EXTENSION 2 TEST 4-3-02
COMPLEX NUMBERS and POLYNOMIALS.

Name______________________Class________

Instructions: Show all necessary working throughout the test on A4 paper.

Begin a new page for each question.

Time allowed: 85 minutes

1. (a) If $z_1=-\sqrt{3}+i$ and $z_2=3+3i$ find

   (i) $|z_1z_2|$ [2]

   (ii) $\arg(z_1z_2)$ [2]

   Hence write $(z_1z_2)^5$ in the form $r\,cis\theta$. [2]

   (b) If $\arg(z-1)=\frac{\pi}{4}$ find the locus of z in Cartesian form. [2]

   (c) If $z=x+iy$ write $\frac{z-2i}{2z-1}$ in the form $\frac{a+ib}{c}$. [2]

   Hence find the locus of z if $\mathrm{Re}(\frac{a+ib}{c})=0$. [2]

   (d) Factorise x^6-1 over the complex number field. [3]

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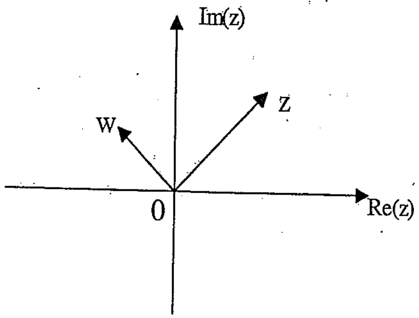
2. (a) If z and w are complex numbers as shown on the Argand Diagram,



indicate on separate Argand Diagrams :

   (i) $z+w$ [2]

   (ii) $w-z$ [2]

   (iii) iz [1]

   (b) Solve the equation: $z^2+4z-1+12i=0$ [4]

   (c) Find the least value of $\arg z$ for which $|z-2i|=1$. [3]

   (d) Sketch the region on the Argand Diagram consisting of the points z for which : $-\frac{\pi}{3}\le\arg z\le\frac{\pi}{3}$, $z+\bar{z}<4$ and $|z|\ge2$. [3]

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3. (a) Prove that the 5 roots of $z^5=-1$ are:
   $z=cis\frac{(2k+1)\pi}{5}$ for $k=0,1,2,3,4.$ [3]

   Hence show that $\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}=\frac{1}{2}$. [2]

   shortcut ?

   (b) If $x=\frac{r}{s}$ is a root of $P(x)=2x^3-3x^2+8x-12$, where r and s are relatively prime, find all the factors of $P(x)$ over the complex number field. [3]

   (c) When the polynomial $P(x)$ is divided by x, $x-1$ and $x+2$ the respective remainders are 1, 2 and 3. Determine what the remainder will be when $P(x)$ is divided by $x(x-1)(x+2)$. [4]

   (d) If the polynomial $P(x)=x^3+qx+r$ has roots α,β and γ form an equation with roots $\frac{1}{\alpha^2},\frac{1}{\beta^2}$ and $\frac{1}{\gamma^2}$. [3]

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4. (a) If α,β and γ are the roots of $x^3-4x^2+5x+3=0$ evaluate $\alpha^3+\beta^3+\gamma^3$. [4]

   (b) If the three roots of the equation $8x^3-36x^2+38x-3=0$ are in arithmetic progression find these roots in ascending order of magnitude. [3]

   (c) Prove that w, a complex cube root of unity, is a repeated root of $P(x)=3x^5+2x^4+x^3-6x^2-5x-4$. Hence find the zeros of $P(x)$ over the complex number field. [4]

   (d) Reduce $\frac{x^5}{(x^2+1)(x-1)^3}$ to partial fractions over the real number field. [4]

EXTENSION 2 TEST COMPLEX NOS / POLYNOMIALS

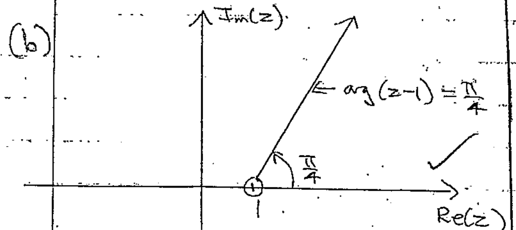
1(a) $z_1 = -\sqrt{3} + i$, $z_2 = 3 + 3i$

