CRANBROOK

MATHEMATICS EXTENSION 2

2007

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

General Instructions

- Reading time – 5 minutes
- Writing time – 3 hours
- All eight questions should be attempted
- Total marks available - 120
- All questions are worth 15 marks
- An approved calculator may be used
- All relevant working should be shown for each question
- Answer each question in a separate 8 page booklet.

Question 1 (15 marks) Marked by SKB **Marks**

(a) Find $\int_1^2 3\sqrt{x-1}\,dx$ **2**

(b) Using the substitution $u = e^x - 1$ or otherwise, **2**

find $\int \frac{2e^x}{e^{2x} - 2e^x + 1}\,dx$

(c) Use integration by parts to evaluate $\int_0^{\frac{\pi}{4}} x\cos 4x\,dx$ **3**

(d) Use the substitution $t = \tan\frac{\theta}{2}$ to find $\int \frac{\sin\theta}{1+\cos\theta}\,d\theta$ **4**

(e) (i) Find the real numbers a, b and c such that **2**

$$\frac{x^3 + 5x^2 + x + 2}{x^2(x^2+1)} \equiv \frac{x+a}{x^2} + \frac{bx+c}{x^2+1}$$

(ii) Find $\int \frac{x^3 + 5x^2 + x + 2}{x^2(x^2+1)}\,dx$ **2**

Question 2 (15 marks) Marked by SKB **Marks**

(a) Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\sin^3 x+\cos^3 x}\,dx$. **2**

(b) Evaluate $\int_0^1 \sqrt{1+x^2}\,dx$. **4**

(c) Find $\int \frac{3x-4}{\sqrt{4+5x-3x^2}}\,dx$ **4**

(d) If $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \sin^2 x\,dx$ for $n \geq 0$, show that $I_n = \frac{n-1}{n+2} I_{n-2}$ for $n \geq 2$. Hence or otherwise evaluate $\int_0^{\frac{\pi}{2}} \cos^4 x \sin^2 x\,dx$. **5**

Question 3 (15 marks) Marked by CJL **Marks**

(a) Let $z = 3-4i$ and $\omega = 2-i$

(i) Find $\frac{1}{z}$ in the form $x+yi$ **1**

(ii) Show that $\operatorname{Im} z + \overline{\omega} + z\omega = -10i$ **2**

(b) If ai is a solution to the equation **2**

$$z^2 + (1-i)z + (2-2i) = 0$$

find the real value of a.

(c) Let $u = 1-i$

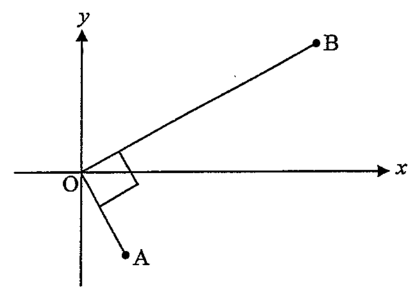
(i) Find $|u|$ and $\arg u$ **2**

(ii) Hence find u^{12}. Express your answer in the form $x+yi$. **2**

(d) Sketch the region on an Argand diagram where the inequalities **3**

$|z-2+i| \leq |z+2-i|$ and $\operatorname{Im} z \geq 0$ both hold.

(e)



The point A on the Argand diagram above corresponds to the complex number z.
Triangle ABO is a right-angled triangle where $OB = 2OA$.

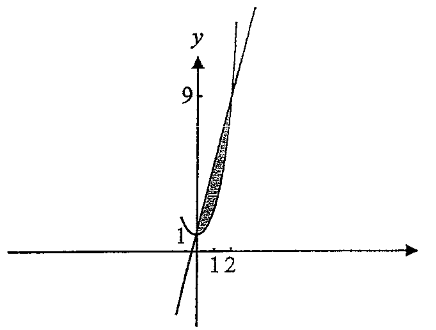
(i) Show that point B corresponds to the complex number $2iz$. **1**

(ii) The point C corresponds to the complex number v and C is situated so that $OACB$ is a rectangle.
Given that $z = x+yi$, $x, y \in R$, find $\overline{v}$ in terms of x and y. **2**

Question 4 (15 marks) Marked by SKB **Marks**

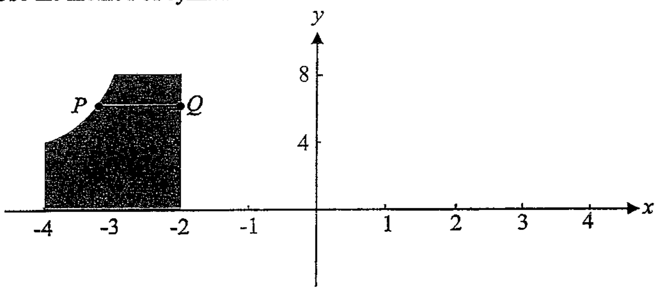
(a) The base of a certain solid S lies on the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$. The cross-section of this solid by planes perpendicular to the $x-$axis are equilateral triangles. By including appropriate views of slices to this solid calculate its volume in exact form. **5**

(b)



The shaded area shown in the diagram above is the area between the graph of $y=4x+1$ and the graph of $y=2x^2+1$. This shaded area is rotated about the y axis to form a solid.
Use the method of cylindrical shells to find the volume of the solid. **4**

(c)



The shaded area is bounded by the lines $x=-4, x=-2, y=8$, by the curve $y=\frac{-8}{x+2}$ and by the x-axis.
The region is rotated about the line $x=3$ until it reaches its original position thus forming a solid. The horizontal line segment PQ forms an annulus as a result of this rotation.

(i) Show that the area of this annulus at height y where $y \geq 4$, is equal to
$$16\pi\left(\frac{4}{y^2}+\frac{5}{y}\right)$$ **2**

(ii) Hence find the volume of the solid. **4**

Question 5 (15 marks) Marked by CJL **Marks**

(a) Let $f(x)=\cos^{-1}x$ for $-1 \leq x \leq 1$ and $g(x)=\sin^{-1}x$ for $-1 \leq x \leq 1$.

(i) Sketch $f(x)$ and $g(x)$ on the same set of axes. **1**

(ii) By differentiating, evaluate $f(x)+g(x)$ **1**

(iii) Hence evaluate $\int_{-1}^{1}(f(x)+g(x))dx$ **1**

(b) The ellipse E has the equation $x^2+\frac{y^2}{4}=1$.

(i) Find the eccentricity and the foci of E. **2**

(ii) Find the length of the major and minor axes of E. **1**

(iii) Write down the equations of the directrices of E. **1**

(iv) Sketch E. **1**

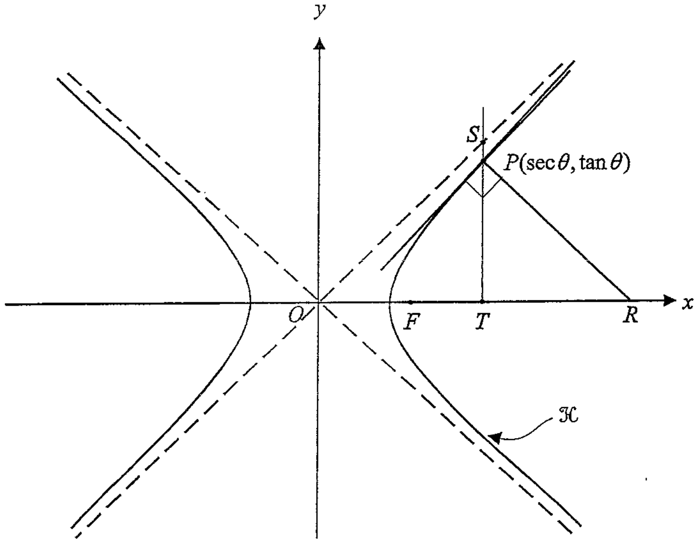
(c) (i) The polynomial equation $p(x)=0$ has a root α of multiplicity 3. **2**
Show that α is a root of $p'(x)=0$ and is of multiplicity 2.

