St Ursula's College
KINGSGROVE

MATHEMATICS EXTENSION 2 – YEAR 12, 2012

ASSESSMENT TASK 3

Name / Student No: ______________ ______________

Teacher: Mr Ratcliffe

Date: Wednesday 13th June, 2012

Time Allowed: 55 minutes

Weighting: 15 %

Total Marks: ____ / 45

INSTRUCTIONS

- Attempt all questions.
- Start each question in a new answer booklet.
- Board approved calculators may be used.
- Write using blue or black pen.
- Diagrams must be drawn in pencil.
- All necessary working should be shown in each question.

TEST STRUCTURE

3 questions of equal value (15marks)

Total Marks: ____ / 45

Question 1 (*Start a new writing booklet*) **(15 marks)** **MARKS**

(a) An ellipse with its centre at the origin has x-intercepts of ± 3 and y-intercepts of ± 2.

(i) Write down the equation of the ellipse. **1**

(ii) Calculate the eccentricity. **1**

(iii) Find the coordinates of the foci. **1**

(iv) Find the equations of the directrices. **1**

(b) The hyperbola with centre (0,0) has asymptotes $y = \pm\frac{5x}{7}$.

(i) Write down the equation of the hyperbola. **1**

(ii) Draw a sketch of the hyperbola showing vertices, foci, asymptotes and directrices. **4**

(c) A rectangular hyperbola has equation $xy = 32$.

(i) Determine the coordinates of the vertices. **1**

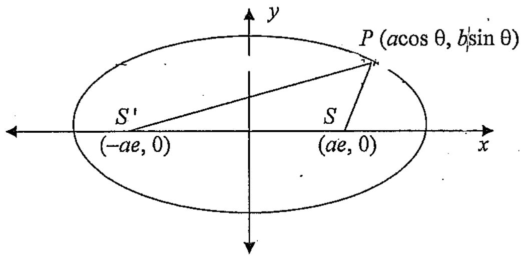
(ii) Determine the coordinates of the foci. **1**

(d) Find the gradient of the normal to the hyperbola $\frac{x^2}{81} - \frac{y^2}{16} = 1$ at the point $(10, \frac{4\sqrt{19}}{9})$. **2**

(e) The equation of the auxiliary circle of an ellipse is $x^2 + y^2 = 36$.
Given that $a{:}b = 2{:}5$, find the equation of the ellipse. **2**

Question 2 (*Start a new writing booklet*) **(15 marks)** **MARKS**

(a) Use the parametric equations given for point P in the diagram below to prove that the sum of the focal lengths (i.e. $PS + PS'$) is equal to $2a$. 3



(b) $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ are variable points on the rectangular hyperbola $xy = 16$.

The tangents at P and Q meet at $R\left(\frac{8pq}{p+q}, \frac{8}{p+q}\right)$.

Given that the equation of the chord of contact PQ is $x + pqy = 4(p+q)$ and PQ passes through the point (7,3), find the equation of the **locus** of R. 4

(c) Find $\int \frac{e^{\sin x}}{\sec x}\,dx$ 2

(d) Find $\int \frac{x^2+x-1}{x^2-x}\,dx$ 2

(e) By completing the square, find $\int \frac{2x+5}{x^2+4x+8}\,dx$ 4

Question 3 (*Start a new writing booklet*) **(15 marks)** **MARKS**

(a) By using two applications of integration by parts, find $\int e^x \sin x\,dx$. 3

(b) Use the substitution $t = \tan\frac{\theta}{2}$ to find $\int \frac{\tan x}{1+\cos x}\,dx$ 3

(c) Use partial fractions to find $\int \frac{x^2-2x-3}{(x+2)(x^2+1)}\,dx$ 3

(d) Prove $\int \operatorname{cosec}^2 x\,dx = -\cot x + c$. 3

(e) If $I_n = \int \tan^n x\,dx$ for $n \geq 0$, show that $I_n = \frac{1}{n-1}\tan^{n-1} x - I_{n-2}$. 3

