CRANBROOK SCHOOL

YEAR 12 MATHEMATICS – EXTENSION 2

Term 1 2004 **Time : 1.5 h / SKB**

All questions are of equal value.
All necessary working should be shown in every question.
Full marks may not be awarded if work is careless or badly arranged.
Approved silent calculators may be used.
Submit your work in four 4 Page Booklets.

1. (15 marks) (Begin a 4 page booklet.) **SKB**

(a) If $z = 4 - 2i$ and $w = 2 + i$ express in the form $x + iy$

(i) $\bar{z}w$ [1]

(ii) $\dfrac{3i}{w}$ [1]

(b) On an Argand diagram shade in the region **R** containing all points representing complex numbers z such that

$|z - 1 + i| < 2$ and $-\dfrac{\pi}{4} \le \arg(z - 1 + i) \le \dfrac{3\pi}{4}$. [3]

(c) Let $\alpha = \sqrt{3} - i$.

(i) Express α in modulus-argument form. [1]

(ii) Show that α is a root of the equation $z^4 - 4z^2 + 16 = 0$. [2]

(iii) Hence find a real quadratic factor of $P(z) = z^4 - 4z^2 + 16$. [2]

(d) By applying de Moivre's Theorem and by also expanding

$(\cos\theta + i\sin\theta)^5$, express $\sin 5\theta$ as a polynomial in $\sin\theta$. [3]

(e) If $\sqrt{5 - 12i} = x + iy$ where x and y are Real and $x > 0$ find x and y. [2]

2. **(15 Marks) (Begin a new 4 page booklet.)** **SKB**

(a) If $P(x)=3x^3-5x^2+12x-20$ factorise $P(x)$ over the complex field C. [2]

(b) Find the values of a, b and c if $(x-1)^3$ is a factor of
$P(x)=x^5+ax^4+bx^3+3x+c$. [3]

(c) When the polynomial $P(x)$ is divided by $x, x+1$ and $x-3$ the respective remainders are 2, 4 and –3. Determine what the remainder will be when $P(x)$ is divided by $x(x+1)(x-3)$. [3]

(d) Express $\dfrac{2x^3+12x^2+28x+26}{(x+1)^2(x+3)}$ in terms of partial fractions over the real number field. [4]

(e) The equation $x^3-x^2-3x+2=0$ has roots α, β and γ. Find the monic polynomial equation with the roots α^2, β^2 and γ^2. [3]

3. **(15 marks) (Begin a new 4 page booklet.)** **skb**

(a) **(i)** If P is any point on the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ and S and S' are the foci prove that $|PS'|-|PS|=2a$. [3]

(ii) Hence or otherwise find PS' and the eccentricity for the hyperbola $\dfrac{x^2}{9}-\dfrac{y^2}{4}=1$ if $PS=5$. [2]

(b) For the ellipse $9x^2+16y^2=144$ find:

(i) the co-ordinates of the foci [2]

(ii) the equations of the directrices [2]

(c) **(i)** Show that the equation of the tangent at $P(4\sec\theta, \frac{16}{\sqrt{11}}\tan\theta)$ to the hyperbola $16x^2-11y^2=256$ is given by
$4x\sec\theta-\sqrt{11}y\tan\theta-16=0$. [3]

(ii) Prove that the tangent at P in (i) cannot also be a tangent to the circle $x^2+y^2-4x-60=0$. [3]

4. **(15 Marks) (Begin a new 4 page booklet.)** **skb**

(a) Consider any point $P(z)$ on the hyperbola $x^2 - y^2 = a^2$ to be of the form $z = x + iy$. The hyperbola $x^2 - y^2 = a^2$ is rotated anticlockwise about the origin through $\frac{\pi}{4}$.

(i) Show that any point $P'(z)$ on the rotated hyperbola is given by $P'(z) = \frac{1}{\sqrt{2}}(1+i)P(z)$. **[1]**

(ii) By letting $P'(z) = u + iv$ prove that the rotated hyperbola has equation $xy = \frac{1}{2}a^2$. **[3]**

(iii) Determine the eccentricity of $xy = \frac{1}{2}a^2$. **[1]**

(b) By using the result $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

(i) Deduce that $8x^3 - 6x - 1 = 0$ has the solution $x = \cos\theta$ where $\cos 3\theta = \frac{1}{2}$. [2]

(ii) Find the roots of $8x^3 - 6x - 1 = 0$ in the form $x = \cos\theta$. [2]

(iii) Hence evaluate $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$. [1]

(c) $P(a\sec\theta, b\tan\theta)$ is a variable point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and the foci are at S and S'. Prove that the tangent at P bisects angle SPS'.