(i) $|z_1 z_2| = |z_1||z_2|$
$= |-\sqrt{3}+i||3+3i|$
$= \sqrt{(\sqrt{3})^2+1^2}\ \sqrt{3^2+3^2}$
$= 2 \cdot 3\sqrt{2}$
$= 6\sqrt{2}$

(ii) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
$= \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) + \tan^{-1}\left(\frac{3}{3}\right)$
$= \frac{5\pi}{6} + \frac{\pi}{4}$
$= \frac{13\pi}{12}$
$= \frac{-11\pi}{12}$

$(z_1 z_2)^5 = \left[6\sqrt{2}\ \text{cis}\left(\frac{-11\pi}{12}\right)\right]^5$
$= 7776 \times 4\sqrt{2}\ \text{cis}\left(\frac{-55\pi}{12}\right)$
$= 31104\sqrt{2}\ \text{cis}\left(\frac{-55\pi}{12} + \frac{48\pi}{12}\right)$
$= 31104\sqrt{2}\ \text{cis}\left(\frac{-7\pi}{12}\right)$

(b)



$m = \tan\frac{\pi}{4} = 1$, $x > 1$, $y > 0$

Locus is: $y - 0 = 1(x-1)$
$\therefore y = x - 1$ for $x > 1,\ y > 0$

(c) If $z = x + iy$

$\frac{z-2i}{2z-1} = \frac{x+iy-2i}{2x+2iy-1} \cdot \frac{(2x-1)-i2y}{(2x-1)-i2y}$

$= \frac{2x^2-x+2y^2-4y+i((y-2)(2x-1)-2x\ldots)}{(2x-1)^2+(2y)^2}$

$= \frac{2x^2-x+2y^2-4y+i[2xy-y-4x+2-2xy]}{(2x-1)^2+4y^2}$

$= \frac{2x^2-x+2y^2-4y+i[2-y-4x]}{(2x-1)^2+4y^2}$

in form $\frac{a+ib}{c}$

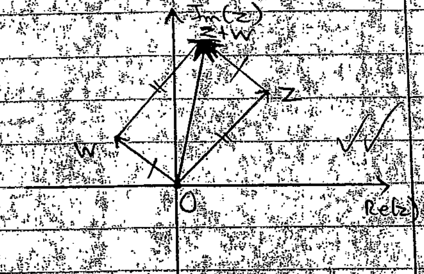
Now if $\text{Re}\left(\frac{a+ib}{c}\right) = 0$
$\therefore 2x^2 - x + 2y^2 - 4y = 0$
$x^2 - \frac{1}{2}x + y^2 - 2y = 0$
$(x-\frac{1}{4})^2 - \frac{1}{16} + (y-1)^2 - 1 = 0$
$\therefore (x-\frac{1}{4})^2 + (y-1)^2 = \left(\frac{\sqrt{17}}{4}\right)^2$
or $|z - \frac{1}{4} - i| = \frac{\sqrt{17}}{4}$ is the locus of z.

(d) $x^6 - 1 = (x^3)^2 - 1$
$= (x^3-1)(x^3+1)$
$= (x-1)(x^2+x+1)(x+1)(x^2-x+1)$

$\therefore x^6 - 1 = (x-1)\left((x+\frac{1}{2})^2+\frac{3}{4}\right)(x+1)\left((x-\frac{1}{2})^2+\frac{3}{4}\right)$
$= (x-1)(x+\frac{1}{2}-\frac{\sqrt{3}i}{2})(x+\frac{1}{2}+\frac{\sqrt{3}i}{2})(x+1)(x-\frac{1}{2}-\frac{\sqrt{3}i}{2})(x-\frac{1}{2}+\frac{\sqrt{3}i}{2})$
over the complex number field

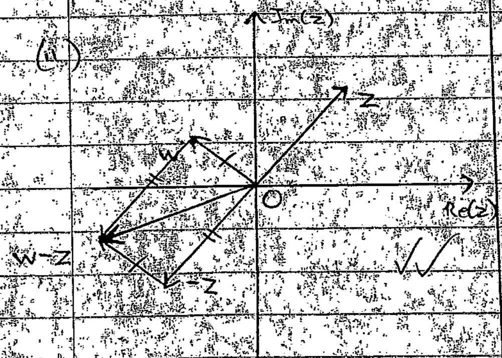
2(a)