(ii) The polynomial $q(x)=x^6+ax^5+bx^4-x^2-2x-1$ has a quadratic factor of x^2+2x+1. Find a and b. **2**

(i) Consider the polynomial **3**
$$r'(x)=1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!} \text{ where } r(0)=1.$$
Show that $r(x)$ has no double roots.

Question 6 (15 marks) Marked by CJL **Marks**

(a)



The point $P(\sec\theta, \tan\theta)$ lies on the hyperbola H with equation $x^2 - y^2 = 1$. A vertical line through P intersects with an asymptote at S and with the x-axis at T as shown. A normal to H at P intersects the x-axis at R. The point F is a foci of H.

(i) Show that the equation of the normal to H at the point P is $y = -\sin\theta\, x + 2\tan\theta$. **2**

(ii) Show that $RS = \sqrt{2}RT$. **3**

(iii) Find the coordinates of the point U which lies on SR such that TU is parallel to the asymptote on which S lies. **2**

(iv) For what values of θ will FU be the perpendicular bisector of SR? **2**

(b) Let $\omega = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$.

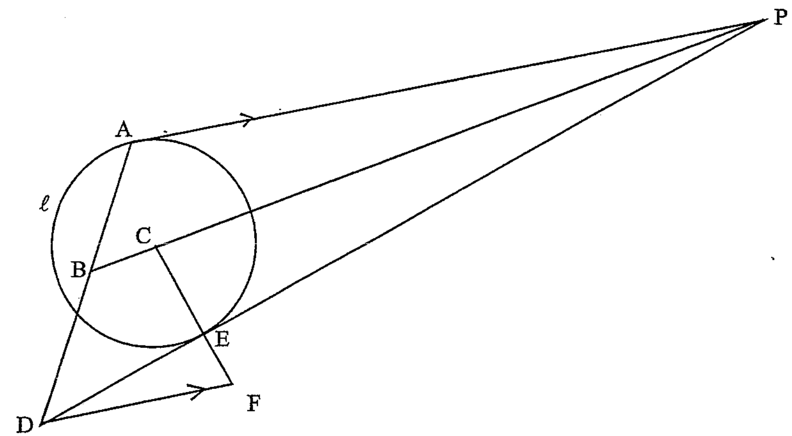
(i) Write down in terms of ω, and the positive integer k, all the solutions of the equation $z^{10} - 1 = 0$. **2**

(ii) Prove that $\omega + \omega^2 + \omega^3 + ... + \omega^{10} = 0$. **2**

(iii) The quadratic equation $x^2 + bx + c = 0$, where b and c are real, has the root $\omega + \omega^4$. Find the other root in terms of ω. **2**

Question 7 (15 marks) Marked by CJL **Marks**

(a)



In the diagram, ℓ is a circle with exterior point P. Tangents from P are drawn to meet ℓ at points A and E. The point C is the centre of ℓ. The line BP passes through C. The line AD passes through B. The line CF passes through E. AP is parallel to DF.

(i) Show that $ACEP$ is a cyclic quadrilateral. **1**

(ii) Use a double angle formula to show that $DE = \dfrac{DF(EP^2 - CE^2)}{CP^2}$ **2**

(iii) Use the sine rule to show that $\dfrac{AB}{BD} = \dfrac{AP}{DP}$ **2**

Marks

(b) (i) Draw the graph of $y = \ln(x+1)$ **1**

(ii) Hence explain why **1**

$$\int_0^n \ln(x+1)\,dx < n\ln(n+1), \qquad n = 1,2,3\ldots$$

(iii) Use integration by parts to show that **3**

$$\int_0^n \ln(x+1)\,dx = \ln(n+1)^{n+1} - n$$

(iv) Hence deduce that $\ln(n+1) < n$ **1**

(v) Show that $\sum_{k=1}^{n} \frac{1}{2}\ln(k+1) = \frac{1}{2}\ln(n+1)!$ **1**

(vi) Use the results from parts (iii) and (v) together with your graph **3**

to deduce that $n! < \left(\frac{n+1}{e}\right)^{2n}(n+1)$

Question 8 (15 marks) **Marked by SKB** **Marks**

(a) (i) For all real, positive numbers a and b, where $a > b$ show that **4**

(α) $a + b > 2\sqrt{ab}$

(β) $b^2 - a^2 < 2\sqrt{ab}(b-a)$

(ii) Hence deduce that $a > c$ given that c is a positive real number and **4**

$$\sqrt{a}(b-a) + \sqrt{c}(c-b) > \frac{c^2 - a^2}{2\sqrt{b}}$$

(b) If $h(n) = n^4 + 6n^2 + 9$

(i) show that $h(n+2) - h(n) = 8(n+1)(n^2 + 2n + 5)$ **3**

(ii) hence prove by mathematical induction that $h(n)$ is divisible by 8 if n is an odd positive integer. **4**

END OF EXAM

STANDARD INTEGRALS

$\int x^n\,dx$	$= \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\; x \neq 0, \text{ if } n < 0$
$\int \frac{1}{x}\,dx$	$= \ln x, \quad x > 0$
$\int e^{ax}\,dx$	$= \frac{1}{a}e^{ax}, \; a \neq 0$
$\int \cos ax\,dx$	$= \frac{1}{a}\sin ax, \; a \neq 0$
$\int \sin ax\,dx$	$= -\frac{1}{a}\cos ax, \; a \neq 0$
$\int \sec^2 ax\,dx$	$= \frac{1}{a}\tan ax, \; a \neq 0$
$\int \sec ax \tan ax\,dx$	$= \frac{1}{a}\sec ax, \; a \neq 0$
$\int \frac{1}{a^2+x^2}\,dx$	$= \frac{1}{a}\tan^{-1}\frac{x}{a}, \; a \neq 0$
$\int \frac{1}{\sqrt{a^2-x^2}}\,dx$	$= \sin^{-1}\frac{x}{a}, a > 0, \; -a < x < a$
$\int \frac{1}{\sqrt{x^2-a^2}}\,dx$	$= \ln\left(x+\sqrt{x^2-a^2}\right), \; x > a > 0$
$\int \frac{1}{\sqrt{x^2+a^2}}\,dx$	$= \ln\left(x+\sqrt{x^2+a^2}\right)$

NOTE : $\ln x = \log_e x, \quad x > 0$

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SOLUTIONS
2007

Question 1

(a) $\displaystyle\int_1^2 3\sqrt{x-1}\,dx = 3\int_0^1 u^{\frac{1}{2}}\frac{du}{dx}dx$

Let $u = x-1$

$\dfrac{du}{dx} = 1$

If $x = 2$, $u = 1$

If $x = 1$, $u = 0$

$\displaystyle = 3\int_0^1 u^{\frac{1}{2}}du$ **(1 mark)**

$\displaystyle = 3\left[\frac{2u^{\frac{3}{2}}}{3}\right]_0^1$

$\displaystyle = 3\left\{\frac{2\times 1^{\frac{3}{2}}}{3} - 0\right\}$

$= 2$

(1 mark)

