Student Number: ____________

St. Catherine's School
Waverley

Mathematics
Extension 2

2013

ASSESSMENT TASK 1

(15%)

Total marks – 46

o Attempt Questions 1–2 and multiple choice

o Marks for each question are indicated on this page

General Instructions

o *Reading time - 3minutes*
• Working time – 57 minutes

o Start each question on a new page in your answer booklet.

o Write using black or blue pen only.

o Board-approved calculators may be used.

o All necessary working must be shown.

o Marks may be deducted for careless or badly arranged work.

TEACHER'S USE ONLY

Question 1	/22
Question 2	/21
Multiple Choice	/3
Total	/46

Multiple choice (3 marks) Circle the correct answer on this page

1) When the polynomial $P(x) = x^4 - 3x^2 + 6x - 2$ is divided by $x^2 - 4$ the remainder is $6x + 2$. The remainder when $P(x)$ is divided by $(x + 2)$ will be:

a) 14 b) $x - 2$ c) 2 d) 10

2) Let $z = x + iy$. If $\frac{z+4}{z}$ is purely imaginary, then which of the following is the locus of z?

a) $x = 0$ b) $x^2 + 4x + y^2 = 0$

c) $y = 0$ d) $x^2 + y^2 = 4$

3) The locus of $|z - 2 + 3i| = |z + 3|$ is a

a) parabola b) straight line

c) circle d) semi-circle

Marks

Question 1 (22 marks)

a) Let $z = \frac{1+2i}{1-3i}$

i) Find the modulus of z. 2

ii) Find the argument of z. 1

iii) Hence find z^8. 2

b) Find the complex number z that satisfies the condition: 2

$$Im(z) + \bar{z} = \frac{1}{1-i}$$

c) Given that $1-3i$ is a root of the equation $2z^3 - 3z^2 + 18z + 10 = 0$, find all the roots this equation. 2

d) The complex number w, where $w \neq 1$, is a root of the equation $z^3 - 1 = 0$.

i) Show that $1 + w + w^2 = 0$ 1

ii) Show that $1 + (1+w)^3 = 0$ 2

iii) Evaluate $1 + \frac{1}{w} + \frac{1}{w^2}$ 1

e) Find the two square roots of $5 + 12i$. 2

f) On an Argand diagram, shade the region specified by the conditions:

$Re(z) \leq 4$ and $|z - 4 + 5i| \leq 3$ 2

g) The complex number z satisfies the equation $Arg(z-3) = \frac{2\pi}{3}$.

i) Sketch the locus of the point P in the argand diagram which represents z. 1

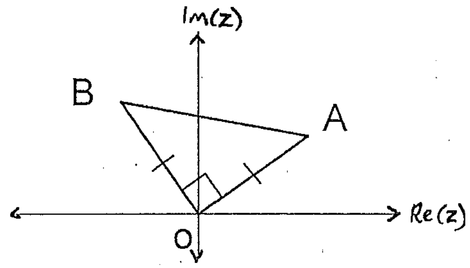
ii) Find the modulus and argument of z when $|z|$ takes its least value. 3

iii) Hence find, in the form $a + ib$, z for which $|z|$ is a minimum. 1

Question 2 (21 marks)

a) The points A and B on the Argand diagram shown correspond to the complex numbers z and w respectively. The triangle OAB is isosceles and $\angle BOA$ is a right angle.

Show that $z^2 + w^2 = 0$. 2



b) For the cubic equation $2x^3 - 6x^2 + 5x - 3 = 0$ with roots $x = \alpha, \beta, \gamma$

α) find the value of:

i) $\alpha^2 + \beta^2 + \gamma^2$ 2

ii) $\alpha^3 + \beta^3 + \gamma^3$ 2

β) find a cubic equation with roots $2\alpha - 1, 2\beta - 1, 2\gamma - 1$. 2

c) If the roots of $P(x) = x^3 + px^2 + qx + r = 0$ are in arithmetic progression,

show that $2p^3 = 9pq - 27r$ 4

d) Sketch $Arg(z-2) - Arg(z) = \frac{\pi}{2}$ on an argand diagram and give its Cartesian equation. 3

e) i) Use De'Moivre's theorem and the expansion of $(cos\theta + isin\theta)^3$ to express $cos3\theta$ and $sin3\theta$ in powers of $cos\theta$ and $sin\theta$. 2

ii) Hence show that $tan3\theta = \frac{3tan\theta - tan^3\theta}{1-3tan^2\theta}$ 2

iii) Show that $tan\frac{\pi}{12}$, $tan\frac{5\pi}{12}$ and $tan\frac{3\pi}{4}$ are the roots of 2

$x^3 - 3x^2 - 3x + 1 = 0.$

END OF PAPER

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Student number.........
Course name.........
Question...1...