(i)



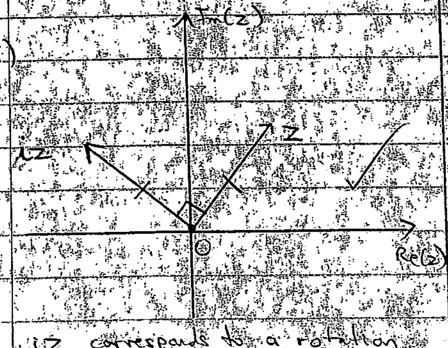
using a parallelogram of vectors $z+w$ is as indicated

(ii)



using a parallelogram of vectors $w-z$ is as indicated

(iii)



iz corresponds to a rotation

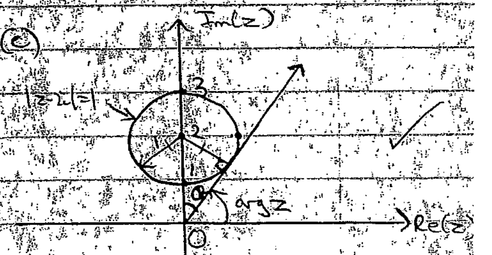
(b) $z^2 + 4z - 1 + 12i = 0$

$z = \frac{-4 \pm \sqrt{16 - 4.1.(12i-1)}}{2}$
$= \frac{-4 \pm \sqrt{16 - 48i + 4}}{2}$
$= \frac{-4 \pm \sqrt{20-48i}}{2}$
$= \frac{-4 \pm 2\sqrt{5-12i}}{2}$
$= -2 \pm \sqrt{5-12i}$

Now $\sqrt{5-12i} = a + ib$, where $a, b \in \mathbb{R}$
$5 - 12i = a^2 - b^2 + 2abi$
Equating real & imag. parts:
$5 = a^2 - b^2$ — (1)
$2ab = -12$ — (2)
From (2), $b = -\frac{6}{a}$ sub. into (1):
$\therefore 5 = a^2 - \frac{36}{a^2}$
$0 = a^4 - 5a^2 - 36$
$0 = (a^2-9)(a^2+4)$
$\therefore a = \pm 3$ ($a \in \mathbb{R}$)
when $a = 3$, $b = -2$; when $a = -3$, $b = 2$
$\sqrt{5-12i} = 3 - 2i$ or $-3 + 2i$

$\therefore z = -2 \pm (3-2i)$ or $-2 \pm (-3+2i)$
ie $z = 1 - 2i$ or $-5 + 2i$

(c)



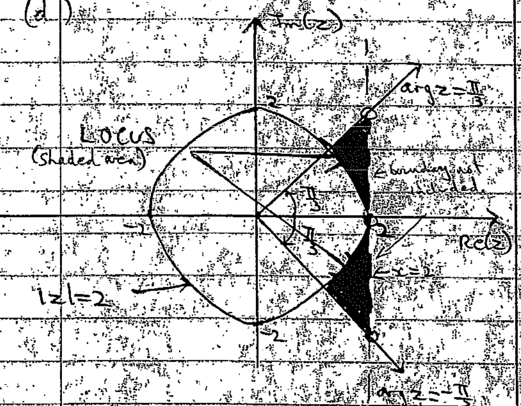
Now the least value of arg z for which $|z-2i|=1$ occurs when arg z is tangential to the circle as shown.

$\sin\theta = \frac{1}{2}$ $\theta = \frac{\pi}{6}$

$\arg z = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$

$\therefore$ Least value of $\arg z = \frac{\pi}{3}$

(d)