[You can assume that the tangent at $P(a\sec\theta, b\tan\theta)$ is given by the equation $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$.] [5]

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

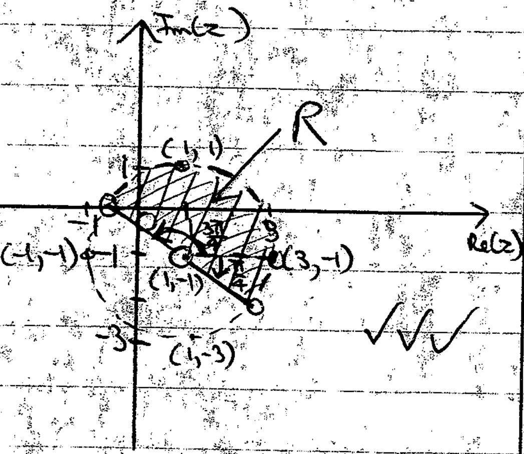
Extension 2 Term 1 2004

1(a) $z = 4-2i$, $w = 2+i$

(i) $\bar{z}w = (4+2i)(2+i)$
$= 8+4i+4i-2$
$= 6+8i$ ✓

(ii) $\frac{3i}{w} = \frac{3i}{2+i} \times \frac{2-i}{2-i}$
$= \frac{6i+3}{5}$
$= \frac{3}{5} + \frac{6}{5}i$ ✓

(b)



(c) $\alpha = \sqrt{3} - i$

(i) $|\alpha| = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$
$\arg \alpha = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = -\frac{\pi}{6}$
$\therefore \alpha = 2\,\text{cis}\left(-\frac{\pi}{6}\right)$ ✓

(ii) Sub $\alpha = 2\,\text{cis}\left(-\frac{\pi}{6}\right)$ into $z^4 - 4z^2 + 16 = 0$
$\therefore LHS = \left(2\,\text{cis}\left(-\frac{\pi}{6}\right)\right)^4 - 4\left(2\,\text{cis}\left(-\frac{\pi}{6}\right)\right)^2 + 16$
$= 16\,\text{cis}\left(-\frac{2\pi}{3}\right) - 16\,\text{cis}\left(-\frac{\pi}{3}\right) + 16$
$= 16\left[\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right] - 16\left[\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right] + 16$
$= 16\left[-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right] - 16\left[\frac{1}{2} - i\frac{\sqrt{3}}{2}\right] + 16$
$= -8 - 8 + 16 - 8i\sqrt{3} + 8i\sqrt{3}$ ✓✓
$= 0 = RHS$

$\Rightarrow \alpha$ is a root of $z^4 - 4z^2 + 16 = 0$

(iii) Let $P(z) = z^4 - 4z^2 + 16$
As $\alpha = \sqrt{3} - i$ is a root and the coefficients of $P(z)$ are real $\Rightarrow$ $\bar{\alpha} = \sqrt{3} + i$ is also a root.
$\therefore P(z) = ([z-\sqrt{3}]+i)([z-\sqrt{3}]-i)\,Q(z)$
$= ((z-\sqrt{3})^2 + 1)\,Q(z)$
$= (z^2 - 2\sqrt{3}z + 4)\,Q(z)$
$\Rightarrow$ a real quadratic factor is $z^2 - 2\sqrt{3}z + 4$ ✓

(d) By de Moivre's Theorem:
$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$
$\therefore LHS = (\cos\theta + i\sin\theta)^5$
$= \cos^5\theta + 5\cos^4\theta(i\sin\theta) + 10\cos^3\theta(i\sin\theta)^2 + 10\cos^2\theta(i\sin\theta)^3 + 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5$ ✓

Equating corr. imaginary terms of LHS & RHS
$\sin 5\theta = 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$
$= 5(1 - 2\sin^2\theta + \sin^4\theta)\sin\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$
$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

(e) $\sqrt{5-12i} = x + iy$ $(x, y \in R,\ x > 0)$
$\therefore 5 - 12i = x^2 - y^2 + i2xy$
equating real and imag. terms: ✓
$5 = x^2 - y^2$ —①
$-6 = xy$ —② from ② $y = \frac{-6}{x}$ sub in ①
$\therefore 5 = x^2 - \frac{36}{x^2}$ $\therefore x^4 - 5x^2 - 36 = 0$
$\therefore (x^2 - 9)(x^2 + 4) = 0$

$\therefore x = 3 \quad (x > 0,\ x \in \mathbb{R})$

$\therefore y = -2$

$\therefore (x, y) = (3, -2).$

2 (a) $P(x) = 3x^3 - 5x^2 + 12x - 20$

$= 3x(x^2+4) - 5(x^2+4)$

$= (3x-5)(x^2+4)$

$= (3x-5)(x+2i)(x-2i)$ over $\mathbb{C}$

(b) $P(x) = x^5 + ax^4 + bx^3 + 3x + c$ has $x=1$ as a 3-fold root

$\therefore P'(x) = 5x^4 + 4ax^3 + 3bx^2 + 3$ has $x=1$ as a 2-fold root

and $P''(x) = 20x^3 + 12ax^2 + 6b$ has $x=1$ as a single root

$\Rightarrow P(1) = 1 + a + b + 3 + c = 0$

$\therefore a + b + c = -4$ —(1)

