+4)(a^2-1) = 0$

as $a \in R$ then

$a^2 = 1$

$a = \pm 1$

$\Longrightarrow b = \pm 2$

$\therefore \sqrt{-3+4i} = \pm(1+2i)$

(c) $\overrightarrow{AB}$

$= (6+8i) - (10+2i)$

$= -4+6i$

Now $\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$

$\overrightarrow{AC} = 2 \operatorname{cis} 45^\circ \times \overrightarrow{AB}$

$= 2\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)(-4+6i)$

$= \sqrt{2}(1+i)(-4+6i)$

$-4+6i-4i-6$

$= \sqrt{2}(-10+2i)$

$\therefore \overrightarrow{OC} = (10+2i) + \sqrt{2}(-10+2i)$

$= 10(1-\sqrt{2}) + 2i(1+\sqrt{2})$