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Question 1 : (11 Marks)

(a) (i) Express $-1+\sqrt{3}i$ in modulus-argument form. 3

(ii) Hence evaluate $(-1+\sqrt{3}i)^9$

(b) (i) Sketch on an Argand diagram, the locus of P which satisfies 4

$$|z-(6+8i)|=6$$

(ii) Hence find the maximum value of $|z|$ and the maximum value of $\arg z$

(c) (i) If $1-2i$ is a root of the equation $z^2-(3+i)z+c=0$, show that the value of c is $8-i$ 4

(ii) Given that the other root of this equation is $2+3i$, find the two square roots of $-24+10i$.

Question 2 : Start a new page (14 Marks)

(a) Sketch the locus of z on an Argand diagram given that $z^2-(\bar{z})^2=i$. 3

(b) (i) Use De Moivre's Theorem to write down the solutions to the equation $z^5=1$ in modulus-argument form. 5

(ii) By grouping the roots of this equation in conjugate pairs, show that:

$$z^5-1=(z-1)(z^2-2z\cos\frac{2\pi}{5}+1)(z^2-2z\cos\frac{4\pi}{5}+1)$$

(iii) Let $A=\cos\frac{2\pi}{5}$ and $B=\cos\frac{4\pi}{5}$

Hence, or otherwise, show that $A=\cos\frac{2\pi}{5}=\frac{-1+\sqrt{5}}{4}$.

(c) (i) By expanding $(z+z^{-1})^4$ where $z=\cos\theta+i\sin\theta$, show that 6

$$\cos^4\theta=\frac{\cos 4\theta+4\cos 2\theta+3}{8}$$

4 UNIT

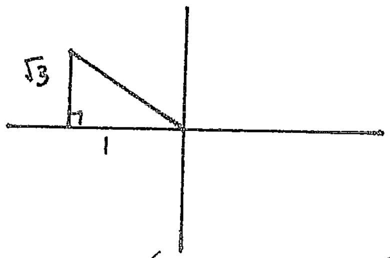
ASSESSMENT TASK 1

27.3.00

SOLUTIONS

QUESTION 1: $\boxed{11}$

(a) (i)



$$-1+\sqrt{3}i = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

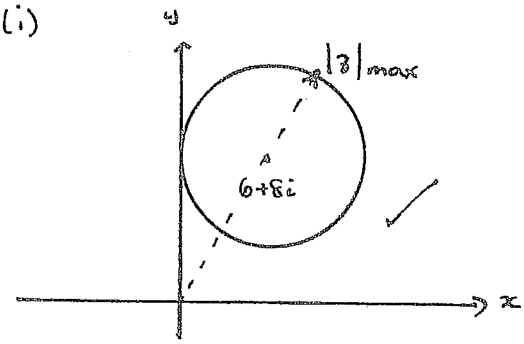
(ii) $(-1+\sqrt{3}i)^9$

$= 2^9(\cos 6\pi + i\sin 6\pi)$

$= 2^9(1+0)$

$= 512$

(b) (i)