$P'(1) = 5 + 4a + 3b + 3 = 0$

$\therefore 4a + 3b = -8$ —(2)

$P''(1) = 20 + 12a + 6b = 0$

$\therefore 12a + 6b = -20$ —(3)

(2)$\times 2$: $8a + 6b = -16$ —(2A)

(3)$-$(2A): $4a = -4 \quad \therefore a = -1$

sub $a=-1$ into (2) $\therefore -4 + 3b = -8$

$\therefore b = -\frac{4}{3}$

sub $a=-1$, $b=-\frac{4}{3}$ into (1)

$\therefore c = -4 + 2\frac{1}{3} = -\frac{5}{3}$

$\therefore (a, b, c) = \left(-1, -\frac{4}{3}, -\frac{5}{3}\right)$

(c) Since $x(x+1)(x-3)$ is cubic the remainder when dividing $P(x)$ by $x(x+1)(x-3)$ will be quadratic

$\therefore P(x) = x(x+1)(x-3)R(x) + ax^2 + bx + c$

Now $P(0) = 2 = c$ —(1)

$P(-1) = 4 = a - b + c$ —(2)

$P(3) = -3 = 9a + 3b + c$ —(3)

from (1) $c = 2$ sub. into (2) and (3):

$\therefore 4 = a - b + 2 \quad \therefore a - b = 2$ —(4)

and $-3 = 9a + 3b + 2 \quad \therefore 9a + 3b = -5$ —(5)

(4)$\times 3$: $3a - 3b = 6$ —(4A)

(5)$+$(4A): $12a = 1 \quad \therefore a = \frac{1}{12}$

sub $a = \frac{1}{12}$ into (4) $\therefore b = -1\frac{11}{12}$

$\therefore$ Remainder is: $\frac{1}{12}x^2 - \frac{23}{12}x + 2$

(d) $\dfrac{2x^3+12x^2+28x+26}{(x+1)^2(x+3)} = 2 + \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x+3}$

$\therefore 2x^3 + 12x^2 + 28x + 26 = 2(x+1)^2(x+3) + A(x+1)(x+3) + B(x+3) + C(x+1)^2$

let $x = -1 \quad \therefore -2 + 12 - 28 + 26 = 2B$

$\therefore B = 4$

$x = -3 \quad \therefore 2(-27) + 108 - 84 + 26 = 4C$

$\therefore C = -1$

$x = 0 \quad \therefore 26 = 6 + 3A + 3B + C$

$\therefore 20 = 3A + 12 - 1$

$\therefore A = 3$

$\therefore \dfrac{2x^3+12x^2+28x+26}{(x+1)^2(x+3)} = 2 + \dfrac{3}{x+1} + \dfrac{4}{(x+1)^2} - \dfrac{1}{x+3}$

in terms of partial fractions.

(e) $P(x) = x^3 - x^2 - 3x + 2$ has roots α, β, γ. For roots $\alpha^2, \beta^2, \gamma^2$

let $y = x^2 \quad \therefore x = \sqrt{y}$ sub into $P(x)$ eqn

$\therefore (\sqrt{y})^3 - (\sqrt{y})^2 - 3(\sqrt{y}) + 2 = 0$

$\therefore y^{3/2} - 3y^{1/2} = y - 2$

$\therefore \left(y^{3/2} - 3y^{1/2}\right)^2 = (y-2)^2$

$\therefore y^3 - 6y^2 + 9y = y^2 - 4y + 4$

$\therefore y^3 - 7y^2 + 13y - 4 = 0$

i.e. $x^3 - 7x^2 + 13x - 4 = 0$ is the req'd monic equation when reverting to the variable x.

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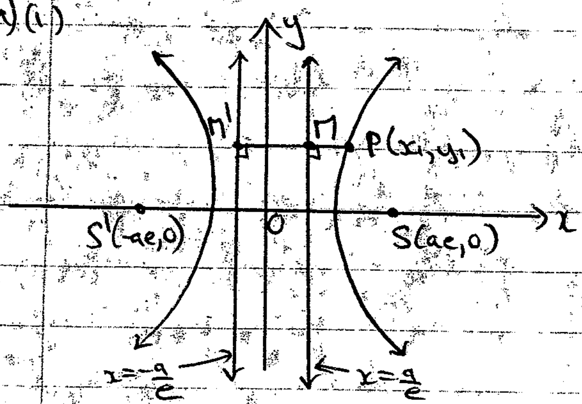
3) (a) (i)



From the definition of a hyperbola

$$\frac{|PS|}{|PM|} = e = \frac{|PS'|}{|PM'|}$$ where M, M' are as shown.