Multiple choice answers ① d ② b ③ b

4 page writing booklet

1a) i $z = \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i}$

$= \frac{1+3i+2i-6}{1+9}$

$= \frac{-5+5i}{10}$

$= \frac{-1+i}{2}$

$= -\frac{1}{2}+\frac{1}{2}i$

$|z| = \frac{1}{\sqrt{2}}$

$= \frac{\sqrt{2}}{2}$

ii $\arg(z) = \frac{3\pi}{4}$

5
5

iii $z^8 = \left(\frac{\sqrt{2}}{2}\right)^8 cis\, \frac{3\pi}{4}\times 8$

$= \frac{1}{16}\, cis\, 6\pi$

$= \frac{1}{16}$

b) $\mathrm{Im}(z) + \bar{z} = \frac{1}{1-i}$

~~$x + x - iy = \frac{1+i}{2}$~~

~~$2x = \frac{1}{2}, \quad -y = \frac{1}{2}$~~

~~$x = \frac{1}{4} \quad y = -\frac{1}{2}$~~

~~$\therefore z = \frac{1}{4} - \frac{1}{2}i$~~

$y + x - iy = \frac{1}{1-i}$

$x+y \quad -iy = \frac{1}{2}+\frac{1}{2}i$

$x+y = \frac{1}{2} \quad -y = \frac{1}{2}$

$y = -\frac{1}{2}$

$\therefore x = 1$

$\therefore z = 1 - \frac{1}{2}i$

$\frac{2}{2}$

DO NOT TEAR ANY PAGES FROM THIS BOOKLET

1c) If $1-3i$ is a root, $1+3i$ is also a root since coefficients are real

$(x-(1-3i))(x-(1+3i))$

$= x^2 - x(1-3i) - x(1+3i) + (1-3i)(1+3i)$

$= x^2 - 2x + 1 + 9$

$= x^2 - 2x + 10$

$\therefore 2z^3 - 3z^2 + 18z + 10 = 0$

$(z^2-2z+10)(2z+1) = 0$

$z = 1-3i,\ 1+3i,\ -\frac{1}{2}$

$\frac{2}{2}$

d) i $z^3 - 1 = 0$

$(z-1)(z^2+z+1) = 0$

$\therefore z = 1$ or $1+z+z^2 = 0$

since w is a root

$1+w+w^2 = 0$

ii $LHS = 1 + (1+w)^3$

$= 1 + (1 + 3w + 3w^2 + w^3)$

$= 1+1+3w+3w^2+1$

$= 3+3w+3w^2$

$= 3(1+w+w^2)$

$= 0$

$= RHS$

$\therefore 1+(1+w)^3 = 0$

$\frac{2}{2}$

d) iii $1+\frac{1}{w}+\frac{1}{w^2}$

$= \frac{w^2+w+1}{w^2}$

$= \frac{0}{w^2}$ ✓ 1/1

$= 0$

e) let $z^2 = 5+12i$

$(x+iy)^2 = 5+12i$

$x^2+2xyi-y^2 = 5+12i$

$x^2-y^2 = 5$ $\quad 2xy = 12$

$y = \frac{6}{x}$

$x^2-\left(\frac{6}{x}\right)^2 = 5$

$x^4-5x^2-36=0$

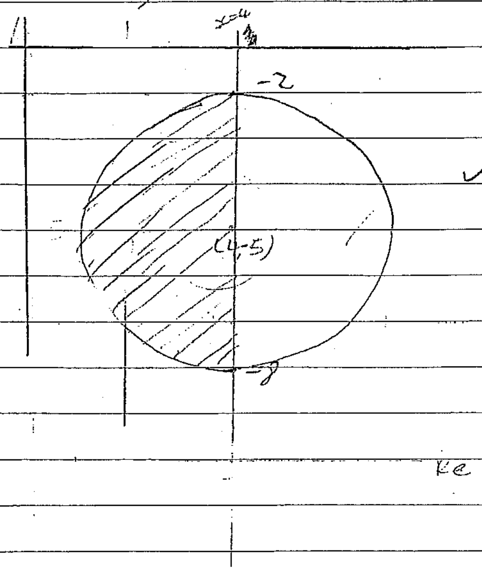
