

CRANBROOK
MATHEMATICS EXTENSION 1

2006

HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

General Instructions

- Reading time – 5 minutes
- Writing time – 2 hours
- All seven questions should be attempted
- Total marks available – 84
- All questions are worth 12 marks
- An approved calculator may be used
- All relevant working should be shown for each question
- Complete your work in 7 separate 4 page booklets

Question 1 (12 marks)	Marks
(a) Evaluate $\lim_{x \rightarrow 0} \frac{2 \sin 2x}{x}$.	2
(b) The polynomial $P(x) = 2x^3 + ax^2 + x + 2$ has a factor $(2x + 1)$. Find the value of a .	2
(c) Solve the inequality $\frac{1}{x-2} \geq 2$.	2
(d) Using the substitution $u = x + 2$, find $\int \frac{x}{3} \sqrt{x+2} \, dx$.	2
(e) Divide the interval AB externally in the ratio $4:3$, where A is the point $(2, -1)$ and B is $(1, -3)$.	2
(f) Find the obtuse angle between the lines $3x - y + 5 = 0$ and $2x + 3y - 1 = 0$. Give your answer correct to the nearest degree.	2

Question 2 (12 marks)

Marks

(a) Differentiate $\cos^{-1}(\sin x)$.

2

(b) Find:

(i) $\int \frac{2}{\sqrt{1-9x^2}} dx$

2

(ii) $\int \sin^2 \frac{x}{2} dx$.

2

(c) If $\log_{16} 2 = \log_x 15$, find x .

2

(d) Consider the function $f(x) = \frac{1}{2} \cos^{-1}(1-3x)$.

(i) State the domain and range of $f(x)$.

2

(ii) Hence sketch the graph of $y = f(x)$.

2

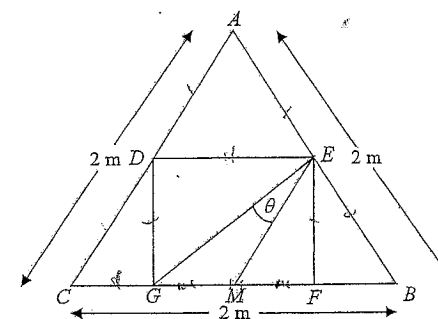
Question 3 (12 marks)

Marks

(a) Use the definition $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ to find $f'(x)$ where $f(x) = x^2 - 2x$.

2

(b)



NOT TO SCALE

The equilateral triangle ABC has side lengths of 2 metres. The points D and E are the mid-points of AC and AB respectively and GF lies on CB . Also, $DEFG$ is a rectangle. Point M is the midpoint of GF . $\angle GEM = \theta$.

(i) Show that $GE = \frac{\sqrt{7}}{2}$ metres.

1

(ii) Show that $\cos \theta = \frac{5\sqrt{7}}{14}$.

2

(iii) Hence show that the area of $\triangle GEM$ is $\frac{\sqrt{3}}{8}$ square metres.

1

Question 3 continues on the next page.

Question 3 (cont'd)

Marks

- (c) (i) Show that $f(x) = e^x - x^3 + 1$ has a zero between 4.4 and 4.6. 1
- (ii) Find an approximation, correct to 1 decimal place, for this zero using the method of halving the interval twice. 2
- (d) (i) Express $\sqrt{3} \cos x - \sin x$ in the form $R \cos(x + \alpha)$ where $0 < \alpha < \frac{\pi}{2}$ and $R > 0$. 2
- (ii) Hence, solve $\sqrt{3} \cos x - \sin x = 1$ for $0 \leq x \leq \frac{\pi}{2}$. 1

Question 4 (12 marks)

Marks

- (a) Prove $\tan^{-1} \frac{2}{3} + \cos^{-1} \frac{2}{\sqrt{5}} = \tan^{-1} \frac{7}{4}$. 3
- (b) Find $\int \frac{\cos x \sin x}{\sqrt{2 - \sin^2 x}} dx$ using the substitution $u = \sin x$. 3
- (c) Sketch the curve $y = x + \frac{4}{x}$ showing clearly all the stationary points and asymptotes. Hence find the values of k such that $x + \frac{4}{x} = k$ where there are no real roots. 3
- (d) Use the method of mathematical induction to prove that $(1+1) + (2+3) + (3+5) + \dots + (n+(2n-1)) = \frac{1}{2}n(3n+1)$ where n is a positive integer. 3

Question 5 (12 marks)

Marks

- (a) A particle with displacement x and velocity v , is moving in simple harmonic motion such that $\ddot{x} = -12x$. Initially, it is stationary at the point where $x = -4$.

- (i) Show that

$$v^2 = 12(16 - x^2).$$

- (ii) By assuming the general form for x or otherwise, find x as a function of t .

- (b) The daily growth rate of a population, P , of a species of mosquito is proportional to the excess of the population over 5000.

$$\text{i.e. } \frac{dP}{dt} = k(P - 5000).$$

If $P = 5000 + Ae^{kt}$ satisfies this differential equation find:

- (i) The values of A and k (in exact form) if initially the population was 5002 and after 6 days it was 25000.
- (ii) The mosquito population after 10 days to the nearest whole mosquito.

