CRANBROOK SCHOOL

YEAR 12 MATHEMATICS – EXTENSION 1

Term 1 2002 **Time : 1.5 h / SKB, MJB and HRK**

All questions are of equal value.
All necessary working should be shown in every question.
Full marks may not be awarded if work is careless or badly arranged.
Approved silent calculators may be used.
Submit your work in three 8 Page Booklets.

1. **(20marks) (Begin an 8 page booklet.)** **skb**

(a) An equatorial arc AB of length 4800π km subtends an angle θ at O, the centre of the Earth. If the Earth's radius is 6400 km find:

(i) the measure of θ in exact form. [1]

(ii) the area of the minor segment cut off by AB in scientific notation correct to 3 significant figures. [2]

(b) **(i)** Sketch $y = 3\sin(2x + \frac{\pi}{4})$ for $0 \le x \le \pi$. [2]

(ii) Find the area bounded by the curve $y = 3\sin(2x + \frac{\pi}{4})$, the x-axis and lines $x = \frac{3\pi}{8}$ and $x = \frac{7\pi}{8}$. [3]

(c) Evaluate $\lim\limits_{\theta \to 0} \dfrac{3\sin^2\theta + 15\sin\theta}{\theta\sin\theta + 5\theta}$ [3]

(d) Find the equation of the tangent to the curve $y = e^{\tan 2x}$ at (0,1). [3]

(e) Evaluate $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^2 x \, dx$, by using the substitution $u = \cos x$. [3]

(f) The area bounded by the curve $y = \cos 2x$, the $x-$axis and lines $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$ is rotated about the x-axis. Find the volume of the solid generated in exact form. [3]

2. (20marks) [Begin a new 8 page booklet]

(a) **(i)** Divide $P(x) = 5x^4 - 2x^3 + 5x^2 - 3x + 4$ by $A(x) = x - 2$ and hence

(ii) Express $P(x)$ as $A(x).Q(x) + R(x)$. [4]

(b) Without dividing, find the remainder when $P(x) = 7x^4 - 5x^2 + 11x - 2$ is divided by $x + 3$. [2]

(c) **(i)** Find all of the factors of $P(x) = x^4 + 6x^3 + 9x^2 + 4x$

(ii) Use the result from part **(i)** to sketch the polynomial. [5]

(d) The polynomial $P(x) = ax^3 - bx^2 + cx - 8$ has zeros at **2** and **-1**. In addition $P(3) = 28$.

Find the values of a, b and c. [4]

(e) If α, β and γ are the roots of $2x^3 + 8x^2 - x + 6 = 0$, find the values of

(i) $\alpha\beta + \alpha\gamma + \beta\gamma$

(ii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

(iii) $\alpha^2 + \beta^2 + \gamma^2$. [5]

3. (20marks) [Begin a new 8 page booklet]

(a) Show that $9^{n+2} - 4^n$ is divisible by 5 for all integers $n \geq 1$. [7]

(b) Prove that the sequence 6, 12, 24, 48,.............. is geometric, but that the sequence log 6, log 12, log 24, log 48,.............. is arithmetic.

Find the eighth term of each sequence. [4]

(c) Into a pension fund, a man contributes \$500 annually on each of his birthdays from his 21st to his 64th inclusive. Find the accumulated value of these contributions on his 65th birthday, assuming that the money is invested at 10% per annum compound interest. **[4]**

(d) The first term of an infinite geometric series is 3. Each term is double the sum of all the terms that follow.

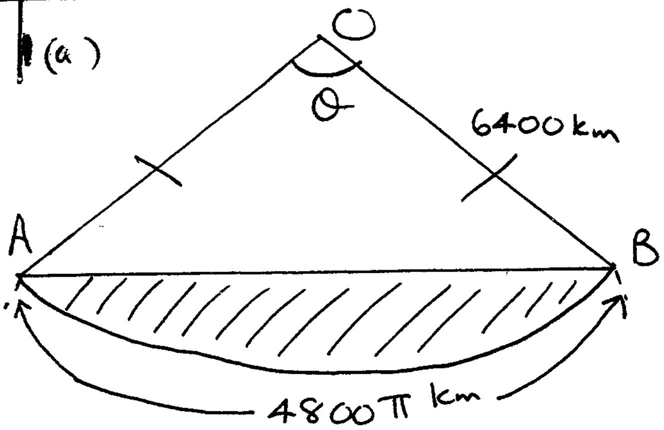
Find :