$(x^2-9)(x^2+4)=0$ ✓ 2/2

$x = \pm 3$

$y = \pm 2$

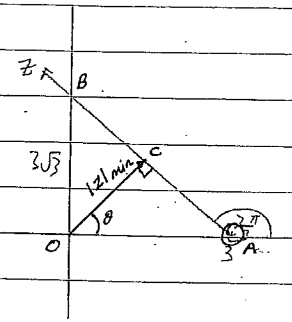
$\therefore z = 3+2i, \; -3-2i$

f)



✓ 2/2

g) i



1/1

(ii) $m_{AB} = \tan\frac{2\pi}{3} = -\sqrt{3}$

$\therefore$ Eqn. of AB is

$y-0 = -\sqrt{3}(x-3)$

$\Rightarrow \sqrt{3}x+y+3\sqrt{3}=0$

$\therefore |z| = \frac{|\sqrt{3}(0)+0+3\sqrt{3}|}{(\sqrt{3})^2+1^2}$

$= \frac{3\sqrt{3}}{2}$

$\arg z = \theta \Rightarrow \theta = \frac{\pi}{6}$

(iii) $\vec{OC} = r\,\text{cis}\,\theta$

$= \frac{3\sqrt{3}}{2}\,\text{cis}\,\frac{\pi}{6} = \frac{3\sqrt{3}}{2}\left[\frac{\sqrt{3}}{2}+i\cdot\frac{1}{2}\right]$

$= \frac{9}{4}+\frac{3\sqrt{3}}{4}i$

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Question.... 3 ! 2

4 page writing booklet

1g) iii $z = \frac{3\sqrt{3}}{4} \text{cis} \frac{\pi}{6}$

$= \frac{3\sqrt{3}}{4}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$ 1/1

$= \frac{9}{8} + \frac{3\sqrt{3}}{8}i$

2a) $w = iz$

$\therefore$ LHS $= z^2 + w^2$

$= z^2 + (zi)^2$

$= z^2 - z^2$ 2/2

$= 0$

$=$ RHS

b) i $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= (3)^2 - 2(\frac{5}{2})$

$\alpha+\beta+\gamma = 3$ $\quad = 4$ 2/2

$\alpha\beta+\beta\gamma+\alpha\gamma = \frac{5}{2}$

$\alpha\beta\gamma = \frac{3}{2}$

ii since α, β, γ are roots $P(\alpha) = P(\beta) = P(\gamma) = 0$

$2\alpha^3 - 6\alpha^2 + 5\alpha - 3 = 0$ — ①

$2\beta^3 - 6\beta^2 + 5\beta - 3 = 0$ — ②

$2\gamma^3 - 6\gamma^2 + 5\gamma - 3 = 0$ — ③

①+②+③ $2(\alpha^3+\beta^3+\gamma^3) = 6(\alpha^2+\beta^2+\gamma^2) - 5(\alpha+\beta+\gamma) + 9$ 2/2

$\alpha^3+\beta^3+\gamma^3 = \frac{1}{2}(24 - 15 + 9)$

$= 9$

2b) β let $x = 2\alpha - 1$

$\alpha = \frac{x+1}{2}$ 2/2

$2\left(\frac{x+1}{2}\right)^3 - 6\left(\frac{x+1}{2}\right)^2 + 5\left(\frac{x+1}{2}\right) - 3 = 0$