THE END OF THE ASSESSMENT MARK

(1) (a) (i) $\frac{x^2}{9} + \frac{y^2}{4} = 1$

(ii) $b^2 = a^2(1-e^2)$

$4 = 9(1-e^2)$

$9e^2 = 5$

$e = \frac{\sqrt{5}}{3}$

(iii) Foci $(\pm\sqrt{5}, 0)$

(iv) Directrices. $x = \pm\frac{9}{\sqrt{5}}$

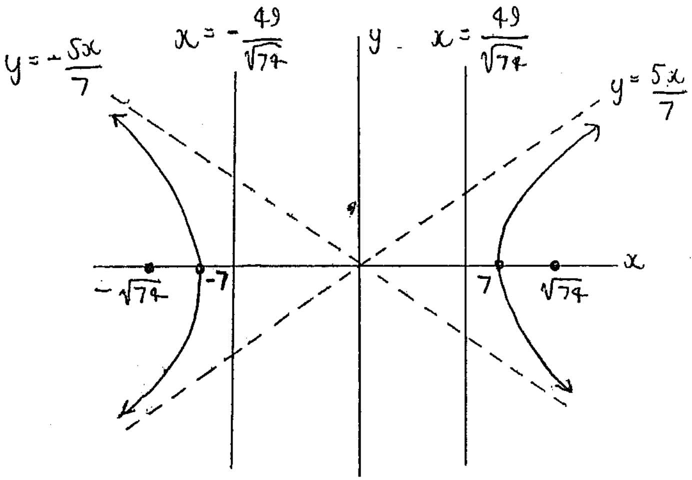
1mk for each correct answer

(b) (i) $\frac{x^2}{49} - \frac{y^2}{25} = 1$

1mk correct answer

(ii) $e = \frac{\sqrt{74}}{7}$; vertices $(\pm 7, 0)$

foci $(\pm\sqrt{74}, 0)$; direct. $x = \pm\frac{49}{\sqrt{74}}$



1mk each correct feature

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(1) (c) $xy = 32 \Rightarrow \frac{1}{2}a^2 = 32$

$a = \pm 8$ (i) vertices $(\pm 4\sqrt{2}, \pm 4\sqrt{2})$ — 1mk vertices

$e = \sqrt{2}$ (ii) foci $(\pm 8, \pm 8)$ — 1mk foci

(d) $\frac{x^2}{81} - \frac{y^2}{16} = 1$ at $x = 10$, $y = \frac{4\sqrt{19}}{9}$

$16x^2 - 81y^2 = 1296$

$32x - 162y \cdot y' = 0$

$y' = \frac{32x}{162y} = \frac{16x}{81y}$

1mk for y'

At $\left(10, \frac{4\sqrt{19}}{9}\right)$, $y' = 160 \div 36\sqrt{19}$

$= \frac{40}{9\sqrt{19}}$

$\therefore$ Normal has gradient of $\frac{-9\sqrt{19}}{40}$

1mk for correct gradient

(e) Aux. circle : $x^2 + y^2 = 36$ $\therefore a^2 = 36$

$\therefore a = 6$

Since $a : b = 2 : 5$ $\therefore b = \frac{5a}{2}$

$\therefore b = \frac{5(6)}{2} = 15$

1mk for $b = 15$

$\therefore$ equation is $\frac{x^2}{36} + \frac{y^2}{225} = 1$

1mk for correct equation

Marking Guidelines

(2) (a) Let P be the point $(a\cos\theta, b\sin\theta)$ and S and S' be the foci.

$PS^2 = (a\cos\theta - ae)^2 + b^2\sin^2\theta$

$= a^2\cos^2\theta - 2a^2e\cos\theta + a^2e^2 + b^2\sin^2\theta$

$= a^2\cos^2\theta - 2a^2e\cos\theta + a^2 - b^2 + b^2\sin^2\theta$ (since $a^2e^2 = a^2 - b^2$)

$= a^2\cos^2\theta - 2a^2e\cos\theta + a^2 - b^2\cos^2\theta$ (since $1 - \sin^2\theta = \cos^2\theta$)

$= a^2e^2\cos^2\theta - 2a^2e\cos\theta + a^2$ (since $a^2 - b^2 = a^2e^2$)

$= a^2(1 - e\cos\theta)^2$

$\therefore PS = a(1 - e\cos\theta)$ taking the positive square root

Similarly…

$PS' = a(1 + e\cos\theta)$

$\therefore$ By addition, $PS + PS' = 2a$

1mk some progress
2mks sig. progress
3 mks complete proof

(b) (ii) Equation of chord of contact PQ is given by…

$\frac{8x}{p+q} + \frac{8pqy}{p+q} = 32$

$\therefore x + pqy = 4(p+q)$

At $(7,3)$ $x + pqy = 4(p+q)$ becomes $7 + 3pq = 4(p+q)$

$\therefore pq = \frac{4(p+q) - 7}{3} \quad \ldots (1)$ ← 1mk

Since R is $\left(\frac{8pq}{p+q}, \frac{8}{p+q}\right)$

$\therefore x = \frac{8pq}{p+q}$, $y = \frac{8}{p+q} \Rightarrow p+q = \frac{8}{y}$