(i) the common ratio **[4]**

(ii) the sum of this series **[1]**

YR 12 Extension 1 Examination 10-4-02

(a)



(i) $l = r\theta$

$\therefore 4800\pi = 6400\theta \qquad \therefore \theta = \frac{3\pi}{4}$ ✓

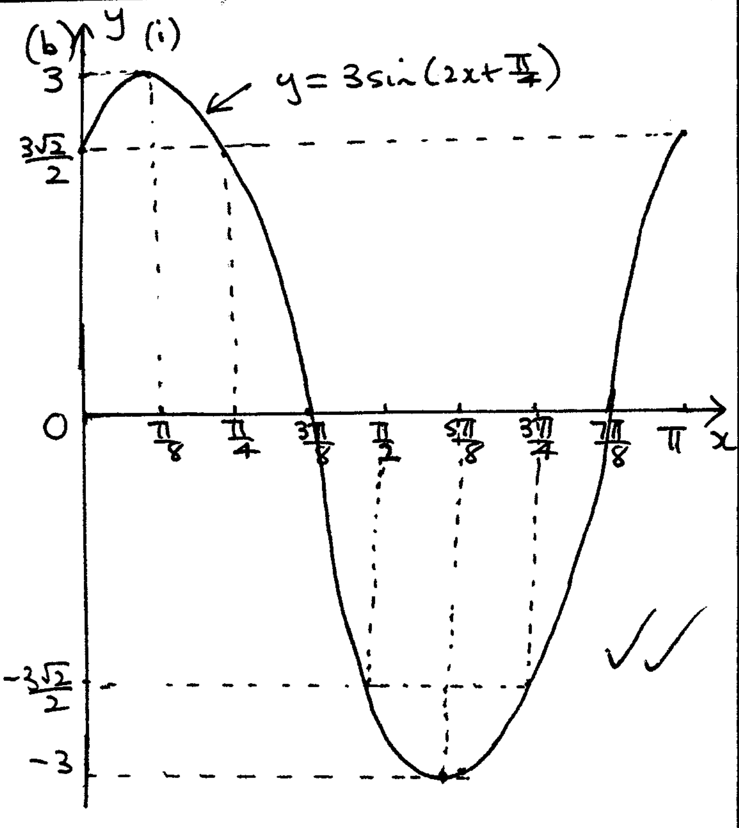
(ii) Area of minor segment shown

$= \frac{1}{2}r^2(\theta - \sin\theta)$

$= \frac{1}{2} \times 6400^2 \left(\frac{3\pi}{4} - \sin\frac{3\pi}{4}\right)$ ✓

$= 3.38 \times 10^7 \text{ km}^2$ (3 sig. fig). ✓

(b) (i)



✓✓

Period $= \frac{2\pi}{2} = \pi$

(ii) Req'd Area $= \left| \int_{\frac{3\pi}{8}}^{\frac{7\pi}{8}} 3\sin\left(2x+\frac{\pi}{4}\right)dx \right|$

$= \frac{3}{2}\left| \left[-\cos\left(2x+\frac{\pi}{4}\right)\right]_{\frac{3\pi}{8}}^{\frac{7\pi}{8}} \right|$ ✓

$= \frac{3}{2}\left|\cos 2\pi - \cos\pi\right|$ ✓

$= \frac{3}{2}\left|1 - -1\right|$

$= 3 \text{ units}^2$ ✓

(c) $\lim_{\theta \to 0} \frac{3\sin^2\theta + 15\sin\theta}{\theta\sin\theta + 5\theta}$

$= \lim_{\theta \to 0} \frac{3\sin\theta\cancel{(\sin\theta+5)}}{\theta\cancel{(\sin\theta+5)}}$ ✓