2

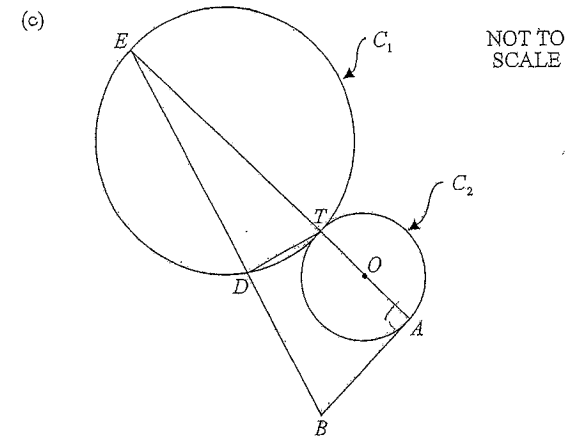
2

2

2

Question 5 (cont'd)

Marks



Two circles C_1 and C_2 touch at T . The line AE passes through O , the centre of C_2 , and through T . The point A lies on C_2 and E lies on C_1 . The line AB is a tangent to C_2 at A , D lies on C_1 and BE passes through D . The radius of C_1 is R and the radius of C_2 is r .

- (i) Find the size of $\angle EDT$, giving reasons for your answer. 1
- (ii) If $DE = 2r$ find an expression for the length of EB in terms of r and R . 3

Question 6 (12 marks)

Marks

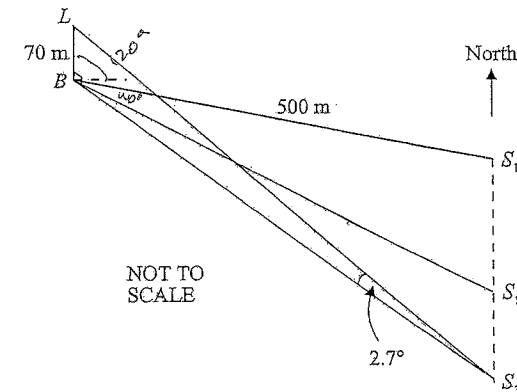
- (a) Prove that $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$ 2
- (b) A particle moves in a straight line so that its velocity v at a position x is given by $v = 9 + 4x^2$. It starts from a fixed point where $x = 0$ at a velocity of 9 ms^{-1} . 1
- (i) Find its acceleration, \ddot{x} as a function of the displacement, x . 3
- (ii) Express its displacement, x , as a function of time, t . 1
- (iii) Find the displacement after $\frac{\pi}{24}$ seconds from its starting point. 2
- (c) (i) Find the equation of the normal at $P(at^2, 2at)$ on the parabola $y^2 = 4ax$. 2
- (ii) The normal intersects with the x -axis at point Q . Find the coordinates of Q and hence find the coordinates of R where R is the midpoint of PQ . 2
- (iii) Hence find the Cartesian equation of the locus of R . 1

Question 7 (12 marks)

Marks

- (a) A life saver sits at point L , 70 m above sea level, in a lookout on a cliff. He spots a surfer lying on his board at point S_1 ; on a bearing of 120° in relation to the life savers position, and a distance of 500 m from point B which is located at sea level directly below the life saver.

Twenty minutes later the life saver notes that the surfer is still lying on his surfboard but is now located at point S_2 on a bearing of 160° in relation to the life savers position and has moved in a line due south.



The life saver raises the alarm and twenty-three minutes after his initial sighting a rescue boat is launched from point B . From point S_3 , which is the point at which the rescue boat reaches the surfer, the angle of elevation of the life saver in the tower is 2.7° .

Assume that the rescue boat has travelled at a constant speed and in a straight line to reach the surfer and that the surfer has drifted at a constant speed in a direction due south since the initial sighting.

- (i) Show that the speed at which the surfer was drifting south was 2.82 km/h (correct to 2 decimal places). 2
- (ii) Find the speed at which the rescue boat was moving. Express your answer in km/h correct to 2 decimal places. 3

Question 7 (cont'd)

Marks

(b)

In the air polluted city of Thaams, a resident asthma sufferer is told by his doctor that he can only go outside his home when the pollution reading is less than 60 parts per million. For each day in Thaams the pollution is lowest at 5am with a reading of 10 parts per million and highest at 2pm with a reading of 90 parts per million.

- (i) Assuming that the level of pollution readings is simple harmonic and of the form $\ddot{x} = -n^2(x-b)$, where $x=b$ is the centre of motion and $x=a$ is the amplitude, show that $x = b - a \cos nt$ satisfies this equation for simple harmonic motion. 2
- (ii) Hence determine the earliest time interval after 5am in the day that the asthma sufferer is **not** to go outside his home (to nearest minute). 5

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln \left(x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln \left(x + \sqrt{x^2 + a^2} \right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

**MATHEMATICS EXTENSION 1
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION
SOLUTIONS
2006**

Question 1 (12 marks)

(a)
$$\lim_{x \rightarrow 0} \frac{2 \sin 2x}{x} = 4 \lim_{x \rightarrow 0} \frac{\sin 2x}{2x}$$

$$= 4 \times 1 \text{ since } \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$= 4$$

2 marks	Correct answer
1 mark	Obtaining $4 \times \lim_{x \rightarrow 0} \frac{\sin 2x}{2x}$