Now $|PS'| - |PS| = e|PM'| - e|PM|$

$= e\left|x_1 + \frac{a}{e}\right| - e\left|x_1 - \frac{a}{e}\right|$

$= ex_1 + a - ex_1 + a$

$= 2a.$

(ii) For $\frac{x^2}{9} - \frac{y^2}{4} = 1$

$a = 3,\ b = 2$

Now $b^2 = a^2(e^2 - 1)$

$\therefore 4 = 9(e^2 - 1)$

$\therefore e^2 = 1\frac{4}{9}$

$\therefore$ eccentricity, $e = \frac{\sqrt{13}}{3}$ $(e > 1)$

Now as $|PS'| - |PS| = 2a$

$\therefore$ if $PS = 5$, $a = 3$

$\therefore |PS'| - 5 = 6$ $\quad \therefore |PS'| = 11$

$\therefore PS' = 11$ $\quad (PS' > 0)$

(b) $9x^2 + 16y^2 = 144$

$\therefore \frac{x^2}{16} + \frac{y^2}{9} = 1$

$\therefore a = 4,\ b = 3$

$b^2 = a^2(1-e$

$\therefore 9 = 16(1-e$

$\therefore e = \frac{\sqrt{7}}{4}$

as $0 < e <$

(i) Foci are: $(\pm ae, 0)$

$= (\pm\sqrt{7}, 0)$

(ii) Directrices are: $x = \pm\frac{a}{e}$

$\therefore x = \pm\frac{4}{\sqrt{7}/4}$

$\therefore x = \frac{\pm 16}{\sqrt{7}}$

(c) (i) $16x^2 - 11y^2 = 256$

$\therefore 32x - 22y\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{16x}{11y}$

At $P\left(4\sec\theta, \frac{16\tan\theta}{\sqrt{11}}\right)$ $\frac{dy}{dx} = \frac{64\sec\theta}{\frac{176}{\sqrt{11}}\tan\theta}$

$\therefore \frac{dy}{dx} = \frac{4\sqrt{11}}{11\sin\theta}$

$= \frac{4}{\sqrt{11}\sin\theta}$

$= m_{\text{tangent}}$

$\therefore$ Eqn of req'd tangent is:

$y - \frac{16\tan\theta}{\sqrt{11}} = \frac{4}{\sqrt{11}\sin\theta}(x - 4\sec\theta)$

$\therefore \sqrt{11}\sin\theta\, y - 16\tan\theta\sin\theta = 4x - 16\sec\theta$

$\div\cos\theta$

$\Rightarrow \sqrt{11}\tan\theta\, y - 16\tan^2\theta = 4x\sec\theta - 16\sec^2\theta$

$\therefore 4x\sec\theta - \sqrt{11}y\tan\theta = 16(\sec^2\theta - \tan^2\theta)$

$\therefore 4x\sec\theta - \sqrt{11}y\tan\theta - 16 = 0$

(ii) For circle $x^2+y^2-4x-60=0$

$\therefore (x-2)^2-4+y^2-60=0$

$\therefore (x-2)^2+y^2=64$

ie circle's centre is $(2,0)$ and radius 8 units.

Now if the tangent at P in (i) is also a tangent to this circle then the radius of circle would equal the perp distance from the centre of the circle to the tangent.

$\therefore$ If a tangent:

$$8=\frac{|2\times 4\sec\theta+0-16|}{\sqrt{(4\sec\theta)^2+(-\sqrt{11}\tan\theta)^2}}$$

$$=\frac{|8\sec\theta-16|}{\sqrt{16\sec^2\theta+11\tan^2\theta}}$$

$$\therefore 1=\frac{|\sec\theta-2|}{\sqrt{16\sec^2\theta+11(\sec^2\theta-1)}}$$

$$=\frac{|\sec\theta-2|}{\sqrt{27\sec^2\theta-11}}$$

$\therefore 27\sec^2\theta-11=\sec^2\theta-4\sec\theta+4$

$\therefore 26\sec^2\theta+4\sec\theta-15=0$

$$\therefore \sec\theta=\frac{-4\pm\sqrt{16-4.26.-15}}{52}$$

$$=\frac{-4\pm\sqrt{1576}}{52}$$

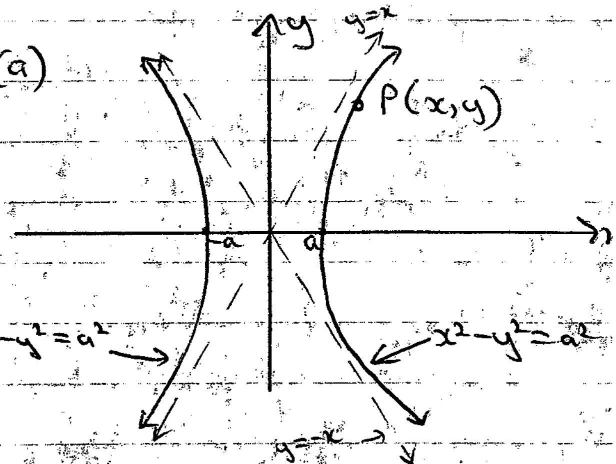
$\doteqdot -0.84$ or 0.69

But $\sec\theta\le -1$ or $\sec\theta\ge 1$

$\Rightarrow$ No real sol'n for θ

$\therefore$ tangent in (i) cannot also be a tangent to the given circle

4 (a)



(i) If the hyperbola $x^2-y^2=a^2$ is rotated about the origin through $\frac{\pi}{4}$ this increases the argument of any point $P(z)$ by $\frac{\pi}{4}$.