(b) $\displaystyle\int\frac{2e^x}{e^{2x}-2e^x+1}dx = \int\frac{2e^x}{(e^x-1)^2}dx$

Let $u = e^x - 1$

$\dfrac{du}{dx} = e^x$

$\displaystyle = 2\int\frac{du}{dx}\cdot u^{-2}dx$ **(1 mark)**

$\displaystyle = 2\int u^{-2}du$

$\displaystyle = \frac{2\ u^{-1}}{-1} + c$

$\displaystyle = \frac{-2}{u} + c$

$\displaystyle = \frac{-2}{e^x - 1} + c$

(1 mark)

(c) The parts formula states that

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx.$$

For $\displaystyle\int_0^{\frac{\pi}{4}} x\cos(4x)dx$, let $u = x$ and $\dfrac{dv}{dx} = \cos(4x)$

So, $\dfrac{du}{dx} = 1$ and $v = \dfrac{1}{4}\sin(4x)$

(1 mark)

So $\displaystyle\int_0^{\frac{\pi}{4}} x\cos(4x)dx = \left[\frac{x}{4}\sin(4x)\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}}\frac{1}{4}\sin(4x)dx$ **(1 mark)**

$\displaystyle = \left\{\frac{\pi}{16}\sin\pi - 0\right\} - \frac{1}{4}\left[-\frac{1}{4}\cos(4x)\right]_0^{\frac{\pi}{4}}$

$\displaystyle = \frac{\pi}{16}\times 0 + \frac{1}{16}\{\cos\pi - \cos 0\}$

$\displaystyle = \frac{1}{16}(-1-1)$

$\displaystyle = -\frac{1}{8}$

(1 mark)

(d) We have $\displaystyle\int\frac{\sin\theta}{1+\cos\theta}d\theta$.

Using the substitution $t = \tan\dfrac{\theta}{2}$

We have $\sin\theta = \dfrac{2t}{1+t^2}$

$\cos\theta = \dfrac{1-t^2}{1+t^2}$ **(1 mark)**

So, $\displaystyle\frac{\sin\theta}{1+\cos\theta} = \frac{2t}{1+t^2} \div \left(1 + \frac{1-t^2}{1+t^2}\right)$

$\displaystyle = \frac{2t}{1+t^2} \div \frac{1+t^2+1-t^2}{1+t^2}$

$\displaystyle = \frac{2t}{1+t^2} \div \frac{2}{1+t^2}$

$\displaystyle = \frac{2t}{1+t^2} \times \frac{1+t^2}{2}$

$= t$

So, $\int \frac{\sin\theta}{1+\cos\theta}d\theta = \int t\cdot\frac{d\theta}{dt}dt$ where $\frac{d\theta}{dt}=\frac{2}{1+t^2}$ **(1 mark)**

$$= \int t\cdot\frac{2}{1+t^2}dt$$

$$= \int\frac{2t}{1+t^2}dt \quad \textbf{(1 mark)}$$

$$= \ln\left(1+t^2\right)+c$$

$$= \ln\left(1+\tan^2\frac{\theta}{2}\right)+c \quad \textbf{(1 mark)}$$

(e) (i) Let $\frac{x^3+5x^2+x+2}{x^2\left(x^2+1\right)} \equiv \frac{x+a}{x^2}+\frac{bx+c}{x^2+1}$

$$\equiv \frac{(x+a)(x^2+1)+x^2(bx+c)}{x^2\left(x^2+1\right)}$$

$$\equiv \frac{x^3+x+ax^2+a+bx^3+cx^2}{x^2(x^2+1)}$$

$$\equiv \frac{x^3(b+1)+x^2(a+c)+x+a}{x^2\left(x^2+1\right)}$$

True iff $x^3+5x^2+x+2 \equiv x^3(b+1)+x^2(a+c)+x+a$ **(1 mark)**

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Put $x=0$, $2=a$

Put $x=1$, $9=b+1+2+c+1+2$

$3=b+c \quad -(A)$

Put $x=-1$, $5=-b-1+2+c-1+2$

$3=-b+c \quad -(B)$

$(A)+(B)$ $6=2c$

$c=3$

In (A) $3=b+3$

$b=0$

So, $a=2,\ b=0,\ c=3$ **(1 mark)**

(ii) Now $\int\frac{x^3+5x^2+x+2}{x^2\left(x^2+1\right)}dx = \int\left(\frac{x+2}{x^2}+\frac{3}{x^2+1}\right)dx$ from part (i)

$$= \int\frac{x}{x^2}dx+\int\frac{2}{x^2}dx+\int\frac{3}{x^2+1}dx$$

$$= \frac{1}{2}\ln x^2-\frac{2}{x}+3\tan^{-1}x+c$$

$$= \ln|x|-\frac{2}{x}+3\tan^{-1}x+c$$

(1 mark) for first two terms

(1 mark) for $3\tan^{-1}x$ term **Total 15 marks**

Question 2

(a) $I=\int_0^{\frac{\pi}{2}}\frac{\cos^3x}{\sin^3x+\cos^3x}\,dx$

$$=\int_0^{\frac{\pi}{2}}\frac{\cos^3(\frac{\pi}{2}-x)}{\sin^3(\frac{\pi}{2}-x)+\cos^3(\frac{\pi}{2}-x)}\,dx$$

$$=\int_0^{\frac{\pi}{2}}\frac{\sin^3x}{\cos^3x+\sin^3x}\,dx \quad \textbf{[1 mark]}$$

$$\therefore 2I=\int_0^{\frac{\pi}{2}}\frac{\cos^3x+\sin^3x}{\sin^3x+\cos^3x}\,dx$$

$$\therefore I=\frac{1}{2}\int_0^{\frac{\pi}{2}}1\,dx$$

$$=\frac{1}{2}[x]_0^{\frac{\pi}{2}}$$

$$=\frac{\pi}{4} \quad \textbf{[1 mark]}$$

(b) $I=\int_0^1\sqrt{1+x^2}\,dx$

Let $x=\tan\theta$ when $x=0,\ \theta=0$

$\therefore\frac{dx}{d\theta}=\sec^2\theta$ $x=1,\ \theta=\frac{\pi}{4}$

$$I=\int_0^{\frac{\pi}{4}}\sqrt{1+\tan^2\theta}\ \sec^2\theta\,d\theta \quad \textbf{[1 mark]}$$

$$I=\int_0^{\frac{\pi}{4}}\sec^3\theta\,d\theta$$

$$=\int_0^{\frac{\pi}{4}}\sec\theta\sec^2\theta\ d\theta$$

Let $u=\sec\theta$ $dv=\sec^2\theta\ d\theta$

$\therefore\frac{du}{d\theta}=\sec\theta\tan\theta$ $v=\tan\theta$

$$=[\sec\theta\tan\theta]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\tan\theta\sec\theta\tan\theta\ d\theta$$

$$=[\sec\theta\tan\theta]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\tan\theta\sec\theta\tan\theta\ d\theta \quad \textbf{[1 mark]}$$

$$=\left[\sqrt{2}.1-0\right]-\int_0^{\frac{\pi}{4}}\tan^2\theta\sec\theta\ d\theta$$

$$=\sqrt{2}-\int_0^{\frac{\pi}{4}} (\sec^2\theta-1)\sec\theta \ d\theta$$

$$=\sqrt{2}-\int_0^{\frac{\pi}{4}} \sec^3\theta \ d\theta+\int_0^{\frac{\pi}{4}} \sec\theta \ d\theta$$

$$\therefore 2I=\sqrt{2}+\left[\ln\left|\sec\theta+\tan\theta\right|\right]_0^{\frac{\pi}{4}}$$ [1 mark]

$$\therefore 2I=\sqrt{2}+\left[\ln(\sqrt{2}+1)-\ln 1\right]$$

$$\therefore I=\frac{1}{2}[\sqrt{2}+\left[\ln(\sqrt{2}+1)\right]]$$ [1 mark]