$\frac{x^3+3x^2+3x+1}{4} - \frac{3(x^2+2x+1)}{2} + \frac{5x+5}{2} - 3 = 0$

$x^3 + 3x^2 + 3x + 1 - 6(x^2+2x+1) + 10x + 10 - 12 = 0$

$x^3 - 3x^2 + x - 7 = 0$

c) let the roots of $P(x) = x^3 + px^2 + qx + r = 0$

be $a-d,\ a,\ a+d$

$a - d + a + a + d = -p$

$3a = -p$

$a = \frac{-p}{3}$ — ①

$a(a-d) + a(a+d) + (a-d)(a+d) = q$

$a^2 - ad + a^2 + ad + a^2 - d^2 = q$

$3a^2 - d^2 = q$ — ②

sub ① into ② $3\left(\frac{p^2}{9}\right) - d^2 = q$

$\frac{p^2}{3} - d^2 = q$ — ③

$a(a-d)(a+d) = -r$ 4/4

$a(a^2 - d^2) = -r$

$a^2 - d^2 = \frac{-r}{a}$

$d^2 = a^2 + \frac{r}{a}$ — ④

from ③ $d^2 = \frac{p^2}{3} - q$ — ⑤

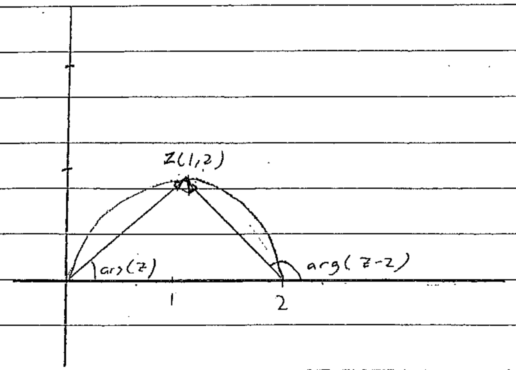
④ = ⑤ $a^2 + \frac{r}{a} = \frac{p^2}{3} - q$

$\frac{p^2}{9} - \frac{3r}{p} = \frac{p^2}{3} - q$

$p^3 - 27r = 3p^3 - 9pq$

$2p^3 = 9pq - 27r$

2d)



$\arg(z) + \frac{\pi}{2} = \arg(z-2)$ (ext. angle of Δ)

$\arg(z-2) - \arg(z) = \frac{\pi}{2}$

$\therefore$

$(x-1)^2 + y^2 = 1$

$\cancel{y^2 = \sqrt{1-(x-1)^2}}$

$y = \sqrt{1-(x-1)^2}$ ✓

$\cancel{= \sqrt{1-x^2+2x-1}}$

$\cancel{= \sqrt{2x-x^2}}$

2/3

e) i $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$ (De Moivre's Theorem)

$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

equate real parts $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ ✓

$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$

$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$

$\therefore \cos 3\theta = 4\cos^3\theta - 3\cos\theta$ ✓ 2/2

equate imaginary parts $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

$= 3\sin\theta(1-\sin^2\theta) - \sin^3\theta$

$\therefore \sin 3\theta = 3\sin\theta - 4\sin^3\theta$

(2)(e)(ii) $\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta}$ 2/2

$= \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\sin^2\theta\cos\theta}$

$= \frac{3\frac{\sin\theta}{\cos\theta} - \frac{\sin^3\theta}{\cos^3\theta}}{\frac{\cos^3\theta}{\cos^3\theta} - 3\frac{\sin^2\theta}{\cos^2\theta}}$

$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

2e) iii $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

let $\tan 3\theta = 1$

~~when $\tan 3\theta = 1$~~

$\cancel{3\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}}$

$\cancel{\theta = \frac{\pi}{12}}$

$3\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}$

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}$

$1 - 3\tan^2\theta = 3\tan\theta - \tan^3\theta$ 2/2

$\tan^3\theta - 3\tan^2\theta - 3\tan\theta + 1 = 0$

if $x = \tan\theta$

$x^3 - 3x^2 - 3x + 1 = 0$

$\tan^3\theta - 3\tan^2\theta - 3\tan\theta + 1 = 0$ only if $\tan 3\theta = 1$

$\therefore \tan\frac{\pi}{12}, \tan\frac{5\pi}{12}, \tan\frac{3\pi}{4}$ are the roots of $x^3 - 3x^2 + 3x + 1 = 0$