$\therefore P'(z)=(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})P(z)$

$=(\frac{1}{\sqrt2}+i\frac{1}{\sqrt2})P(z)$

$=\frac{1}{\sqrt2}(1+i)P(z)$

(ii) Let $P'(z)=u+iv$

$\therefore u+iv=\frac{1}{\sqrt2}(1+i)(x+iy)$

$=\frac{1}{\sqrt2}(x-y+i[x+y])$

$=\frac{x-y}{\sqrt2}+i\frac{(x+y)}{\sqrt2}$

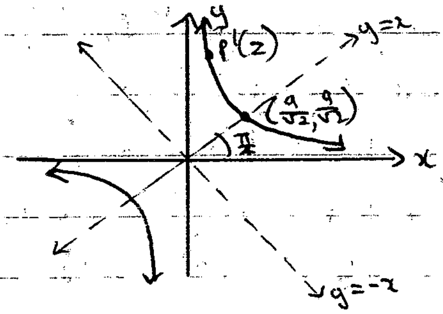
$\therefore u=\frac{x-y}{\sqrt2}$, $v=\frac{x+y}{\sqrt2}$

$\therefore uv=\frac{x^2-y^2}{2}$

But $P(x,y)$ lies on $x^2-y^2=a^2$

$\therefore uv=\frac{a^2}{2}$

ie



(ie $(a,0)\rightarrow(\frac{a}{\sqrt2},\frac{a}{\sqrt2})$)

$\therefore$ Rotated hyperbola has equation $xy=\frac{a^2}{2}$.

(iii) As rotations preserve shape and distances the eccentricity is also preserved.

$\therefore$ For $x^2 - y^2 = a^2$

$$\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$$

As $b^2 = a^2(e^2-1)$ and $a=b$

$\therefore a^2 = a^2(e^2-1)$

$\therefore 1 = e^2 - 1 \quad \therefore e = \sqrt{2} \ (e>1)$

$\therefore$ Eccentricity of $xy = \frac{1}{2}a^2$ is also $\sqrt{2}$. ✓

(b) (i) As $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$\therefore 2\cos 3\theta = 8\cos^3\theta - 6\cos\theta$

$\therefore 2\cos 3\theta - 1 = 8\cos^3\theta - 6\cos\theta - 1$

$\therefore 2\cos 3\theta - 1 = 0$ has the same solutions as $8\cos^3\theta - 6\cos\theta - 1 = 0$

i.e. $8x^3 - 6x - 1 = 0$ has the solution $x = \cos\theta$ where $\cos 3\theta = \frac{1}{2}$. ✓✓

(ii) Now if $\cos 3\theta = \frac{1}{2}$

$\therefore 3\theta = \frac{\pi}{3}, 2\pi - \frac{\pi}{3}, \frac{\pi}{3} + 2\pi, 4\pi - \frac{\pi}{3}, \frac{\pi}{3} + 4\pi, 6\pi - \frac{\pi}{3}$ for $0 \le 3\theta < 6\pi$

$\therefore 3\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \frac{17\pi}{3}$

$\therefore \theta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{17\pi}{9}$

$\therefore x = \cos\frac{\pi}{9}, \cos\frac{5\pi}{9}, \cos\frac{7\pi}{9}, \cos\frac{11\pi}{9}, \cos\frac{13\pi}{9}, \cos\frac{17\pi}{9}$.

$\therefore x = \cos\frac{\pi}{9}, \cos\left(\pi - \frac{4\pi}{9}\right), \cos\left(\pi - \frac{2\pi}{9}\right), \cos\left(\pi + \frac{2\pi}{9}\right), \cos\left(\pi + \frac{4\pi}{9}\right), \cos\left(2\pi - \frac{\pi}{9}\right)$

$\therefore x = \cos\frac{\pi}{9}, -\cos\frac{4\pi}{9}, -\cos\frac{2\pi}{9}, -\cos\frac{2\pi}{9}, -\cos\frac{4\pi}{9}, \cos\frac{\pi}{9}$

i.e. $x = \cos\frac{\pi}{9}, -\cos\frac{2\pi}{9}, -\cos\frac{4\pi}{9}$ are all the roots of $8x^3 - 6x - 1 = 0$. ✓✓