(c)

$$I=\int \frac{3x-4}{\sqrt{4+5x-3x^2}}\,dx$$

$$=\int \frac{-\frac{1}{2}(5-6x)-\frac{3}{2}}{\sqrt{4+5x-3x^2}}\,dx$$ [1 mark]

$$=-\frac{1}{2}\int \frac{(5-6x)}{\sqrt{4+5x-3x^2}}\,dx-\frac{3}{2}\int \frac{1}{\sqrt{-3[x^2-\frac{5}{3}x-\frac{4}{3}]}}\,dx$$

$$=-\frac{1}{2}\sqrt{4+5x-3x^2}\times 2-\frac{3}{2}\int \frac{1}{\sqrt{-3[(x-\frac{5}{6})^2-\frac{73}{36}]}}\,dx$$ **[1 mark]**

$$=-\sqrt{4+5x-3x^2}-\frac{3}{2\sqrt{3}}\int \frac{1}{\sqrt{[(\frac{\sqrt{73}}{6})^2-(x-\frac{5}{6})^2]}}\,dx$$ **[1 mark]**

$$=-\sqrt{4+5x-3x^2}-\frac{\sqrt{3}}{2}\sin^{-1}\left[\frac{6x-5}{\sqrt{73}}\right] + c$$ **[1 mark]**

(d)

$$I_n=\int_0^{\frac{\pi}{2}} \cos^n x\sin^2 x \ dx \text{ for } n\geq 0$$

$$=\int_0^{\frac{\pi}{2}} \cos^n x\,(1-\cos^2 x) \ dx$$

$$=\int_0^{\frac{\pi}{2}} \cos^n x-\cos^{n+2}x \ dx$$

$$=U_n-U_{n+2}$$ (equation 1) [1 mark]

Now $U_n=\int_0^{\frac{\pi}{2}}\cos^n x\,dx$

$$=\int_0^{\frac{\pi}{2}}\cos^{n-1}x\cos x\,dx$$

Let $u=\cos^{n-1}x$ $\quad dv=\cos x\,dx$

$\therefore \frac{du}{dx}=(n-1)\cos^{n-2}x.-\sin x \quad v=\sin x$

$$=\left[\cos^{n-1}x\sin x\right]_0^{\frac{\pi}{2}}+(n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}x\sin^2 x\,dx$$ **[1 mark]**

$$=[0-0]+(n-1)I_{n-2}$$

$$=(n-1)I_{n-2}$$

$\therefore U_n=(n-1)I_{n-2}$ and $U_{n+2}=(n+2-1)I_{n+2-2}$

$=(n+1)I_n$ **[1 mark]**

But $I_n=U_n-U_{n+2}$ (from equation 1)

$\therefore I_n=(n-1)I_{n-2}-(n+1)I_n$

$\therefore I_n(1+(n+1))=(n-1)I_{n-2}$

$\therefore I_n=\frac{(n-1)}{(n+2)}I_{n-2}, n\geq 2$ **[1 mark]**

Let $I_4=\int_0^{\frac{\pi}{2}}\cos^4 x\sin^2 x\,dx$

$$\therefore I_4=\frac{4-1}{4+2}I_2$$

$$=\frac{3}{6}.\frac{1}{4}I_0$$

$$=\frac{1}{8}\int_0^{\frac{\pi}{2}}\sin^2 x\,dx$$

$$=\frac{1}{8}\int_0^{\frac{\pi}{2}}\frac{1}{2}[1-\cos 2x]\,dx$$

$$=\frac{1}{16}\left[x-\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$

$=\frac{1}{16}[(\frac{\pi}{2}-0)-0]$

$=\frac{\pi}{32}$ **[1 mark]**

Question 3

(a) (i)

$$z=3-4i$$
$$\frac{1}{z}=\frac{1}{3-4i}$$
$$=\frac{1}{3-4i}\times\frac{3+4i}{3+4i}$$
$$=\frac{3+4i}{9+16}$$
$$=\frac{3+4i}{25}$$
$$=\frac{3}{25}+\frac{4}{25}i$$

(1 mark)

(ii) $z=3-4i$ and $\omega=2-i$

To show: $\operatorname{Im} z+\overline{\omega}+z\omega=-10i$

Left side $=\operatorname{Im} z+\overline{\omega}+z\omega$

$=-4+2+i+(3-4i)(2-i)$

$=-2+i+6-3i-8i-4$

$=-10i$

$=$ right side

as required

(1 mark) for correct expansion of brackets
(1 mark) for the rest

(b) If ai is a solution to the equation

$$z^2+(1-i)z+(2-2i)=0$$

then, $(ai)^2+(1-i)ai+2-2i=0$

So, $-a^2+ai+a+2-2i=0$

$-a^2+a+2+i(a-2)=0+0i$

Equating real and imaginary parts, we require that

$-a^2+a+2=0$ AND $a-2=0$

$(-a+2)(a+1)=0$ $a=2$

$a=2$ or $a=-1$

(1 mark)

The result $a=2$ holds for both equations and so this is the answer required.

(1 mark)

(c) (i) $u = 1 - i$

$|u| = \sqrt{1^2 + (-1)^2}$

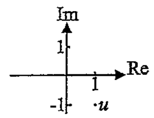
$= \sqrt{2}$ (1 mark)

$\arg u = \tan^{-1}\left(\frac{-1}{1}\right)$

$= \tan^{-1}(-1)$

Now u is a 4^{th} quadrant complex number.

So $\arg u = -\frac{\pi}{4}$ (1 mark)



(ii) $u^{12} = (1-i)^{12}$

$= \left\{\sqrt{2}cis\left(-\frac{\pi}{4}\right)\right\}^{12}$ from part (i)

$= \left(\sqrt{2}\right)^{12} cis\left(12 \times -\frac{\pi}{4}\right)$ De Moivre's Theorem **(1 mark)**

$= 64cis(-3\pi)$

$= 64cis(-\pi)$

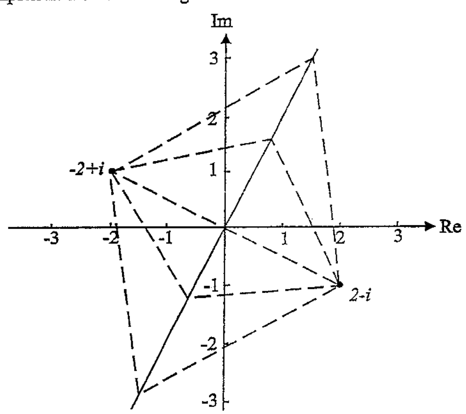
$= 64(\cos(-\pi) + i\sin(-\pi))$

$= 64(-1 + i \times 0)$

$= -64 + 0i$

(1 mark)

(d) For the inequation $|z-2+i| \le |z+2-i|$ we are looking for the locus of points for which the distance from the complex number $2 - i$ is less than or equal to the distance from the complex number $-2 + i$. Mark each of these two complex numbers on the Argand plane. Mark the midpoint of the line joining these two points. Draw a straight line, which passes through this midpoint and runs at right angles to the line joining he two complex numbers. The diagram below shows this.