For $z + \bar{z} = 4$

let $z = x+iy$

$\therefore x+iy+x-iy = 4$

$\therefore 2x = 4$ ie $x = 2$

3(a) $z^5 = -1$

$\therefore |z^5| = |-1|$

$\therefore |z|^5 = 1$

$\therefore |z| = 1$

$\Rightarrow$ roots lie on the unit circle, centre the origin

Let $z = r(\cos\theta + i\sin\theta)$

$z^5 = r^5(\cos\theta + i\sin\theta)^5 = -1$

ie $\cos 5\theta + i\sin 5\theta = -1$, $r = 1$

$\therefore \cos 5\theta = -1$, $\sin 5\theta = 0$

$\therefore 5\theta = \pi, 3\pi, 5\pi, 7\pi, 9\pi$ for $0 \le \theta < 2\pi$

$\theta = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}$

ie $z = \text{cis}\left[\frac{(2k+1)\pi}{5}\right]$ for $k = 0, 1, 2, 3, 4$

Let roots be z_1, z_2, z_3, z_4 and z_5

$z_1 = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$

$z_2 = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}$

$z_3 = \cos\pi + i\sin\pi = -1$

$z_4 = \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5} = \cos\left(-\frac{3\pi}{5}\right) + i\sin\left(-\frac{3\pi}{5}\right)$
$= \cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5}$

$z_5 = \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5} = \cos\left(-\frac{\pi}{5}\right) + i\sin\left(-\frac{\pi}{5}\right)$
$= \cos\frac{\pi}{5} - i\sin\frac{\pi}{5}$

Now sum of roots $= \sum \alpha_i = 0$

$\therefore 2\left(\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}\right) - 1 = 0$

$\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$

(b) Now if $x = \frac{r}{s}$ is a root of $P(x)$ then $s|2$ and $r|-12$ ($s>0$ by convention)

Factors of 2 are: 1, 2

Factors of −12 are: $\pm1, \pm2, \pm3, \pm4, \pm6, \pm12$

$\frac{r}{s} = \frac{\pm1}{1}, \frac{\pm1}{2}, \frac{\pm2}{1}, \frac{\pm2}{2}, \frac{\pm3}{1}, \frac{\pm3}{2}, \frac{\pm4}{1}, \frac{\pm4}{2}, \frac{\pm6}{1}, \frac{\pm6}{2}, \frac{\pm12}{1}$ or $\frac{\pm12}{2}$

Now $P\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 - 3\left(\frac{3}{2}\right)^2 + 8\left(\frac{3}{2}\right) - 12$
$= \frac{54}{8} - \frac{27}{4} + 12 - 12$
$= 0$

$\Rightarrow x = \frac{3}{2}$ is a zero and $2x-3$ is a factor of $P(x)$

$\therefore P(x) = (2x-3)R(x)$, where $R(x)$ is of degree 2

$P(x) = (2x-3)(x^2 + 0x + 4)$

$\therefore P(x) = (2x-3)(x^2+4)$
$= (2x-3)(x-2i)(x+2i)$
over the complex number field

(c) Since $x(x-1)(x+2)$ is cubic, the remainder on dividing $P(x)$ by $x(x-1)(x+2)$ will be quadratic.

ie $P(x) = x(x-1)(x+2)R(x) + ax^2 + bx + c$

Now $P(0) = 1 = c$ — (1)

$P(1) = 2 = a+b+c$ — (2)

$P(-2) = 3 = 4a - 2b + c$ — (3)

From (1) $c = 1$ sub into (2) and (3)

$1 = a + b$ — (2A)

$2 = 4a - 2b$ — (3A)

(2A)×2: $2 = 2a + 2b$ — (2B)

(3A)+(2B): $4 = 6a$ $\therefore a = \frac{2}{3}$

$\therefore b = \frac{1}{3}$

$\therefore$ Remainder is $\frac{2}{3}x^2 + \frac{1}{3}x + 1$

(d) $P(x) = x^3 + qx + r$ has roots α, β & γ

For roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ let $y = \frac{1}{x}$

$\therefore x = \frac{1}{y}$ sub into $P(x)$ eqn

$\left(\frac{1}{y}\right)^3 + q\left(\frac{1}{y}\right) + r = 0$

$\frac{1}{y^3} + \frac{q}{y} + r = 0$

$1 + qy^2 + ry^3 = 0$

ie $rx^3 + qx^2 + 1 = 0$

is an equation, when reverting to the variable x, for roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$

4(a) Let $P(x) = x^3 - 4x^2 + 5x + 3$

As α is a root of $P(x)$: $\alpha^3 - 4\alpha^2 + 5\alpha + 3 = 0$ — (1)

β " $\therefore \beta^3 - 4\beta^2 + 5\beta + 3 = 0$ — (2)

γ " $\gamma^3 - 4\gamma^2 + 5\gamma + 3 = 0$ — (3)