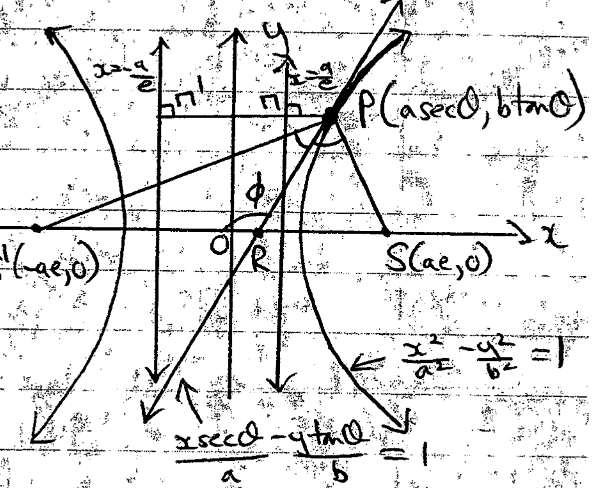
(iii) Now product of roots of $8x^3 - 6x - 1 = 0$

$= -\frac{-1}{8}$

$\therefore \cos\frac{\pi}{9} \cdot -\cos\frac{2\pi}{9} \cdot -\cos\frac{4\pi}{9} = \frac{1}{8}$

$\therefore \cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{8}$. ✓

(c)



TO PROVE: The tangent at P bisects $\angle SPS'$.

SOL'N: Let tangent at P $\left(\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1\right)$ intersect x-axis at R.

$\therefore y = 0 \quad \therefore x = \frac{a}{\sec\theta} \quad \therefore R = \left(\frac{a}{\sec\theta}, 0\right)$

Now from definition of hyperbola: ✓

$$\frac{|PS|}{|PM|} = e = \frac{|PS'|}{|PM'|}$$

$\therefore |PS| = e|PM| = e\left|a\sec\theta - \frac{a}{e}\right| = a(e\sec\theta - 1)$

and $|PS'| = e|PM'| = e\left|a\sec\theta + \frac{a}{e}\right| = a(e\sec\theta + 1)$

Now $RS = ae - \frac{a}{\sec\theta} = a\left(e - \frac{1}{\sec\theta}\right) = a\left(\frac{e\sec\theta - 1}{\sec\theta}\right)$

$RS' = \frac{a}{\sec\theta} + ae = a\left(\frac{e\sec\theta + 1}{\sec\theta}\right)$ ✓

Let $\angle PRS' = \phi$

$\therefore$ In $\Delta PRS'$ by the Sine Rule:

$$\frac{PS'}{\sin\phi} = \frac{RS'}{\sin\angle RPS'}$$

$$\therefore \frac{\cancel{a(e\sec\theta+1)}}{\sin\phi} = \frac{\cancel{a(e\sec\theta+1)}}{\sec\theta\,.\sin\angle RPS'}$$

$$\therefore \sin\phi = \sec\theta\,\sin\angle RPS'$$

$$\therefore \sin\angle RPS' = \frac{\sin\phi}{\sec\theta} \quad \text{— (1)}$$

Similarly by the sine rule in ΔPRS:

$$\frac{PS}{\sin(\pi-\phi)} = \frac{RS}{\sin\angle RPS}$$

$$\therefore \frac{\cancel{a(e\sec\theta-1)}}{\sin\phi} = \frac{\cancel{a(e\sec\theta-1)}}{\sec\theta\,\sin\angle RPS}$$

$$\therefore \sin\angle RPS = \frac{\sin\phi}{\sec\theta} \quad \text{— (2)}$$

But (1) = (2) $\therefore \sin\angle RPS = \sin\angle RPS'$

ie $\angle RPS = \angle RPS'$

$\Rightarrow$ the tangent at P bisects $\angle SPS'$.

Alternative Solutions:

1 (e) (ii) Sub $\alpha = \sqrt{3} - i$ into $z^4 - 4z^2 + 16 = 0$

$\therefore$ LHS $= (\sqrt{3} - i)^4 - 4(\sqrt{3} - i)^2 + 16$

$= (\sqrt{3})^4 + 4(\sqrt{3})^3(-i) + 6(\sqrt{3})^2(-i)^2$
$+ 4\sqrt{3}(-i)^3 + (-i)^4 - 4(3 - 2\sqrt{3}i - 1) + 16$

$= 9 - 12\sqrt{3}i - 18 + 4\sqrt{3}i + 1 - 12 + 8\sqrt{3}i$
$+ 4 + 16$

$= 9 - 18 + 1 - 12 + 4 + 16$

$= 0$

$=$ RHS

$\Rightarrow \alpha$ is a root of $z^4 - 4z^2 + 16 = 0$

or Sub $\alpha = \sqrt{3} - i$ into $z^4 - 4z^2 + 16 = 0$

$\therefore$ LHS $= (\sqrt{3} - i)^4 - 4(\sqrt{3} - i)^2 + 16$

$= ((\sqrt{3} - i)^2)^2 - 4(3 - 2\sqrt{3}i - 1) + 16$

$= (2 - 2\sqrt{3}i)^2 - 4(2 - 2\sqrt{3}i) + 16$

$= 4 - 8\sqrt{3}i - 12 - 8 + 8\sqrt{3}i + 16$

$= 0$

$=$ RHS

$\Rightarrow \alpha$ is a root of $z^4 - 4z^2 + 16 = 0$.