$$(x-6)^2 + (y-8)^2 = 36 \quad (*)$$

(ii) max arg $= \frac{\pi}{2}$

Sub $y = \frac{4}{3}x$ into $(*)$

$x^2 - 12x + \cancel{36} + \frac{16}{9}x^2 - 21\frac{1}{3}x + 64 = \cancel{36}$

$9x^2 - 108x + 16x^2 - 192x + 576 = 0$

$25x^2 - 300x + 576 = 0$

$$x = \frac{300 \pm \sqrt{300^2 - 4.25.576}}{50}$$

$= 9\frac{3}{5}$ or $2\frac{2}{5}$

$\therefore$ When $x = 9\frac{3}{5}$, $y = 12\frac{4}{5}$

$$\therefore |z|_{max} = \sqrt{(9\tfrac{3}{5})^2 + (12\tfrac{4}{5})^2}$$

$= 16$

(c) $(1-2i)^2 - (3+i)(1-2i) + c = 0$

$1 - 4i - 4 - 3 + 5i - 2 + c = 0$

$\therefore c = 8 - i$

(ii)
$$z = \frac{(3+i) \pm \sqrt{(3+i)^2 - 4.(8-i)}}{2}$$

$$= \frac{(3+i) \pm \sqrt{9+6i-1-32+4i}}{2}$$

$$= \frac{3+i \pm \sqrt{-24+10i}}{2}$$

But the 2 roots are $1-2i$ and $2+3i$

$$\therefore 1-2i = \frac{3+i \pm \sqrt{-24+10i}}{2}$$

$2-4i = 3+i \pm \sqrt{-24+10i}$

$-1-5i = \pm\sqrt{-24+10i}$

(OR,

$$2+3i = \frac{3+i \pm \sqrt{-24+10i}}{2}$$

$4+6i = 3+i \pm \sqrt{-24+10i}$

$1+5i = \pm\sqrt{-24+10i}$)

$\therefore$ The two square roots are:

$\pm(1+5i)$

(iii) Let $A = \cos\frac{2\pi}{5}$ and $B = \cos\frac{4\pi}{5}$

$$\therefore z^5 - 1 = (z-1)(z^2 - 2Az + 1)(z^2 - 2Bz + 1)$$

$$= (z-1)\left(z^4 - \cancel{2Bz^3} + \cancel{z^2} - \cancel{2Az^3} + \cancel{4AB}z^2 - 2Az + \cancel{z^2} - 2Bz + 1\right)$$

$$= (z-1)\left(z^4 - 2(A+B)z^3 + 2(2AB+1)z^2 - 2(A+B)z + 1\right)$$

$$= z^5 - 2(A+B)z^4 + 2(2AB+1)z^3 - 2(A+B)z^2 + z$$
$$\quad - z^4 + 2(A+B)z^3 - 2(2AB+1)z^2 + 2(A+B)z - 1$$

$$= z^5 - z^4\left(1 + 2(A+B)\right) + z^3\left(2(A+B) + 2(2AB+1)\right) - z^2\left(2(A+B) + 2(2AB+1)\right)$$
$$+ z\left(1 + 2(A+B)\right) - 1$$

✓

Equating co-efficients:

z^4: $\quad 1 + 2(A+B) = 0 \quad (1)$

z^3: $\quad 2(A+B) + 2(2AB+1) = 0 \quad (2)$

From (1):

$$B = -\frac{1}{2} - A$$

Sub into (2):

$$\cancel{A} - \frac{1}{2} - \cancel{A} + 2A\left(-\frac{1}{2} - A\right) + 1 = 0$$

$$\frac{1}{2} - A - 2A^2 = 0$$

$$2A^2 + A - \frac{1}{2} = 0$$

$$4A^2 + 2A - 1 = 0$$

$$A = \frac{-2 \pm \sqrt{4 + 4.4.1}}{8}$$

$$= \frac{-2 \pm \sqrt{20}}{8}$$

$$= \frac{-1 \pm \sqrt{5}}{4}$$

But $\cos\frac{2\pi}{5} > 0$ ✓

$$\therefore A = \cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{4}$$

QUESTION 2: $\boxed{14}$

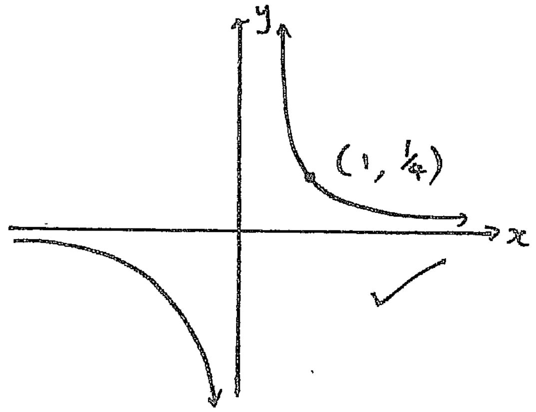
(a) Let $z = x + iy$

$\therefore (x+iy)^2 - (x-iy)^2 = i$ ✓

$\cancel{x^2} + 2xiy - \cancel{y^2} - \cancel{x^2} + 2xiy + \cancel{y^2} = i$

$4xiy = i$

$\therefore y = \frac{1}{4x}$ ✓



✓

(b) (i) $z_1 = 1$ $\quad z_2 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$ $\quad z_3 = \cos\frac{-2\pi}{5} + i\sin\frac{-2\pi}{5}$