(b) $P(x) = 2x^3 + ax^2 + x + 2$
 Now, $P\left(-\frac{1}{2}\right) = 0$
 so $P\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^3 + a\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) + 2 = 0$

$$-\frac{1}{4} + \frac{1}{4}a - \frac{1}{2} + 2 = 0$$

$$\frac{1}{4}a = -\frac{5}{4}$$

$$a = -5$$

2 marks	Correct answer
1 mark	Obtaining correct expression for $P\left(-\frac{1}{2}\right)$ and then equating it to zero

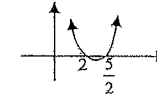
(c)
$$\frac{1}{x-2} \geq 2, \quad x \neq 2$$

$$x-2 \geq 2(x-2)^2$$

$$2(x-2)^2 - (x-2) \leq 0$$

$$(x-2)[2(x-2)-1] \leq 0$$

$$(x-2)(2x-5) \leq 0$$



so $2 < x \leq \frac{5}{2}$ since $x \neq 2$

2 marks	Correct answer
1 mark	Obtaining $(x-2)[2(x-2)-1] \leq 0$

(d)
$$\int_{\frac{x}{3}}^x \sqrt{x+2} \, dx = \frac{1}{3} \int (u-2)u^{\frac{1}{2}} \, du \quad \text{where } u = x+2, \quad \frac{du}{dx} = 1 \text{ and } x = u-2$$

$$= \frac{1}{3} \int (u^{\frac{3}{2}} - 2u^{\frac{1}{2}}) \, du$$

$$= \frac{1}{3} \left(\frac{2}{5} u^{\frac{5}{2}} - \frac{4}{3} u^{\frac{3}{2}} \right)$$

$$= \frac{2}{3} \left\{ \frac{1}{5} (x+2)^{\frac{5}{2}} - \frac{2}{3} (x+2)^{\frac{3}{2}} \right\}$$

2 marks	Correct answer
1 mark	Obtaining $\frac{1}{3} \int (u^{\frac{3}{2}} - 2u^{\frac{1}{2}}) \, du$

(e) $A(2, -1), B(1, -3)$
 $4: -3$
 $\left(\frac{4(1) - 3(2)}{4-3}, \frac{4(-3) - 3(-1)}{4-3} \right)$
 $= (-2, -9)$

2 marks	Correct answer
1 mark	One correct coordinate

(f) $3x - y + 5 = 0, 2x + 3y - 1 = 0$

$$m_1 = 3, \quad m_2 = -\frac{2}{3}$$

$$\tan \theta = \frac{3 + \frac{2}{3}}{1 + 3\left(-\frac{2}{3}\right)} \quad \text{OR} \quad \tan \theta = \left| \frac{3 + \frac{2}{3}}{1 + 3\left(-\frac{2}{3}\right)} \right|$$

$$= -3\frac{2}{3} \quad \theta = 75^\circ$$

$\theta = 105^\circ$ so the obtuse angle is 105°

2 marks	Correct answer
1 mark	Obtaining $m_1 = 3$ and $m_2 = -\frac{2}{3}$

Question 2 (12 marks)

(a) $\frac{d}{dx}(\cos^{-1}(\sin x)) = \frac{-\cos x}{\sqrt{1 - \sin^2 x}}$

$$= \frac{-\cos x}{|\cos x|}$$

$$= \begin{cases} -1, & \text{if } \cos x > 0 \\ 1, & \text{if } \cos x < 0 \end{cases}$$

2 marks	Correct answer
1 mark	Obtaining $\frac{d}{dx}(\cos^{-1}(\sin x)) = \frac{-\cos x}{\sqrt{1 - \sin^2 x}}$

(b) (i) $\int \frac{2dx}{\sqrt{1-9x^2}} = \frac{2}{3} \int \frac{3dx}{\sqrt{1-(3x)^2}}$

$$= \frac{2}{3} \sin^{-1}(3x) + c$$

2 marks	Correct answer
1 mark	Obtaining $\frac{2}{3} \int \frac{3dx}{\sqrt{1-(3x)^2}}$

(ii) $\int \sin^2 \frac{x}{2} dx = \int \left(\frac{1}{2} - \frac{1}{2} \cos x\right) dx$

$$= \frac{1}{2}x - \frac{1}{2} \sin x + c$$

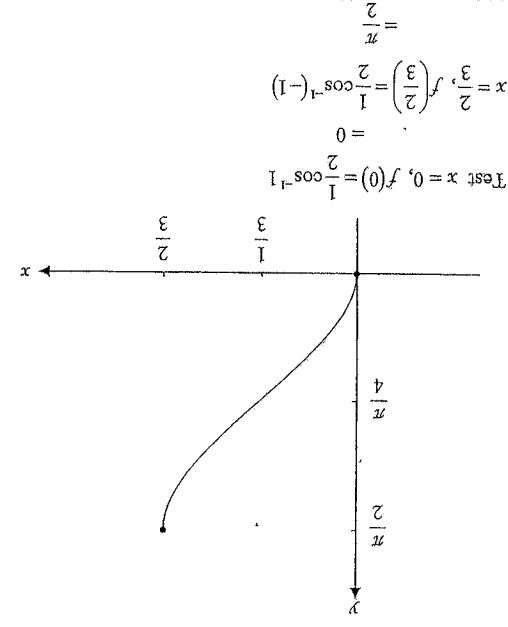
2 marks	Correct answer
1 mark	Obtaining $\int \left(\frac{1}{2} - \frac{1}{2} \cos x\right) dx$