2 (a) $P(x) = 3x^3 - 5x^2 + 12x - 20$

Let $x = \frac{r}{s}$ be a rational root

where $s \mid 3$ $(s > 0)$, $r \mid -20$

$\therefore \frac{r}{s} = \frac{\pm 1}{1}, \frac{\pm 2}{1}, \frac{\pm 4}{1}, \frac{\pm 5}{1}, \frac{\pm 10}{1}, \frac{\pm 20}{1}$

or $\frac{\pm 1}{3}, \frac{\pm 2}{3}, \frac{\pm 4}{3}, \frac{\pm 5}{3}, \frac{\pm 10}{3}, \frac{\pm 20}{3}$

Let $x = \frac{5}{3}$ Now $P\left(\frac{5}{3}\right) = 0$

$\Rightarrow 3x - 5$ is a factor of $P(x)$

$\therefore P(x) = (3x - 5)(x^2 + 4)$ (by inspection)

$= (3x - 5)(x - 2i)(x + 2i)$ over $\mathbb{C}$.

2(e) As $x^3 - x^2 - 3x + 2 = 0$

has roots α, β, γ

$\therefore \alpha + \beta + \gamma = 1$ —— (1)

$\alpha\beta + \alpha\gamma + \beta\gamma = -3$ —— (2)

$\alpha\beta\gamma = -2$ —— (3)

If roots are $\alpha^2, \beta^2, \gamma^2$:

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= 1^2 - 2(-3)$

$= 7$

$\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 = (\alpha\beta + \alpha\gamma + \beta\gamma)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)$

$= (-3)^2 - 2(-2)(1)$

$= 13$

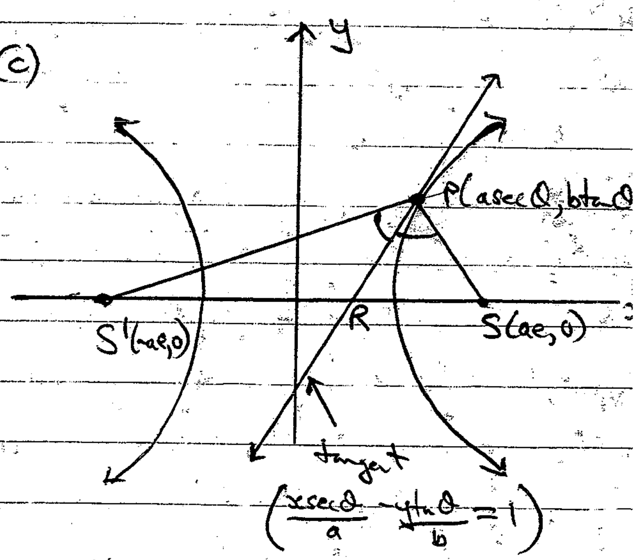
$\alpha^2\beta^2\gamma^2 = (\alpha\beta\gamma)^2$

$= (-2)^2$

$= 4$

$\Rightarrow$ monic polynomial eq'n with roots $\alpha^2, \beta^2, \gamma^2$ is: $x^3 - 7x^2 + 13x - 4 = 0$

4 (c)



From geometry of figure the tangent at P will make two acute angles S'PR and SPR.

Now $m_{PS} = \dfrac{b\tan\theta}{a(\sec\theta - e)}$

$m_{PS'} = \dfrac{b\tan\theta}{a(\sec\theta + e)}$

$m_{PR} = \dfrac{b\sec\theta}{a\tan\theta}$, $b^2 = a^2(e^2-1)$

Now $\tan\angle SPR = \left|\dfrac{m_{PS} - m_{PR}}{1 + m_{PS}\cdot m_{PR}}\right|$

$$= \left|\frac{\dfrac{b\tan\theta}{a(\sec\theta - e)} - \dfrac{b\sec\theta}{a\tan\theta}}{1 + \dfrac{b\tan\theta}{a(\sec\theta - e)}\cdot\dfrac{b\sec\theta}{a\tan\theta}}\right|$$

$$= \left|\frac{ab\tan^2\theta - ab\sec\theta(\sec\theta - e)}{a^2(\sec\theta - e)\tan\theta + b^2\tan\theta\sec\theta}\right|$$

$$= \left|\frac{ab(\tan^2\theta - \sec^2\theta) + abe\sec\theta}{\sec\theta\tan\theta(a^2+b^2) - a^2e\tan\theta}\right|$$

$$= \left|\frac{-ab + abe\sec\theta}{\dfrac{\sin\theta}{\cos^2\theta}\left(a^2 + a^2(e^2-1)\right) - a^2e\tan\theta}\right|$$

$$= \left|\frac{ab\left(\dfrac{e}{\cos\theta} - 1\right)}{\dfrac{\sin\theta}{\cos^2\theta}(a^2e^2) - a^2e\dfrac{\sin\theta}{\cos\theta}}\right|$$