$= 3 \lim_{\theta \to 0} \frac{\sin\theta}{\theta}$ ✓

$= 3 . 1$

$= 3$ ✓

(d) $y = e^{\tan 2x}$

$\therefore \frac{dy}{dx} = e^{\tan 2x} . 2\sec^2 2x$ ✓

At $(0,1)$ $\frac{dy}{dx} = e^0 . 2 = 2 = m_{\text{tangent}}$ ✓

$\therefore$ Eqn of req'd tangent is: $y - 1 = 2(x - 0)$

$\therefore \; 2x - y + 1 = 0$ ✓

(e) $I = \int_0^{\frac{\pi}{2}} \sin^3 x \cos^2 x \, dx$ ✓

$= \int_0^{\frac{\pi}{2}} \sin x(1-\cos^2 x)\cos^2 x \, dx$

let $u = \cos x$ when $x = 0$ $u = 1$

$\therefore \frac{du}{dx} = -\sin x$ $\quad x = \frac{\pi}{2}$ $u = 0$

$\therefore I = \int_1^0 (1-u^2)u^2 . -du$ ✓

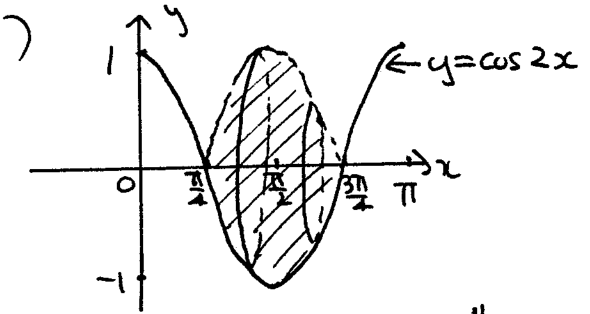
$= \int_0^1 u^2 - u^4 \, du$

$= \left[\frac{u^3}{3} - \frac{u^5}{5}\right]_0^1$ ✓

$= \left[\left(\frac{1}{3} - \frac{1}{5}\right) - 0\right]$

$= \frac{2}{15}$ ✓

(f)



Volume $= \pi \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} y^2 \, dx$ ✓

$= \pi \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos^2 2x \, dx$

$\cos 2x = 2\cos^2 x - 1$
$\therefore \cos 4x = 2\cos^2 2x -$
$\therefore \cos^2 2x = \frac{1}{2}[1+$

$= \frac{\pi}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 1 + \cos 4x \, dx$ ✓

$= \frac{\pi}{2}\left[x + \frac{\sin 4x}{4}\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$

$= \frac{\pi}{2}\left[\left(\frac{3\pi}{4} + 0\right) - \left(\frac{\pi}{4} + 0\right)\right]$ ✓

$= \frac{\pi^2}{4} \text{ units}^3$

ii $= (5x^3+8x^2+21x+39)(x-2) + 82$ ✓

$$
\begin{array}{l}
x-2\ )\ 5x^4-2x^3+5x^2-3x+4 \\
\quad 5x^4-10x^3 \\
\qquad 8x^3+5x^2 \\
\qquad 8x^3-16x^2 \\
\qquad\quad 21x^2-3x \\
\qquad\quad 21x^2-42x \\
\qquad\qquad 39x+4 \\
\qquad\qquad 39x-78 \\
\qquad\qquad\quad 82
\end{array}
$$

(4)

b $P(-3) = 7(-3)^4 - 5(-3)^2 + 11(-3) - 2$ ✓
$= 487$ ✓ (2)

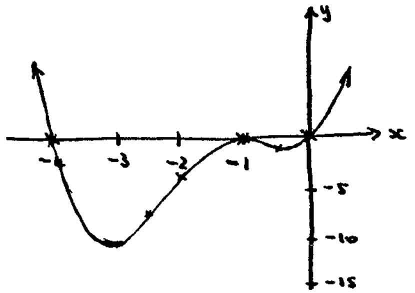
c i $P(x) = x^4 + 6x^3 + 9x^2 + 4x$
$= x(x^3 + 6x^2 + 9x + 4)$ ✓
$= x(x+1)(x^2+5x+4)$ ✓
$= x(x+1)(x+4)(x+1)$ ✓

For $P(x) = 0$
$x = 0, -1, -1, -4$

$R(x) = x^3+6x^2+9x+4$
$R(1) = 1+6+9+4$
$R(-1) = -1+6-9+4 = 0$
$\therefore x+1$ factor

$$
\begin{array}{l}
\qquad\qquad x^2+5x+4 \\
x+1\ )\ x^3+6x^2+9x+4 \\
\qquad x^3+x^2 \\
\qquad\quad 5x^2+9x \\
\qquad\quad 5x^2+5x \\
\qquad\qquad 4x+4
\end{array}
$$

ii $P(0) = 0$ $P(-2) = -4$ $P(-3) = -12$ $P(-2.5) \doteqdot -8$ $P(-.5) \doteqdot -1$



✓✓ (5)

d $P(x) = ax^3 - bx^2 + cx - 8$
$P(2) = 8a - 4b + 2c - 8 = 0 \to 4a - 2b + c = 4$ — ①
$P(-1) = -a - b - c - 8 = 0 \to a + b + c = -8$ — ②
$P(3) = 27a - 9b + 3c - 8 = 28 \to 9a - 3b + c = 12$ ✓ — ③