(1)+(2)+(3):

$\alpha^3 + \beta^3 + \gamma^3 - 4(\alpha^2 + \beta^2 + \gamma^2) + 5(\alpha + \beta + \gamma) + 9 = 0$ (*)

Now $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$ — (4)

But $\sum \alpha_i = \alpha + \beta + \gamma = \frac{-(-4)}{1} = 4$

$\sum \alpha_i\alpha_j = \alpha\beta + \alpha\gamma + \beta\gamma = \frac{5}{1} = 5$

$\prod \alpha_i = \alpha\beta\gamma = \frac{-3}{1} = -3$

$\therefore$ (*) becomes

$\alpha^3 + \beta^3 + \gamma^3 = 4(\alpha^2 + \beta^2 + \gamma^2) - 5(\alpha + \beta + \gamma) - 9$
$= 4\left[4^2 - 2(5)\right] - 5(4) - 9$ (using (4))
$= 4[6] - 20 - 9$
$= -5$

(b) If roots are in A.P. let the roots be $a-d, a, a+d$.

Now $\sum \alpha_i = \frac{36}{8} = \frac{9}{2} = 3a$ — (1)

$\prod \alpha_i = \frac{3}{8} = a(a^2 - d^2)$ — (2)

From (1) $a = \frac{3}{2}$ sub into (2)

$\frac{3}{8} = \frac{3}{2}\left(\frac{9}{4} - d^2\right)$

$\therefore \frac{1}{4} = \frac{9}{4} - d^2$ $\therefore d^2 = 2$

$\therefore d = \pm\sqrt{2}$

$\therefore$ Roots are: $\frac{3}{2} - \sqrt{2}, \frac{3}{2}, \frac{3}{2} + \sqrt{2}$ in ascending order of magnitude.

(c) $P(x) = 3x^5 + 2x^4 + x^3 - 6x^2 - 5x - 4$

If $P(x)$ has a repeated root

the $P'(x)$ has a 1 fold root.

Now $P'(x) = 15x^4 + 8x^3 + 3x^2 - 12x - 5$

let $x = \omega$

$\therefore P'(\omega) = 15\omega^4 + 8\omega^3 + 3\omega^2 - 12\omega - 5$

$= 15\omega + 8 + 3\omega^2 - 12\omega - 5$

$= 3\omega + 3\omega^2 + 3$

$= 3(1 + \omega + \omega^2)$

$= 0$

(as $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$ for ω, a complex cube root of unity).

Also $P(\omega) = 3\omega^5 + 2\omega^4 + \omega^3 - 6\omega^2 - 5\omega - 4$

$= 3\omega^2 + 2\omega + 1 - 6\omega^2 - 5\omega - 4$

$= -3\omega^2 - 3\omega - 3$

$= -3(\omega^2 + \omega + 1)$

$= 0$

$\therefore$ As $P(\omega) = P'(\omega) = 0$

$\Rightarrow \omega$ is a repeated root of $P(x)$

But as the coefficients of $P(x)$ are all real, the the complex conjugate ω^2 of ω will also be a repeated root.

$\therefore P(x) = (x-\omega)^2(x-\omega^2)^2 R(x)$

where $R(x)$ is of degree 1.

Let other root be α.

Now product of roots $= \Pi \alpha_i = \frac{4}{3}$

$\therefore \omega \cdot \omega \cdot \omega^2 \cdot \omega^2 \cdot \alpha = \frac{4}{3}$

$\therefore \omega^6 \alpha = \frac{4}{3}$

$\therefore \alpha = \frac{4}{3}$ (as $\omega^6 = (\omega^3)^2 = 1$)

$\Rightarrow$ zeros of $P(x)$ are:

$x = \omega$ (repeated), ω^2 (repeated) and $\frac{4}{3}$ over the complex number field

(d) For $\frac{x^5}{(x^2+1)(x-1)^2}$ ← if to the power of 2!