(c) $\log_{16} 2 = \log_x 15 \quad \therefore \log_{16} 16^{\frac{1}{4}} = \log_x 15 \quad \therefore \frac{1}{4} = \log_x 15$

$$\therefore x^{\frac{1}{4}} = 15 \quad \therefore x = 15^4 = 50625$$

2 marks	Correct answer
1 mark	Giving $\frac{1}{4} = \log_x 15$

2 marks	Correct graph
1 mark	Correct shape without endpoints marked or point of inflection labelled

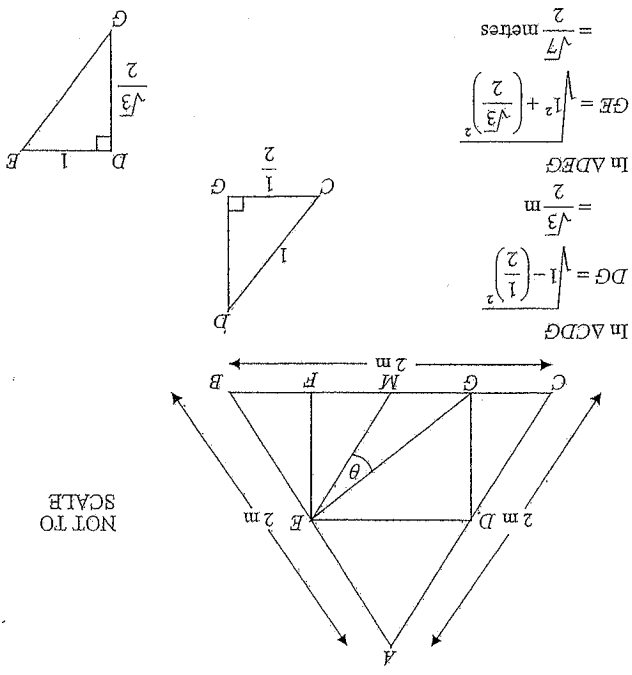


2 marks	Correct answer
1 mark	Correct domain or correct range

- (d) $f(x) = \frac{1}{2} \cos^{-1}(1-3x)$
- Domain: $-1 \leq 1-3x \leq 1$
 $-2 \leq -3x \leq 0$
 $\frac{2}{3} \geq x \geq 0$
 so $0 \leq x \leq \frac{2}{3}$
- Range: $0 \leq 2y \leq \pi$
 $0 \leq y \leq \frac{\pi}{2}$

(ii)

1 mark	Correct answer
--------	----------------



2 marks	Correctly derived derivative
1 mark	Correctly substituting into $f'(x)$

- (a) $f(x) = x^2 - 2x$
- $$f(x+h) = (x+h)^2 - 2(x+h)$$
- $$= x^2 + 2hx + h^2 - 2x - 2h$$
- $$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$
- $$= \lim_{h \rightarrow 0} \frac{x^2 + 2hx + h^2 - 2x - 2h - (x^2 - 2x)}{h}$$
- $$= \lim_{h \rightarrow 0} \frac{2hx + h^2 - 2h}{h}$$
- $$= \lim_{h \rightarrow 0} (2x + h - 2)$$
- $$= 2x - 2$$

(i) (b)

In $\triangle ADG$

$$DG = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2} \text{ m}$$

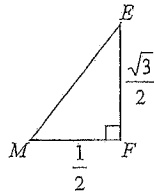
In $\triangle DGE$

$$GE = \sqrt{1^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{\sqrt{7}}{2} \text{ metres}$$

(ii) In $\triangle EFM$

$$EM = \sqrt{\frac{1}{4} + \frac{3}{4}}$$

$$= 1$$

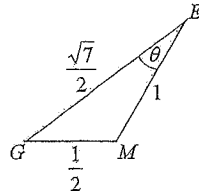


In $\triangle GEM$,

$$\left(\frac{1}{2}\right)^2 = \frac{7}{4} + 1 - 2 \times \frac{\sqrt{7}}{2} \times 1 \cos \theta$$

$$-\frac{5}{2} = -\sqrt{7} \cos \theta$$

$$\cos \theta = \frac{5\sqrt{7}}{14} \text{ as required}$$



2 marks	Correct answer
1 mark	Finding EM and attempting to use the cosine rule

(iii) Area = $\frac{1}{2} \times GE \times EM \sin \theta$

$$= \frac{1}{2} \times \frac{\sqrt{7}}{2} \times 1 \sin \theta$$

$$= \frac{\sqrt{7} \cdot \sqrt{21}}{56}$$

$$= \frac{7\sqrt{3}}{56}$$

$$= \frac{\sqrt{3}}{8} \text{ m}^2 \text{ as required.}$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\sin \theta = \sqrt{1 - \frac{175}{196}}$$

$$= \frac{\sqrt{21}}{14}$$

1 mark	Correct answer
--------	----------------

(c) (i) $f(x) = e^x - x^3 + 1$

$$f(4.4) = -2.7331\dots$$

$$f(4.6) = 3.1483\dots$$

The sign of the function changes between $x = 4.4$ and $x = 4.6$ and hence the zero must lie between $x = 4.4$ and 4.6 .

1 mark	Correct answer
--------	----------------

(ii) By the method of halving the interval $x_1 = \frac{4.4 + 4.6}{2} = 4.5$

Now $f(4.5) = -0.10786\dots < 0$

\therefore As $f(4.5)$ and $f(4.6)$ have opposite signs then the zero is in the interval $4.5 < x < 4.6$.