① − ② $3a - 3b = 12$ — ④
③ − ② $8a - 4b = 20$ — ⑤

④ × 4: $12a - 12b = 48$ — ⑥
⑤ × 3: $24a - 12b = 60$ — ⑦
⑦ − ⑥: $12a = 12$
$a = 1$

In ④ $3 - 3b = 12$
$-3b = 9$ ✓
$b = -3$

In ① $4 + 6 + c = 4$
$c = -6$

check in ②. $a + b + c = 1 - 3 - 6 = -8 =$ RHS. ✓

(4)

$\therefore a = 1, b = -3$ and $c = -6$ ✓

i $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = -\frac{1}{2}$

ii $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$ ✓ $= \frac{-\frac{1}{2}}{-\frac{6}{2}} = -\frac{1}{2} \times -\frac{2}{6} = \frac{1}{6}$ ✓

iii $\alpha^2 + \beta^2 + \gamma^2 \Rightarrow (\alpha+\beta+\gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \alpha\gamma)$ ✓
$\therefore \alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$
$= \left(-\frac{8}{2}\right)^2 - 2 \times \left(-\frac{1}{2}\right)$
$= 16 + 1 = 17$ ✓

(5)

2d Errors with signs caused difficulties. Also $P(3) = 28$ not 0 as taken by a significant n

Sloppy, careless writing and setting out caused errors also

2cii The failure to recognise that $x = -1$ is a double point caused problems.

$\therefore$ Prove true for $n=1$

$9^3 - 4^1 = 725$

$= 5(145)$

ie divisible by 5 ✓

$\therefore$ true for $n=1$

assume true for $n=k$

ie assume

$9^{k+2} - 4^k = 5M$ ✓

where M is an integer.

ie $4^k = 9^{k+2} - 5M$

Prove true for $n = k+1$

ie $9^{k+1+2} - 4^{k+1} = 5Q$ ✓

Q an integer

$LHS = 9^{k+3} - 4^1 \times 4^k$ ✓

$= 9^{k+3} - 4(9^{k+2} - 5M)$

using assumption

$= 9^1(9^{k+2}) - 4(9^{k+2}) + 20M$ ✓

$= 5(9^{k+2}) + 20M$

$= 5(9^{k+2} + 4M)$ ✓

ie divisible by 5

$\therefore$ If true for $n=k$ it is true for $n=k+1$

But it is true for $n=1$

$\therefore$ it is true for $n=1+1=2$

& since true for $n=2$ it is true for $n=2+1=3$ ✓

& so on

$\therefore$ Result is proven by Mathematical Induction.

d) $T_1 = 2\left(\frac{a}{1-r} - a\right)$ ✓ ✓

each term is twice S_∞ of all that follow ie all except a ✓

$\therefore 3 = 2\left(\frac{3}{1-r} - 3\right)$ ✓

$\therefore r = \frac{1}{3}$

✓ = 1 MARK (b)

$\frac{12}{6} = \frac{24}{12} = \frac{48}{24} = 2$

$\therefore$ GP, $r=2$ ✓

$\log 12 - \log 6 = \log \frac{12}{6} = \log 2$

$\log 24 - \log 12 = \log \frac{24}{12} = \log 2$

$\log 48 - \log 24 = \log \frac{48}{24} = \log 2$

$\therefore$ AP $d = \log 2$ ✓

$T_8 = ar^7$

$= 6 \times 2^7$

$= 768$ ✓

$T_8 = a + (n-1)d$

$= \log 6 + 7\log 2$

$= \log 6 + \log 2^7$

$= \log(6 \times 2^7)$

$= \log 768$ ✓

c) Let A_n = amount to which nth contribution accrues

$A_1 = 500 \times 1.1^{44}$

$A_2 = 500 \times 1.1^{43}$

$A_3 = 500 \times 1.1^{42}$ ✓

$\vdots$

$A_{44} = 500 \times 1.1^1$

$Total = 500[1.1^1 + \dots 1.1^{44}]$ ✓

$= 500\left(\frac{1.1(1.1^{44}-1)}{1.1-1}\right)$ ✓

$= \$358952.42$ ✓

$S = \frac{a}{1-r}, |r|<1$

$= \frac{3}{1-\frac{1}{3}} = 4\frac{1}{2}$ ✓