$\therefore (x^2+1)(x-1)^2 = (x^2+1)(x^2-2x+1)$

$= x^4 - 2x^3 + 2x^2 - 2x + 1$

$$\begin{array}{r|l} & x+2 \\ x^4-2x^3+2x^2-2x+1 & x^5 \\ & -(x^5-2x^4+2x^3-2x^2+x) \\ & 2x^4-2x^3+2x^2-x \\ & -(2x^4-4x^3+4x^2-4x+2) \\ & 2x^3-2x^2+3x-2 \end{array}$$

$\therefore$ By partial fractions:

$\frac{x^5}{(x^2+1)(x-1)^2} = x+2 + \frac{2x^3-2x^2+3x-2}{(x^2+1)(x-1)^2}$

$= x+2 + \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$

Now $\frac{2x^3-2x^2+3x-2}{(x^2+1)(x-1)^2} = \frac{(Ax+B)(x-1)^2 + C(x^2+1)(x-1) + D(x^2+1)}{(x^2+1)(x-1)^2}$

$\therefore 2x^3 - 2x^2 + 3x - 2 = (Ax+B)(x-1)^2 + C(x^2+1)(x-1) + D(x^2+1)$

let $x=1$ $\therefore 1 = 2D$ $\therefore D = \frac{1}{2}$

let $x=0$ $\therefore -2 = B - C + D$ $\therefore B - C = -2\frac{1}{2}$

let $x=2$ $\therefore 12 = 2A + B + 5C + 5D$

$\therefore 9\frac{1}{2} = 2A + B + 5C$ — (2)

Equating corr. coeffs of x^3: $2 = A + C$ — (3)

From (1) $B = C - 2\frac{1}{2}$ sub into (2) $\therefore 12 = 2A + 6C$ — (4)

$\therefore 6 = A + 3C$ — (4)

(4) − (3): $4 = 2C$ $\therefore C = 2$, $A = 0$, $B = -\frac{1}{2}$

$\frac{x^5}{(x^2+1)(x-1)^2} = x + 2 - \frac{1}{2(x^2+1)} + \frac{2}{x-1} + \frac{1}{2(x-1)^2}$

4(d) $\frac{x^5}{(x^2+1)(x-1)^3} = a + \frac{bx+c}{x^2+1} + \frac{d}{x-1} + \frac{e}{(x-1)^2} + \frac{f}{(x-1)^3}$

$= \frac{a(x^2+1)(x-1)^3 + (bx+c)(x-1)^3 + d(x^2+1)(x-1)^2 + e(x^2+1)(x-1) + f(x^2+1)}{(x^2+1)(x-1)^3}$

$\therefore x^5 = a(x^2+1)(x-1)^3 + (bx+c)(x-1)^3 + d(x^2+1)(x-1)^2 + e(x^2+1)(x-1) + f(x^2+1)$

Equating c.c. of x^5 terms: $1 = a$ — (1)

let $x=0$ $\therefore 0 = -a - c + d - e + f$

$1 = -c + d - e + f$ — (2)

let $x=1$ $\therefore 1 = 2f$ $\therefore f = \frac{1}{2}$

$\therefore \frac{1}{2} = -c + d - e$ — (2A)

let $x=2$ $\therefore 32 = 5a + 2b + c + 5d + 5e + 5f$

$\therefore 24\frac{1}{2} = 2b + c + 5d + 5e$ — (3)

let $x=-1$ $\therefore -1 = -16a + 8b - 8c + 8d - 4e + 2f$

$\therefore 14 = 8b - 8c + 8d - 4e$ — (4)

let $x=3$ $\therefore 243 = 80a + 24b + 8c + 40d + 20e + 10f$

$\therefore 158 = 24b + 8c + 40d + 20e$ — (5)

(2A) + (3): $25 = 2b + 6d + 4e$ — (6)

(4) + (5): $172 = 32b + 48d + 16e$ — (7)

(5) + 8×(2A): $162 = 24b + 48d + 12e$ — (8)

(7) − 16×(6): $-228 = -48d - 48e$ — (9)

(8) − 12×(6): $-138 = -24d - 36e$ — (10)

(9) − 2×(10): $48 = 24e$ $\therefore e = 2$ sub into (9)

$\therefore d = 2\frac{3}{4}$ sub into (6)

$\therefore b = \frac{1}{4}$

$\therefore c = \frac{1}{4}$

$\therefore \frac{x^5}{(x^2+1)(x-1)^3} = 1 + \frac{x+1}{4(x^2+1)} + \frac{11}{4(x-1)} + \frac{2}{(x-1)^2} + \frac{1}{2(x-1)^3}$