Now $x_2 = \frac{4.5 + 4.6}{2} = 4.55 = 4.6$ (1 d.p.)

\therefore The zero is 4.6 to 1 decimal place by using the halving the interval method twice.

2 marks	Correct answer
1 mark	Using the correct method but with an arithmetic mistake

(d) (i) $R \cos(x + \alpha) = R \cos x \cos \alpha - R \sin x \sin \alpha$ for all x

Now, $\sqrt{3} \cos x - \sin x = R \cos x \cos \alpha - R \sin x \sin \alpha$

so, $\sqrt{3} = R \cos \alpha$

and $1 = R \sin \alpha$

So, $\tan \alpha = \frac{1}{\sqrt{3}}$

$$\alpha = \frac{\pi}{6}$$

Also $R^2 \cos^2 \alpha + R^2 \sin^2 \alpha = 4$

$$R = 2, R > 0$$

So $\sqrt{3} \cos x - \sin x = 2 \cos\left(x + \frac{\pi}{6}\right)$

2 marks	Correct answer
1 mark	Correctly finding R or α

(ii) $\sqrt{3} \cos x - \sin x = 1$

$$2 \cos\left(x + \frac{\pi}{6}\right) = 1 \quad \text{from part (i)}$$

$$\cos\left(x + \frac{\pi}{6}\right) = \frac{1}{2}$$

$$x + \frac{\pi}{6} = \frac{\pi}{3}$$

$$x = \frac{\pi}{6}$$

1 mark	Correct answer
--------	----------------

Question 4 (12 marks)

(a) Prove $\tan^{-1} \frac{3}{2} + \cos^{-1} \frac{\sqrt{5}}{2} = \tan^{-1} \frac{4}{7}$

$$\tan \left(\tan^{-1} \frac{3}{2} + \cos^{-1} \frac{\sqrt{5}}{2} \right) = \frac{4}{7}$$

let $A = \tan^{-1} \frac{3}{2}$ and $B = \cos^{-1} \frac{\sqrt{5}}{2}$

so $\tan A = \frac{3}{2}$ and $\cos B = \frac{\sqrt{5}}{2}$

so $\tan B = \frac{1}{2}$

$LHS = \tan(A+B)$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= \frac{\frac{3}{2} + \frac{1}{2}}{1 - \frac{3}{2} \times \frac{1}{2}}$$

$$= \frac{2}{1 - \frac{3}{4}}$$

$$= \frac{6}{7}$$

$$= \frac{3}{2}$$

$$= \frac{4}{7}$$

$= RHS$

3 marks	Correct answer
2 marks	Obtaining $\tan A = \frac{3}{2}$ and $\tan B = \frac{1}{2}$
1 mark	Obtaining $\tan \left(\tan^{-1} \frac{3}{2} + \cos^{-1} \frac{\sqrt{5}}{2} \right) = \frac{4}{7}$

let $u = \sin x$ so $\frac{dx}{du} = \cos x$

$$\frac{dx}{du} = \cos x$$

$$\int \frac{\sqrt{2 - \sin^2 x}}{\cos x \sin x} dx$$

$$= \int \frac{\sqrt{2 - u^2}}{\cos x \times u} \frac{dx}{du}$$

$$= \int \frac{u(2 - u^2)^{\frac{1}{2}}}{u} du$$

$$= -\frac{1}{2} \int -2u(2 - u^2)^{\frac{1}{2}} du$$

$$= -\frac{1}{2} \frac{1}{\frac{1}{2}} (2 - u^2)^{\frac{3}{2}}$$

$$= -\sqrt{2 - u^2}$$

$$= -\sqrt{2 - \sin^2 x} + c$$

(b)

$$\frac{d^2 y}{dx^2} = \frac{x^2}{8}$$

$$\frac{dy}{dx} = 1 - \frac{x}{4}$$

$$y = x + \frac{x^2}{4}$$

(c)

$$y = x + \frac{x^2}{4}$$

$$\frac{d^2 y}{dx^2} = \frac{x^2}{8}$$

Stationary points occur when $\frac{dy}{dx} = 0$

When $1 - \frac{x^2}{4} = 0$

$$x^2 = 4$$

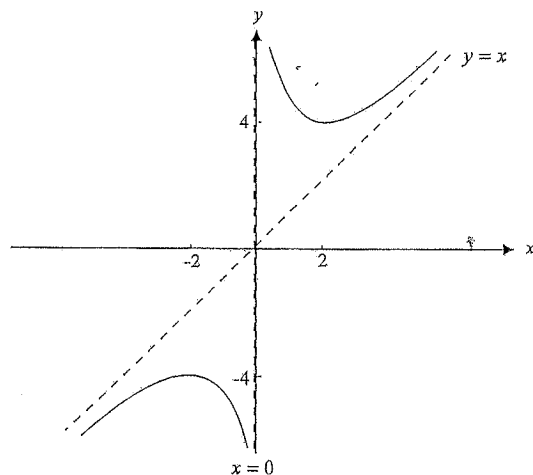
$$x = 2, y = 2 + \frac{2^2}{4} = 4$$

$$x = -2, y = -2 + \frac{(-2)^2}{4} = -4$$

At the point $(-2, -4)$, $\frac{d^2 y}{dx^2} > 0$ so we have a max. at this point.