$x = \frac{8pq}{\frac{8}{y}}$

$x = pqy$ ← 1mk

$x = \left[\frac{4(p+q) - 7}{3}\right]y$

But $p+q = \frac{8}{y}$

$\therefore x = \left[\frac{4\left(\frac{8}{y}\right) - 7}{3}\right]y$ ← 1mk

$3x = 32 - 7y$

$\therefore 3x + 7y - 32 = 0$ ← 1mk

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(2) (c) $\int \frac{e^{\sin x}}{\sec x}\,dx = \int e^{\sin x}\cdot\cos x\,dx$

let $u = \sin x$ $\therefore du = \cos x\,dx$

$\therefore I = \int e^u\,du$

$= e^u + c$

$= e^{\sin x} + c$

1mk working toward.
2mks correct integral

(d) $\int \frac{x^2 + x - 1}{x^2 - x}\,dx = \int \frac{x^2 - x + 2x - 1}{x^2 - x}\,dx$

$= \int \frac{x^2 - x}{x^2 - x}\,dx + \int \frac{2x - 1}{x^2 - x}\,dx$

$= x + \ln(x^2 - x) + c$

As above

(e) $\int \frac{2x+5}{x^2 + 4x + 8}\,dx$

$= \int \frac{2x+5}{(x+2)^2 + 4}\,dx$

Let $u = x + 2 \Rightarrow du = dx$
$x = u - 2$

$\therefore I = \int \frac{2u + 1}{u^2 + 4}\,du$

$= \int \frac{2u}{u^2+4}\,du + \int \frac{1}{u^2+4}\,du$

$= \ln(u^2 + 4) + \frac{1}{2}\tan^{-1}\frac{u}{2} + c$

$= \ln(x^2 + 4x + 8) + \frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right) + c$

1mk correct perfect square
2mks further progre
3mks partial solution.
4mks complete sol.

Marking Guidelines

(3) (a) $\int e^x \sin x \, dx$	
$= e^x \sin x - \int e^x \cos x \, dx$	1mk for one applic.
$= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$	2mks two applications
$\therefore 2\int e^x \sin x \, dx = (\sin x - \cos x)e^x + c$	
$\therefore \int e^x \sin x \, dx = \frac{1}{2}(\sin x - \cos x)e^x + c$	3mks correct integral
(b) $\int \frac{\tan x}{1+\cos x} \, dx$	
Now $t = \tan\frac{x}{2} \Rightarrow \cos x = \frac{1-t^2}{1+t^2}$	
$\tan x = \frac{2t}{1-t^2}$	
$\frac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$	
$= \frac{t^2+1}{2}$	1mk some progress
$\therefore dx = \frac{2dt}{t^2+1}$	2mks sig. progress
$I = \int \frac{2t}{1-t^2} \cdot \frac{t^2+1}{2} \cdot \frac{2dt}{t^2+1}$	3mks correct integral
$= \int \frac{2t}{1-t^2} \, dt$	
$= -\ln(1-t^2) + c$	
$= -\ln\left(1 - \tan^2\frac{x}{2}\right) + c$	

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(3) (c) $\int \frac{x^2-2x-3}{(x+2)(x^2+1)} \, dx$	
Let $\frac{x^2-2x-3}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$	
$\therefore A(x^2+1) + Bx(x+2) + C(x+2) \equiv x^2-2x-3$	1mk for identity
When $x=-2$, $5A = 5 \therefore A = 1$	
$x = 0$, $A + 2C = -3$	
$1 + 2C = -3$	
$\therefore C = -2$	
$x = 1$, $2 + 3B - 6 = -4$	
$\therefore B = 0$	2mks correct value for A, B, C
$I = \int \left(\frac{1}{x+2} + \frac{-2}{x^2+1}\right) dx$	
$= \ln(x+2) - 2\tan^{-1}x + c$	3mks correct integra
(d) $\int \operatorname{cosec}^2 x \, dx = \int (1+\cot^2 x) \, dx$	1mk some progress
$= \int \left(1 + \frac{1}{\tan^2 x}\right) dx$	2mks sig. progress
$= \int \frac{\tan^2 x + 1}{\tan^2 x} \, dx$	
$= \int \frac{\sec^2 x}{\tan^2 x} \, dx$ Let $u = \tan x$, $du = \sec^2 x \, dx$	
$I = \int u^{-2} \, du$	
$= -\frac{1}{u}$	
$= -\frac{1}{\tan x}$	
$= -\cot x + c$	3mks complete proo

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(3) (e) $\int \tan^n dx = \int (\tan^{n-2}x . \tan^2 x)\, dx$

$= \int \tan^{n-2}x\,(\sec^2 x - 1)\,dx$ ← 1mk

$= \int \tan^{n-2}x . \sec^2 x\, dx - \int \tan^{n-2}x\, dx$

$I_n = \int \tan^{n-2}x . \sec^2 x\, dx - I_{n-2}$

2mks further sig. progress

Let $u = \tan x$, $du = \sec^2 x$

$\therefore I_n = \int u^{n-2}\, du - I_{n-2}$

$= \dfrac{u^{n-1}}{n-1} - I_{n-2}$

$= \dfrac{1}{n-1} \tan^{-1} x - I_{n-2}$

3mks complete proof