This straight line gives us those points on the Argand plane for which $|z-2+i| = |z+2-i|$.

The shaded region to the right of this line gives us those points on the Argand plane for which $|z-2+i| < |z+2-i|$.

To check this, choose a point, say $2 + 0i$. Substitute this into

$|z-2+i| < |z+2-i|$

So, $|2+0i-2+i| < |2+0i+2-i|$

$|i| < |4-i|$

$\sqrt{1} < \sqrt{4^2 + (-1)^2}$

which is true. That is the point $2 + 0i$,on the right hand side of the "borderline", satisfies the inequation.

To establish the Cartesian equation of the "borderline" algebraically, we have.

$|z-2+i| = |z+2-i|$

$|x+yi-2+i| = |x+yi+2-i|$

$|x-2+i(y+1)| = |x+2+i(y-1)|$

$\sqrt{(x-2)^2 + (y+1)^2} = \sqrt{(x+2)^2 + (y-1)^2}$

$(x-2)^2 + (y+1)^2 = (x+2)^2 + (y-1)^2$

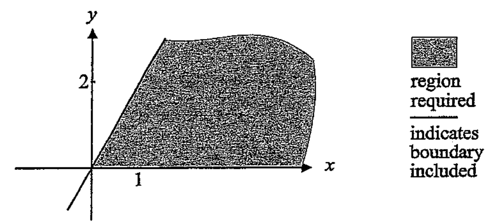
$x^2 - 4x + 4 + y^2 + 2y + 1 = x^2 + 4x + 4 + y^2 - 2y + 1$

$-8x + 4y = 0$

$y = 2x$

The inequation $\text{Im } z \ge 0$ describes all the points on the Argand plane, which lie on or above the real axis.

The required region where both inequalities hold is shown on the diagram below.



(1 mark) for boundary given by $y = 2x$

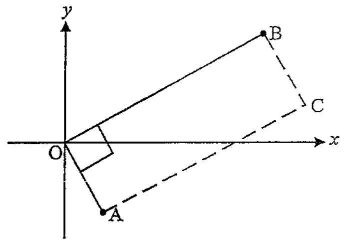
(1 mark) for boundary given by $y = 0$

(1 mark) for correct shading

(e) (i) The point A corresponds to the complex number z. By rotating OA anticlockwise about the origin through 90° and stretching it by a factor of 2 we obtain OB.
Algebraically this is equivalent to multiplying z by i (for the rotation) and by 2 (for the scale factor).
So B corresponds to the complex number $2iz$ as required.

(1 mark)

(ii)



Now $OA + OB = OC$

So, the point C corresponds to the complex number $z + 2iz$ from part (i).

So $\quad v = z + 2iz$ **(1 mark)**

Now since $z = x + yi, \quad x, y \in R$

We have $v = x + yi + 2i(x + yi)$

$= x + yi + 2ix - 2y$

$= x - 2y + i(2x + y)$

So $\bar{v} = x - 2y - i(2x + y)$

since the conjugate of a complex number is a reflection in the Real axis.

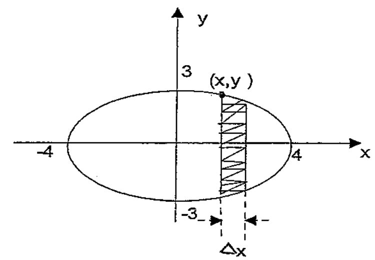
(1 mark)

Total 15 marks

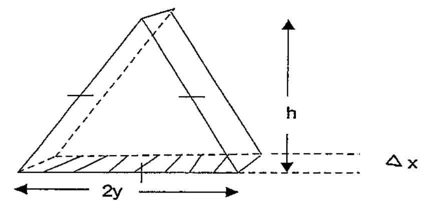
Question 4

(a) Top View: Consider a slice of thickness Δx to $\frac{x^2}{16} + \frac{y^2}{9} = 1$ at (x, y)

[1 mark]



Side View:
[1 mark]



Now area of cross sectional slice $A(x) = \frac{1}{2}.2y.2y.\sin 60^0$

$\therefore A(x) = \sqrt{3}y^2$

$= \sqrt{3}(9[1 - \frac{x^2}{16}])$ **[1 mark]**

Now volume, ΔV, of each slice $= A(x)\Delta x$

$\therefore$Total volume $= \lim_{\Delta x \to 0} \sum_{x=-4}^{4} A(x)\Delta x$

$=9\sqrt{3}\int_{-4}^{4} 1-\frac{x^2}{16}\,dx$

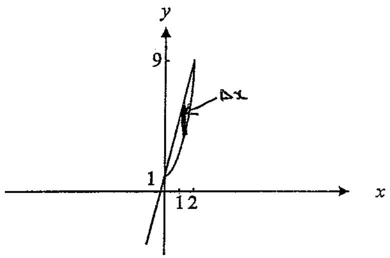
$=18\sqrt{3}\int_{0}^{4} 1-\frac{x^2}{16}\,dx$ **[1 mark]**

$=18\sqrt{3}\left[x-\frac{x^3}{48}\right]_0^4$

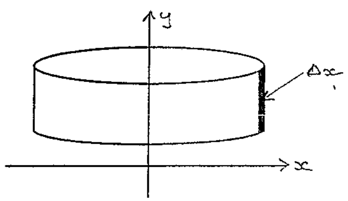
$=18\sqrt{3}\left[(4-\frac{64}{48})-0\right]$

$=48\sqrt{3}$ units^3 **[1 mark]**

(b)



Consider a slice of thickness Δx. This slice when rotated about the y axis represents a thin cylindrical shell of area $2\pi rh$. [1 mark]



$\therefore A(x) = 2\pi x[(4x+1)-(2x^2+1)]$

$= 2\pi x[4x-2x^2]$ [1 mark]

Now volume, ΔV, of each shell $= A(x)\Delta x$

$\therefore$ Total volume $= \lim_{\Delta x\to 0}\sum_{x=0}^{2} A(x)\Delta x$

So $V = 2\pi\int_0^2 (-2x^3+4x^2)\,dx$ **(1 mark)**

$= 2\pi\left[\frac{-2x^4}{4}+\frac{4x^3}{3}\right]_0^2$

$= 2\pi\left\{\frac{-2\times16}{4}+\frac{4\times8}{3}\right\}$

$= 2\pi\left(-8+\frac{32}{3}\right)$

$= 2\pi\times\frac{-24+32}{3}$

$= 2\pi\times\frac{8}{3}$

$= \frac{16\pi}{3}$ cubic units **(1 mark)**

(c) (i) The area of the annulus formed by the rotation of PQ is given by

Area $= \pi$(radius of outer edge of annulus)^2 - (radius of inner edge of annulus)^2

$= \pi\left((3-x)^2-5^2\right)$ **(1 mark)**

$= \pi\left(9-6x+x^2-25\right)$

$= \pi\left(x^2-6x-16\right)$

Now, $y = \frac{-8}{x+2}$

So, $x+2 = \frac{-8}{y}$

$x = \frac{-8}{y}-2$

$x^2 = \frac{64}{y^2}+\frac{32}{y}+4$

So Area $= \pi\left(\frac{64}{y^2}+\frac{32}{y}+4+\frac{48}{y}+12-16\right)$

$= \pi\left(\frac{64}{y^2}+\frac{80}{y}\right)$

$= 16\pi\left(\frac{4}{y^2}+\frac{5}{y}\right)$ square units **(1 mark)**

(ii) For $0 \le y \le 4$, the area of the annulus created by the rotation is given by