At the point $(2, 4)$, $\frac{d^2 y}{dx^2} > 0$ so we have a min. at this point.

Asymptotes $x = 0$, $y = x$



From the graph we see that there are no real roots for $x + \frac{4}{x} = k$ when $-4 < k < 4$.

3 marks	Correct answer and graph
2 marks	Correct graph
1 mark	Finding correct local minimum and local maximum OR correct asymptotes

(d) $(1+1) + (2+3) + (3+5) + \dots + (n + (2n-1)) = \frac{1}{2}n(3n+1)$, n is a positive integer

Step 1

For $n = 1$,

$$\begin{aligned} LHS &= 1+1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} RHS &= \frac{1}{2}(1)(3+1) \\ &= 2 \end{aligned}$$

$$\therefore LHS = RHS$$

Hence true for $n = 1$

Step 2: Assume true for $n = k$,

$$(1+1) + (2+3) + (3+5) + \dots + (k + (2k-1)) = \frac{1}{2}k(3k+1) \quad (*)$$

Prove true for $n = k+1$,

$$LHS = (1+1) + (2+3) + \dots + (k + (2k-1)) + (k+1 + (2(k+1)-1))$$

$$= \frac{1}{2}k(3k+1) + (k+1 + 2k+1) \quad (\text{from } *)$$

$$= \frac{3}{2}k^2 + \frac{1}{2}k + 3k + 2$$

$$= \frac{3}{2}k^2 + \frac{7}{2}k + 2$$

$$= \frac{1}{2}(3k^2 + 7k + 4)$$

$$= \frac{1}{2}(k+1)(3k+4)$$

$$= \frac{1}{2}(k+1)(3(k+1)+1)$$

So it is true for $n = (k+1)$

Step 3 Since it is true for $n = 1, n = 2$ and so on and it is true for $n = k+1$ by assuming true for $n = k$, therefore by mathematical induction, it is true for all n where n is a positive integer.

3 marks	Correct proof
2 marks	Verifies the case for $n = 1$ and attempts to prove that it is true for $n = k+1$
1 mark	Verifies the case for $n = 1$

Question 5 (12 marks)

(a) (i) $\ddot{x} = -12x$

$$\frac{d}{dx} \left(\frac{1}{2} v^2 \right) = -12x$$

$$\frac{1}{2} v^2 = -12 \int x dx$$

$$\frac{1}{2} v^2 = \frac{-12x^2}{2} + c$$

When $v = 0$, $x = -4$

$$0 = -96 + c$$

$$c = 96$$

So, $\frac{1}{2} v^2 = -6x^2 + 96$

$$v^2 = -12x^2 + 192$$

$$v^2 = 12(16 - x^2)$$

as required.

2 marks	Correct answer
1 mark	Obtaining $\frac{1}{2} v^2 = -6x^2 + c$ OR Correct substitution of initial conditions into incorrect function

- (ii) Since the motion starts at the 'negative' end of the path of motion we use the general form $x = -a \cos nt$.
 Now, $\ddot{x} = -12x$ so $n^2 = 12$ and so $n = 2\sqrt{3}$, $n > 0$.
 Also, the particle is stationary at $x = -4$ and since $\ddot{x} = -12x$, the centre of motion is at $x = 0$ so $a = 4$.
 So $x = -4 \cos(2\sqrt{3}t)$.

2 marks	Correct answer
1 mark	Finding a and n correctly OR giving correct general form for x

(b) (i) When $t = 0$ $P = 5002$

$$\therefore 5002 = 5000 + Ae^0$$

$$\therefore A = 2$$

When $t = 6$ $P = 25000$ $\therefore 25000 = 5000 + 2e^{6k}$

$$\therefore k = \frac{1}{6} \ln 10000$$

2 marks	Correct answers
1 mark	Obtaining only $A = 2$ or $k = \frac{1}{6} \ln 10000$

(ii) Now $P = 5000 + 2e^{\left(\frac{1}{6} \ln 10000\right)t}$

When $t = 10$ $P = 5000 + 2e^{\left(\frac{1}{6} \ln 10000\right)10}$

$$\therefore P = 5000 + 2e^{\ln 10000 \frac{10}{6}}$$

$$\therefore P = 5000 + 2(10000)^{\frac{10}{6}}$$

$$\therefore P = 9288178 \text{ (to nearest whole mosquito)}$$

2 marks	Correct answer
1 mark	Showing $P = 5000 + 2e^{\ln 10000 \frac{10}{6}}$

- (c) (i) Since C_1 and C_2 touch at T and AE passes through O , the centre of C_2 , and through T , then AE also passes through the centre of C_1 . So TE is a diameter of C_1 . (When circles touch, the line through the centres passes through the point of contact.)
So $\angle EDT = 90^\circ$ (Angles in a semi-circle are right angles.)

1 mark	Correct answer
--------	----------------

- (ii) $\angle EAB = 90^\circ$ (The tangent to a circle is perpendicular to the radius drawn to the point of contact.)

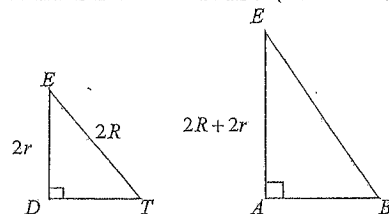
In Δ 's ABE and DTE ,

$$\angle EDT = \angle EAB = 90^\circ \quad (\text{from part (i) and from above})$$

$\angle DET$ is common

$$\angle ETD = \angle EBA \quad (\text{Angles in a triangle add to give } 180^\circ.)$$

So ΔABE is similar to ΔDTE . (Three corresponding angles are equal.)