$z_4 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$ $\quad z_5 = \cos\frac{-4\pi}{5} + i\sin\frac{-4\pi}{5}$ ✓

(ii) $z^5 - 1 = (z-1)[(z-z_2)(z-z_3)][(z-z_4)(z-z_5)]$

$= (z-1)(z^2 - (z_2+z_3)z + z_2z_3)(z^2 - (z_4+z_5)z + z_4z_5)$ ✓

But $z_2 + z_3 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} + \cos\frac{-2\pi}{5} + i\sin\frac{-2\pi}{5}$

$= \cos\frac{2\pi}{5} + \cancel{i\sin\frac{2\pi}{5}} + \cos\frac{2\pi}{5} - \cancel{i\sin\frac{2\pi}{5}}$

$= 2\cos\frac{2\pi}{5}$

Similarly $z_4 + z_5 = 2\cos\frac{4\pi}{5}$ ✓

$z_2 z_3 = \cos\left(\frac{2\pi}{5} - \frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5} - \frac{2\pi}{5}\right)$

$= 1$

Similarly, $z_4 z_5 = 1$

$\therefore z^5 - 1 = (z-1)\left(z^2 - 2z\cos\frac{2\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{5} + 1\right)$

(i) $(z+z^{-1})^4$

$= \left(\cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta)\right)^4$

$= (2\cos\theta)^4$

$= 16\cos^4\theta$ ✓

Also,

$(z+z^{-1})^4 = z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4}$

$= \cos 4\theta + \cancel{i\sin 4\theta} + 4(\cos 2\theta + \cancel{i\sin 2\theta}) + 6$

$\quad + 4(\cos -2\theta + \cancel{i\sin -2\theta}) + \cos -4\theta + \cancel{i\sin -4\theta}$ ✓

$= 2\cos 4\theta + 8\cos 2\theta + 6$

$\therefore 16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$

$\therefore \cos^4\theta = \dfrac{\cos 4\theta + 4\cos 2\theta + 3}{8}$ ✓

(ii) $\displaystyle\int_{\frac{\pi}{12}}^{\pi/8} \cos^4\theta \, d\theta$

$= \displaystyle\frac{1}{8}\int_{\frac{\pi}{12}}^{\pi/8} \cos 4\theta + 4\cos 2\theta + 3 \; d\theta$ ✓

$= \dfrac{1}{8}\left[\dfrac{1}{4}\sin 4\theta + 2\sin 2\theta + 3\theta\right]_{\pi/12}^{\pi/8}$

$= \dfrac{1}{8}\left[\left(\dfrac{1}{4}\sin\dfrac{\pi}{2} + 2\sin\dfrac{\pi}{4} + \dfrac{3\pi}{8}\right)\right.$ ✓

$\quad \left. - \left(\dfrac{1}{4}\sin\dfrac{\pi}{3} + 2\sin\dfrac{\pi}{6} + \dfrac{\pi}{4}\right)\right]$

$= \dfrac{1}{8}\left[\left(\dfrac{1}{4} + \sqrt{2} + \dfrac{3\pi}{8} - \dfrac{\sqrt{3}}{8} - 1 - \dfrac{\pi}{4}\right)\right.$

$= \dfrac{1}{8}\left(\sqrt{2} + \dfrac{\pi}{8} - \dfrac{\sqrt{3}}{8} - \dfrac{3}{4}\right)$

$= \dfrac{1}{64}\left(8\sqrt{2} + \pi - \sqrt{3} - 6\right)$ ✓