$\text{Area} = \pi(7^2 - 5^2)$ **(1 mark)**

$= 24\pi$ square units

So required volume is

$V = 16\pi \int_4^8 \left(\frac{4}{y^2} + \frac{5}{y}\right) dy + 24\pi \int_0^4 1\,dy$ **(1 mark)**

$= 16\pi \left[\frac{4y^{-1}}{-1} + 5\ln y\right]_4^8 + 24\pi[y]_0^4$ **(1 mark)**

$= 16\pi\left\{\left(\frac{-4}{8} + 5\ln 8\right) - \left(\frac{-4}{4} + 5\ln 4\right)\right\} + 24\pi(4-0)$

$= 16\pi\left(-\frac{1}{2} + 5\ln 8 + 1 - 5\ln 4\right) + 96\pi$

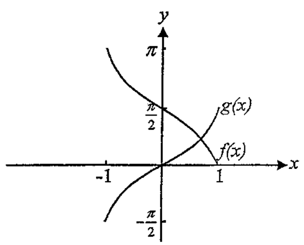
$= 16\pi\left(\ln\frac{8^5}{4^5} + \frac{1}{2}\right) + 96\pi$

$= 16\pi\left(\ln 32 + \frac{1}{2} + 6\right)$

$= 16\pi\left(\frac{13}{2} + \ln 32\right)$ cubic units **(1 mark)**

Question 5

(a) (i)



(1 mark)

(ii) Let $y = f(x) + g(x)$

So $y = \cos^{-1} x + \sin^{-1} x$

$\frac{dy}{dx} = \frac{-1}{1+x^2} + \frac{1}{1+x^2}$

$= 0$

Now, $\int \frac{dy}{dx} dx = c$ where c is a constant.

So, $y = c$

So, $\cos^{-1} x + \sin^{-1} x = c$

Since the function is constant over the domain $0 \le x \le 1$ choose any x value in this domain.

Let $x = 0$, say,

Then $y = \cos^{-1} 0 + \sin^{-1} 0$

$= \frac{\pi}{2}$

So, $f(x) + g(x) = \frac{\pi}{2}$ **(1 mark)**

(iii)

So, $\int_{-1}^{1} (\cos^{-1} x + \sin^{-1} x) dx$

$= \int_{-1}^{1} \frac{\pi}{2} dx$

$= 2 \times \frac{\pi}{2}$

$= \pi$

(1 mark)

(b) (b) (i) $x^2+\frac{y^2}{4}=1$ so, $a=2$ and $b=1$

Now $b^2=a^2(1-e^2)$

$1=4(1-e^2)$

$e=\frac{\sqrt{3}}{2}$ **(1 mark)**

The foci are located at $(0,\pm ae)$, that is at $(0,\sqrt{3})$ and $(0,-\sqrt{3})$. **(1 mark)**

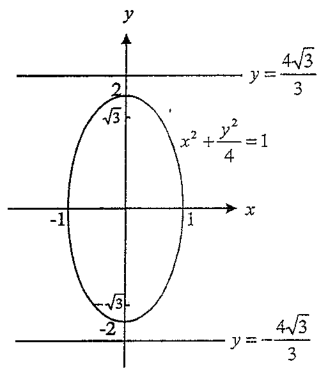
(ii) The length of the major axis is 4 units and the length of the minor axis is 2 units. **(1 mark)**

(iii) The equations of the directrices are given by $y=\pm\frac{a}{e}$

$=\pm 2\div\frac{\sqrt{3}}{2}$

$=\pm\frac{4\sqrt{3}}{3}$ **(1 mark)**

(iv)



(1 mark)

(c) (i) $p(x)=(x-\alpha)^3h(x)$ $h(\alpha)\neq 0$ **(1 mark)**

So, $p'(x)=3(x-\alpha)^2h(x)+(x-\alpha)^3h'(x)$

$=(x-\alpha)^2(3h(x)+(x-\alpha)h'(x))$

So α is a root of $p'(x)=0$ and has a multiplicity of 2. **(1 mark)**

(ii) The polynomial $q(x)$ has a quadratic factor of $x^2+2x+1=(x+1)^2$.
So, $q(x)=0$ has a root of -1 with a multiplicity of 2 or more. Therefore $q'(x)=0$ has a root of -1.

So, $q(-1)=1-a+b-1+2-1=0$

$b=a-1$

and $q'(x)=6x^5+5ax^4+4bx^3-2x-2$ **(1 mark)**

so, $q'(-1)=-6+5a-4b+2-2=0$

$5a-4b=6$

So, $a=2$ and $b=1$ **(1 mark)**

(iii) $r(x)=\int r'(x)\,dx$

$=x+\frac{x^2}{2\times 1!}+\frac{x^3}{3\times 2!}+\ldots+\frac{x^{n+1}}{(n+1)n!}+c$

Since $r(0)=1$,

$r(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\frac{x^{n+1}}{(n+1)!}$ **(1 mark)**

Now let k be a root of $r(x)=0$ and $r'(x)=0$.

So, $r(k)=0$ and $r'(k)=0$.

So, $1+k+\frac{k^2}{2!}+\ldots+\frac{k^n}{n!}+\frac{k^{n+1}}{(n+1)!}=1+k+\frac{k^2}{2!}+\ldots+\frac{k^n}{n!}$

So, $\frac{k^{n+1}}{(n+1)!}=0$

This is only true if $k=0$. **(1 mark)**

But if $k=0$ is a double root, then $r'(k)=0$ and $r(k)=0$. However $r'(k)=1$ and $r(k)=1$ and so k is not a double root.

(1 mark)

Total 15 marks

Question 6

(a) (i) $x=\sec\theta$ $\quad$ $y=\tan\theta$

$\frac{dx}{d\theta}=-1(\cos\theta)^{-2}\times-\sin\theta$ $\quad$ $\frac{dy}{d\theta}=\sec^2\theta$

$=\frac{\sin\theta}{\cos^2\theta}$

So, $\frac{dy}{dx}=\frac{\sec^2\theta\cos^2\theta}{\sin\theta}$

$=\frac{1}{\sin\theta}$

So, the gradient of the tangent at P is $\frac{1}{\sin\theta}$.

So the gradient of the normal at P is $-\sin\theta$. **(1 mark)**

So the equation of the normal is $y-\tan\theta=-\sin\theta(x-\sec\theta)$

$y=-\sin\theta x+2\tan\theta$ as required.

(1 mark)

(ii) If S lies on the asymptote with equation $y=x$ then S is the point $(\sec\theta,\sec\theta)$.

(1 mark)

T is the point $(\sec\theta,0)$.

From (i), $y=-\sin\theta x+2\tan\theta$

When $y=0$, $x=\frac{2\tan\theta}{\sin\theta}$

$=2\sec\theta$

So R is the point $(2\sec\theta,\ 0)$

Since RST is a right angled triangle, we have

$ST^2+RT^2=SR^2$

Now $ST=RT=\sec\theta$

So, $2RT^2=SR^2$

$SR=\sqrt{2}RT$ as required. **(1 mark)**

(iii) STR is an isosceles right angled triangle($ST=\sec\theta, RT=\sec\theta$ from part (ii)) and TU will be a perpendicular bisector of SR and will have a gradient of 1.