Now, $DE = 2r$, $ET = 2R$, and $EA = 2R + 2r$

Since the Δ 's are similar,

$$\frac{EB}{ET} = \frac{EA}{ED}$$

$$\begin{aligned} EB &= 2R \left(\frac{2R + 2r}{2r} \right) \\ &= \frac{2R(R + r)}{r} \end{aligned}$$

3 marks	Correct answer including showing that ΔABE is similar to ΔDTE
2 marks	Showing that ΔABE is similar to ΔDTE
1 mark	Stating that $\angle EAB = 90^\circ$

Question 6 (12 marks)

$$\begin{aligned} \text{(a) LHS} &= \frac{d}{dx}(\cos ex) \\ &= \frac{d}{dx} \left(\frac{1}{\sin x} \right) \\ &= \frac{d}{dx}([\sin x]^{-1}) \\ &= -[\sin x]^{-2} \cos x \\ &= -\frac{1 \cos x}{\sin x \sin x} \\ &= -\cos ex \cot x \\ &= \text{RHS} \end{aligned}$$

2 marks	Correct answer
1 mark	Writing LHS as $-\sin x^{-2} \cos x$

$$\begin{aligned} \text{(b) (i) } v &= 9 + 4x^2 \\ \ddot{x} &= \frac{d}{dx} \left(\frac{1}{2} v^2 \right) \\ &= \frac{d}{dx} \left(\frac{1}{2} (9 + 4x^2)^2 \right) \\ &= (9 + 4x^2)(8x) \\ &= 8x(9 + 4x^2) \end{aligned}$$

1 mark	Correct answer
--------	----------------

(ii) $v = 9 + 4x^2$

$$\frac{dx}{dt} = 9 + 4x^2$$

$$\frac{dt}{dx} = \frac{1}{9 + 4x^2}$$

$$t = \int \frac{1}{9 + 4x^2} dx$$

$$= \frac{1}{4} \int \frac{1}{\frac{9}{4} + x^2} dx$$

$$= \frac{1}{4} \times \frac{2}{3} \tan^{-1}\left(\frac{2x}{3}\right) + c$$

$$= \frac{1}{6} \tan^{-1}\left(\frac{2x}{3}\right) + c$$

$t = 0$ and $x = 0$, so $c = 0$

$$t = \frac{1}{6} \tan^{-1}\left(\frac{2x}{3}\right)$$

$$6t = \tan^{-1}\left(\frac{2x}{3}\right)$$

$$\frac{2x}{3} = \tan 6t$$

$$x = \frac{3}{2} \tan 6t$$

3 marks	Correct answer
2 marks	Obtaining $t = \frac{1}{6} \tan^{-1}\left(\frac{2x}{3}\right)$
1 mark	Obtaining $t = \int \frac{1}{9 + 4x^2} dx$

(iii) $t = \frac{\pi}{24}$, $x = \frac{3}{2} \tan 6\left(\frac{\pi}{24}\right)$

$$= \frac{3}{2} \tan \frac{\pi}{4}$$

$$= 1\frac{1}{2} \text{ m}$$

The displacement is 1.5m after $\frac{\pi}{24}$ seconds from its starting point.

1 mark	Correct answer
--------	----------------

(c) (i) $x = at^2$, $y = 2at$

$$\frac{dx}{dt} = 2at$$

$$\frac{dy}{dt} = 2a$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 2a \times \frac{1}{2at}$$

$$= \frac{1}{t}$$

$m_T = \frac{1}{t}$ so $m_N = -t$ (since $m_1 m_2 = -1$)

Equation of normal at $P(at^2, 2at)$ is given by

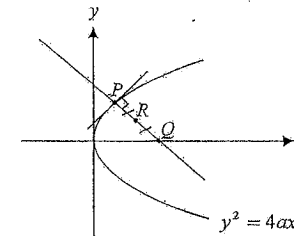
$$y - 2at = -t(x - at^2)$$

$$= -tx + at^3$$

$$tx + y = 2at + at^3$$

2 marks	Correct answer
1 mark	Obtaining the correct gradient of the normal

(ii)



At Q , $y = 0$.

sub $y = 0$ into $tx + y = 2at + at^3$

$$tx = 2at + at^3$$

$$x = a(2 + t^2)$$

So we have $Q(a(2 + t^2), 0)$

and $R\left(\frac{a(2 + t^2) + at^2}{2}, \frac{0 + 2at}{2}\right)$

$$R(a(1 + t^2), at)$$

2 marks	Correct answer
1 mark	Obtaining one correct coordinate of R OR finding the correct coordinates of Q

(iii) At R , $x = a(1+t^2)$ and $y = at$
 so $x = a\left(1 + \frac{y^2}{a^2}\right)$ and $t = \frac{y}{a}$