$$= \left|\frac{ab\cos\theta(e - \cos\theta)}{\sin\theta(a^2e^2) - a^2e\sin\theta\cos\theta}\right|$$

$$= \left|\frac{ab\cos\theta(e - \cos\theta)}{a^2e\sin\theta(e - \cos\theta)}\right|$$

$$= \frac{b}{ae\tan\theta}$$

Similarly $\tan\angle S'PR = \left|\dfrac{m_{PS'} - m_{PR}}{1 + m_{PS'}\cdot m_{PR}}\right|$

$$\therefore \tan\angle S'PR = \left|\frac{\dfrac{b\tan\theta}{a(\sec\theta + e)} - \dfrac{b\sec\theta}{a\tan\theta}}{1 + \dfrac{b\tan\theta}{a(\sec\theta + e)}\cdot\dfrac{b\sec\theta}{a\tan\theta}}\right|$$

$$= \left|\frac{ab(\tan^2\theta - \sec^2\theta) - abe\sec\theta}{\sec\theta\tan\theta(a^2+b^2) + a^2e\tan\theta}\right|$$

$$= \left|\frac{-ab - abe\sec\theta}{\dfrac{\sin\theta}{\cos^2\theta}(a^2e^2) + a^2e\tan\theta}\right|$$

$$= \frac{ab\left(1 + \dfrac{e}{\cos\theta}\right)}{\dfrac{\sin\theta}{\cos^2\theta}(a^2e^2) + a^2e\dfrac{\sin\theta}{\cos\theta}}$$

$$= \frac{ab\cos\theta(\cos\theta + e)}{\sin\theta(a^2e^2) + a^2e\sin\theta\cos\theta}$$

$$= \frac{ab\cos\theta(\cos\theta + e)}{a^2e\sin\theta(e + \cos\theta)}$$

$$= \frac{b}{ae\tan\theta}$$

$$= \tan\angle SPR$$

$\Rightarrow$ tangent at P bisects $\angle SPS'$.

3 c (ii) If the tangent at P in (i) $[4x\sec\theta - \sqrt{11}y\tan\theta - 16 = 0$ is also a tangent to the circle $x^2+y^2-4x-60=0$ then through simultaneous equations the resultant quadratic equation's discriminant will be zero,

ie $x^2+y^2-4x-60=0$

$\therefore y = \pm\sqrt{-x^2+4x+60}$ —— ①

$4x\sec\theta - \sqrt{11}y\tan\theta - 16 = 0$ —— ②

sub ① into ②: $4x\sec\theta \mp \left(\sqrt{-11x^2+44x+660}\right)\tan\theta - 16 = 0$

$\therefore 4x\sec\theta - 16 = \pm\sqrt{-11x^2+44x+660}\,\tan\theta$

$\therefore 16x^2\sec^2\theta - 128x\sec\theta + 256 = -11x^2\tan^2\theta + 44x\tan^2\theta + 660\tan^2\theta$

$\therefore (16\sec^2\theta + 11\tan^2\theta)x^2 - (128\sec\theta + 44\tan^2\theta)x + (256 - 660\tan^2\theta) =$

For unique solution $\Delta = 0$

$\therefore 16384\sec^2\theta + 11264\sec\theta\tan^2\theta + 1936\tan^4\theta$

$- 4(16\sec^2\theta + 11\tan^2\theta)(256 - 660\tan^2\theta) = 0$

$\therefore 30976\tan^4\theta + 11264\sec\theta\tan^2\theta - 11264\tan^2\theta + 42240\sec^2\theta\tan^2\theta = ($

(bs $\div$ 2816) $\therefore 11\tan^4\theta + 4\sec\theta\tan^2\theta - 4\tan^2\theta + 15\sec^2\theta\tan^2\theta = 0$

$\therefore \tan^2\theta\,(11\tan^2\theta + 4\sec\theta - 4 + 15\sec^2\theta) = 0$

$\therefore \tan^2\theta = 0$ or $11(\sec^2\theta - 1) + 4\sec\theta - 4 + 15\sec^2\theta = 0$

or $26\sec^2\theta + 4\sec\theta - 15 = 0$

$\theta = 0, \pi$ $\quad\quad \therefore \sec\theta = \dfrac{-4 \pm \sqrt{1576}}{52}$

But if $\theta = 0$ tangent is $x=4$

and if $\theta = \pi$ tangent is $x=-4$.

But these tangents would intersect the circle at $(4, \pm 2\sqrt{5})$ or $(-4, \pm 2\sqrt{7})$

$\therefore$ Not a unique solution

$\doteqdot -0.84$ or 0.69

But as $\sec\theta \le -1$ or $\sec\theta \geq 1$

$\Rightarrow$ No real solution for θ

$\Rightarrow$ the tangent in (1) cannot also be a tangent to the given circle.