So U is the midpoint of SR. (1 mark)

So, $U=\left(\frac{\sec\theta+2\sec\theta}{2},\frac{\sec\theta}{2}\right)$

$=\left(\frac{3\sec\theta}{2},\frac{\sec\theta}{2}\right)$ (1 mark)

(iv) For FU to be a perpendicular bisector, F must be coincident with T.

F is the point $(ae,0)$ or $(-ae,0)$ where $b^2=a^2(e^2-1)$ and so $e=\sqrt{2}$ **(1 mark)**

So, $\sec\theta=\sqrt{2}$ $\quad$ or $\quad$ $\sec\theta=-\sqrt{2}$

$\cos\theta=\frac{1}{\sqrt{2}}$ $\quad$ $\cos\theta=-\frac{1}{\sqrt{2}}$

$\theta=\frac{\pi}{4},\frac{7\pi}{4}$ $\quad$ $\theta=\frac{3\pi}{4},\frac{5\pi}{4}$ (1 mark)

(b) (i) We have $z^{10}-1=0$

$z^{10}=1$

Let $z=rcis\theta$ $\quad$ $(rcis\theta)^{10}=cis\,0$

$r^{10}cis10\theta=cis\,0$ $\quad$ De Moivre's Theorem

So $r=1$ and $\cos10\theta+i\sin10\theta=\cos0+i\sin0$

So $\cos10\theta=\cos0$ and $\sin10\theta=\sin0$

So $\quad$ $10\theta=0+2k\pi$ $\quad$ k is an integer

$\theta=\frac{2k\pi}{10}$

$\theta=\frac{k\pi}{5}$

So $\quad$ $z=\cos\frac{k\pi}{5}+i\sin\frac{k\pi}{5}$

(1 mark)

Now, $\omega=\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}$

Also, $\omega^2=\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}$

$\omega^3=\cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5}$

.
.
.
.

$\omega^k=\cos\frac{k\pi}{5}+i\sin\frac{k\pi}{5}$

So, all the solutions of the equation $z^{10}-1=0$ may be expressed as ω^k where k is an integer.

(1 mark)

(ii) Method 1

If we consider the left hand side of the equation as a geometric series with $a=\omega$ and $r=\omega$ then,

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$\text{So, } S_{10} = \frac{\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)\left(1-\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)^{10}\right)}{1-\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)}$$

$$= \frac{\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)(1-(\cos 2\pi+i\sin 2\pi))}{1-\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)}$$

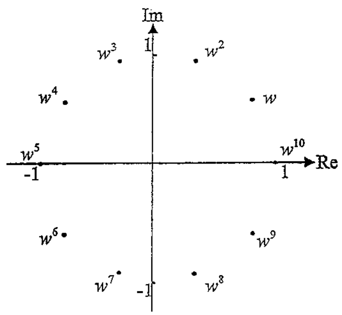
$$= \frac{\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)(1-1)}{1-\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)}$$

$$= 0$$

(2 marks)

Method 2

Geometrically, we can consider the sum of all the 10th roots of unity.



These ten complex numbers are spaced evenly around a circle of radius 1 unit. If we take these complex numbers two at a time, for example ω and ω^6, and apply the parallelogram rule for the addition of complex numbers we come up with

$\omega^1+\omega^6+\omega^2+\omega^7+\omega^3+\omega^8+\omega^4+\omega^9+\omega^5+\omega^{10}$

$=0+0+0+0+0$

$=0$

(iii) Let $\alpha=\omega+\omega^4$

Let β equal the other root.

Since the coefficients of the quadratic equation are real, then according to the conjugate root theorem, the roots must appear in conjugate pairs and so $\beta=\bar{\alpha}$.

(1 mark)

From the diagram shown in the solutions to part (ii),

ω and ω^9 are complex conjugates and ω^4 and ω^6 are complex conjugates

Hence, $\beta=\omega^6+\omega^9$ **(1 mark)**

Question 7

(a) (i) In $\Delta ACP, \angle CAP = 90^\circ$ (A tangent to a circle meets the radii (AC) at 90°).

Similarly in $\Delta CEP, \angle CEP = 90^\circ$

So in quadrilateral $ACEP$, the opposite angles $\angle CAP$ and $\angle CEP$ add to 180° and so the quadrilateral is cyclic. **(1 mark)**

(ii) In $\Delta CEP, \angle CEP = 90^\circ$ (A tangent to a circle meets the radii at 90°).

In $\Delta CAP, \angle CAP = 90^\circ$ (same reason)

Also, $CE = CA$ (both are radii of the circle ℓ)

CP is a common side.

So, ΔCEP is congruent to ΔCAP. (each have a right angle, the hypotenuse is shared and there is one other pair of sides equal in length)

Let $\angle CPE = \theta$ -(A)

So, $\angle CPA = \theta$ and $\angle APE = 2\theta$ (corresponding pairs of angles in congruent triangles are equal)

In $\Delta DEF, \angle DEF = 90^\circ$ (vertically opposite angles are equal).

Also, $\angle EDF = \angle APE = 2\theta$ (alternate angles since AP is parallel to DF)

So, $\cos 2\theta = \dfrac{DE}{DF}$

Also, from (A), $\cos\theta = \dfrac{EP}{CP}$.

Now, $\cos 2\theta = 2\cos^2\theta - 1$ (double angle formula)

So, $\dfrac{DE}{DF} = \dfrac{2EP^2}{CP^2} - 1$ **(1 mark)**

$$\text{So, } \frac{DE}{DF} = \frac{2EP^2 - CP^2}{CP^2}$$

$$= \frac{2EP^2 - (CE^2 + EP^2)}{CP^2} \quad \text{(Pythagoras' Theorem)}$$

$$= \frac{EP^2 - CE^2}{CP^2}$$

$$\text{So, } DE = \frac{DF(EP^2 - CE^2)}{CP^2}$$ **(1 mark)**

(iii) Now $\Delta ACP \equiv \Delta CEP$ from part (ii)

Now, let $\angle APC = \theta$

So, $\angle CPE = \theta$ (1 mark)

Also, let $\angle ABC = \alpha$

So, $\angle CBD = 180^\circ - \alpha$

In ΔABP we have,

$$\frac{\sin\theta}{AB} = \frac{\sin\alpha}{AP}$$

So, $$\frac{\sin\theta}{\sin\alpha} = \frac{AB}{AP}$$

In ΔBDP we have,

$$\frac{\sin\theta}{BD} = \frac{\sin(180^\circ - \alpha)}{DP}$$

Now $\sin(180^\circ - \alpha) = \sin\alpha$

So $$\frac{\sin\theta}{BD} = \frac{\sin\alpha}{DP}$$

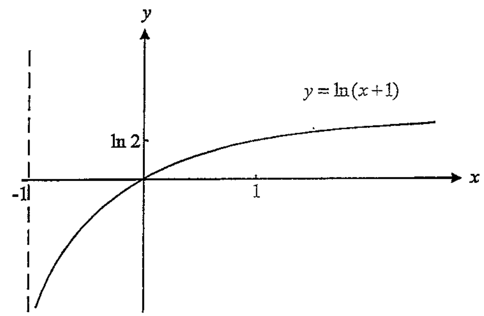
$$\frac{\sin\theta}{\sin\alpha} = \frac{BD}{DP}$$

So, $$\frac{AB}{AP} = \frac{BD}{DP}$$

and so $$\frac{AB}{BD} = \frac{AP}{DP}$$

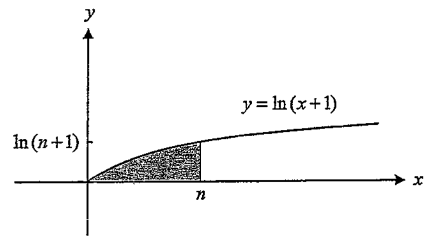
as required. (1 mark)

(b) (i)

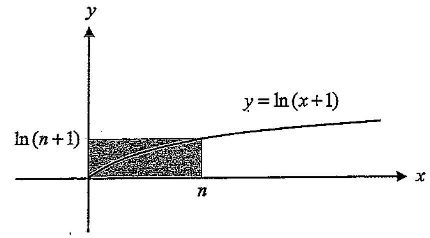


(1 mark)

(ii) The region described by $\int_0^n \ln(x+1)dx, n = 1,2,3....$ is the region bounded by the graph of $y = \ln(x+1)$, the x-axis and the line with equation $x = n$, $n = 1,2,...$



The region described by $n\ln(n+1)$ is a rectangle with side length of n units and $\ln(n+1)$ units.