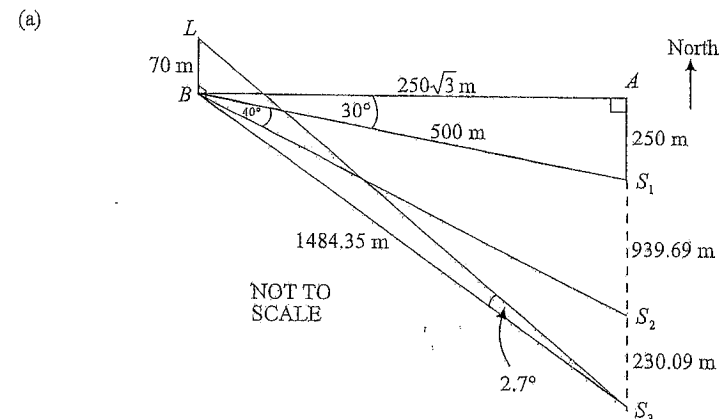
$$\frac{x}{a} = 1 + \frac{y^2}{a^2}$$

$$\frac{y^2}{a^2} = \frac{x}{a} - 1$$

$$y^2 = ax - a^2$$
 so $y^2 = a(x - a)$

1 mark	Correct answer
--------	----------------

Question 7 (12 marks)



(i)

$$\text{In } \triangle ABS_1, AB = 500 \cos 30^\circ = 250\sqrt{3}$$

$$\text{Also, } AS_1 = 500 \sin 30^\circ = 250$$

$$\text{In } \triangle ABS_2, AS_2 = 250\sqrt{3} \tan 70^\circ = 1189.692\dots$$

$$\text{so, } S_1S_2 = AS_2 - AS_1 = 939.692\dots$$

$$\text{Speed of surfer} = \frac{S_1S_2}{20 \text{ mins}} = \frac{0.939692\dots}{0.3} \text{ km/h}$$

$$= 2.82 \text{ km/h (correct to 2 decimal places)}$$

as required

2 marks	Correct answer
1 mark	Making a reasonable attempt to find the distance S_1S_2

$$(ii) \text{ In } \triangle BLS_3, BS_3 = \frac{70}{\tan 2.7^\circ} = 1484.346\dots$$

$$\text{In } \triangle ABS_3, AS_3 = \sqrt{(BS_3)^2 - (250\sqrt{3})^2} = 1419.783\dots$$

$$\text{So } S_2S_3 = 1419.783 - 250 - 939.692 = 230.091$$

The time taken for the surfer to drift from S_2 to S_3 is $\frac{0.230091}{2.82} = 0.08159$ h (to 5 dec. places).

The rescue boat leaves 3 minutes or $\frac{3}{60} = 0.05$ hours after the lifesaver spotted the surfer at point S_2 . Therefore the rescue boat took $0.08159 - 0.05 = 0.03159$ hours to cover the distance of 1.48435 km.

The speed of the rescue boat was therefore $\frac{1.48435}{0.03159} = 46.99$ km/h (correct to 2 decimal places).

3 marks	Correct answer
2 marks	Finding correctly the time taken for the surfer to drift from S_2 to S_3
1 mark	Finding correctly the distance from S_2 to S_3

(b)

$$(i) \ddot{x} = -n^2(x-b) \dots\dots(1)$$

$$\text{If } x = b - a \cos nt \dots\dots(2)$$

$$\therefore \dot{x} = an \sin nt$$

$$\therefore \ddot{x} = an^2 \cos nt \dots\dots(3)$$

Sub (2) and (3) into (1):

$$\text{LHS} = \ddot{x}$$

$$= an^2 \cos nt$$

$$= n^2(a \cos nt)$$

$$= n^2(b-x) \quad (\text{from (2)})$$

$$= -n^2(x-b)$$

$$= \text{RHS}$$

2 mark	Correctly proving the result
1 mark	Partly proving the result

(ii) Using $x = b - a \cos nt$

$$\therefore b = \frac{10+90}{2} = 50 \quad \text{and} \quad a = \frac{90-10}{2} = 40$$

$$\text{Period, } T = \frac{2\pi}{n} \quad \therefore 18 = \frac{2\pi}{n} \quad \therefore n = \frac{\pi}{9}$$

$$\therefore x = 50 - 40 \cos \frac{\pi t}{9}$$

$$\text{If } x = 60 \quad 60 = 50 - 40 \cos \frac{\pi t}{9}$$

$$\therefore -\frac{1}{4} = \cos \frac{\pi t}{9}$$

$$\therefore \frac{\pi t}{9} = \cos^{-1}\left(-\frac{1}{4}\right)$$

$$\therefore \frac{\pi t}{9} = \pi - \cos^{-1}\frac{1}{4} \text{ or } \pi + \cos^{-1}\frac{1}{4} \quad (\text{for } 2^{\text{nd}}, 3^{\text{rd}} \text{ quadrants})$$

$$\therefore t = 5.22\dots \text{ or } 12.776\dots$$

$$\therefore t = 5\text{hr } 13\text{ min } 25.95\dots \text{ s or } 12\text{hr } 46\text{ min } 34.05\dots \text{ s}$$

\therefore the earliest time interval from 5am that the asthma sufferer should not go outside his home is from 10.13am to 5.47pm (to the nearest minute) so that the pollution reading is less than 60 parts per million.

5 marks	Correctly determining the earliest time interval
4 marks	Making only one mistake with the earliest time interval
3 marks	Obtaining the equation $-\frac{1}{4} = \cos \frac{\pi t}{9}$
2 marks	Correctly determining two of the values of a, b and n
1 mark	- Determining the correct value of a, b or n