Clearly from the graphs we see that $\int_0^n \log_e(x+1)dx < n\log_e(n+1)$, $n = 1,2,3...$

(1 mark)

(iii) Consider $\int_0^n \ln(x+1)dx$.

The integration by parts formula states that $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$.

Now, let $u=\ln(x+1)$ and $\frac{dv}{dx}=1$

so $\frac{du}{dx}=\frac{1}{x+1}$ so $v=x$

So, $\int_0^n \ln(x+1)\cdot 1dx$

$=[x\ln(x+1)]_0^n - \int_0^n x\cdot\frac{1}{x+1}dx$ (1 mark)

$=n\ln(n+1)-0-\int_0^n \frac{x}{x+1}dx$

$=n\ln(n+1)-\int_1^{n+1}(u-1)\cdot\frac{1}{u}\cdot\frac{du}{dx}dx$

$=n\ln(n+1)-\int_1^{n+1}(1-u^{-1})du$ (1 mark)

$=n\ln(n+1)-[u-\ln u]_1^{n+1}$

$=n\ln(n+1)-\{n+1-\ln(n+1)-(1-\ln 1)\}$

$=n\ln(n+1)-\{n+1-\ln(n+1)-1\}$

$=n\ln(n+1)-n+\ln(n+1)$

$=\ln(n+1)^n(n+1)-n$

$=\ln(n+1)^{n+1}-n$ **(1 mark)**

$u=x+1$

$\frac{du}{dx}=1$

$x=u-1$

since $x=n$

$u=n+1$

since $x=0$

$u=1$

(iv) From part (iii) we have $\int_0^n \ln(x+1)dx = \ln(n+1)^{n+1}-n$

From part (ii) we have $\int_0^n \ln(x+1)dx < n\ln(n+1)$

So $\ln(n+1)^{n+1}-n < n\ln(n+1)$

$n\ln(n+1)+\ln(n+1)-n < n\ln(n+1)$

So, $\ln(n+1)<n$ as required **(1 mark)**

(v)

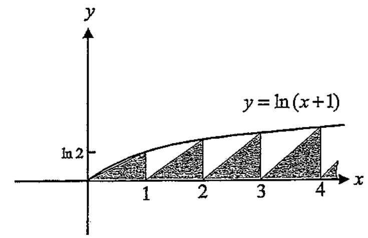
Now $\sum_{k=1}^{n}\frac{1}{2}\ln(k+1)=\frac{1}{2}\ln 2+\frac{1}{2}\ln 3+\ldots+\frac{1}{2}\ln n+\frac{1}{2}\ln(n+1)$

$=\frac{1}{2}\ln(2\times 3\times.....\times n\times(n+1))$

$=\frac{1}{2}\ln(n+1)!$

(1 mark)

(vi)



The shaded region indicated in the diagram above can be described by $\sum_{k=1}^{n}\frac{1}{2}\ln(k+1)$.

Clearly, $\sum_{k=1}^{n}\frac{1}{2}\ln(k+1) < \int_0^n \ln(x+1)dx$ by comparing the graphs. **(1 mark)**

So $\frac{1}{2}\ln(n+1)! < \int_0^n \ln(x+1)dx$ from part (v)

and $\frac{1}{2}\ln(n+1)! < \ln(n+1)^{n+1}-n$ from part (iii)

$\ln(n+1)!-2\ln(n+1)^{n+1} < -2n$

$\ln\frac{(n+1)!}{(n+1)^{2n+2}} < -2n$

$e^{-2n} > \frac{(n+1)!}{(n+1)^{2n+2}}$ **(1 mark)**

So, $(n+1)! < e^{-2n}(n+1)^{2n+2}$

$(n+1)! < (e^{-1})^{2n}(n+1)^{2n}(n+1)^2$

$(n+1)! < \left(\frac{n+1}{e}\right)^{2n}(n+1)^2$

$n! < \left(\frac{n+1}{e}\right)^{2n}(n+1)$ as required

(1 mark)

Total 15 marks

Question 8

(a) (i) (α)

$$(a+b)^2-4ab=a^2+2ab+b^2-4ab$$
$$=(a-b)^2$$
$$>0 \text{ (as } a>b)$$

[1 mark]

$$\therefore (a+b)^2>4ab$$
$$\therefore a+b>2\sqrt{ab}$$

[1 mark]

(β) $b^2-a^2-2\sqrt{ab}(b-a)=(b-a)(b+a)-2\sqrt{ab}(b-a)$

$=(b-a)[b+a-2\sqrt{ab}]$ **[1 mark]**

<0 (as $b-a<0$ and $b+a-2\sqrt{ab}>0$ from (α))

$\therefore b^2-a^2<2\sqrt{ab}(b-a)$ **[1 mark]**

(ii) We have, $\sqrt{a}(b-a)+\sqrt{c}(c-b)>\dfrac{c^2-a^2}{2\sqrt{b}}$

So, $2\sqrt{ab}(b-a)+2\sqrt{bc}(c-b)>c^2-a^2$

So, $c^2-a^2<2\sqrt{ab}(b-a)+2\sqrt{bc}(c-b)$ **(1 mark)**

From part (i) we know that $a>b$ and $b^2-a^2<2\sqrt{ab}(b-a)$ -(A)

Suppose that $b>c$, then, following the pattern, we have

$c^2-b^2<2\sqrt{bc}(c-b)$ -(B)

(1 mark)

Adding (A) and (B) gives $c^2-a^2<2\sqrt{ab}(b-a)+2\sqrt{bc}(c-b)$ which was given.

So, we know now that $b>c$. **(1 mark)**

So if $b>c$ and $a>b$ (from **part** (i)) then $a>c$ as required. **(1 mark)**

(b) (i) $h(n)=n^4+6n^2+9$

$\therefore h(n+2)=(n+2)^4+6(n+2)^2+9$

$\therefore h(n+2)-h(n)=(n+2)^4+6(n+2)^2+9-(n^4+6n^2+9)$ [1 mark]

$=(n+2)^4-n^4+6[(n+2)^2-n^2]$

$=[(n+2)^2-n^2][(n+2)^2+n^2]+6[(n+2)-n][(n+2)+n]$

$=[4n+4][2n^2+4n+4]+6[2][2n+2]$ [1 mark]

$=[4n+4][2n^2+4n+10]$

$=8[n+1][n^2+2n+5]$ [1 mark]

(ii) Step 1: When $n=1$ $h(1)=1^4+6.1^2+9=16$, which is divisible by 8.

$\therefore$ it is true for $n=1$. **[1 mark]**

Step 2: Assume it is true for $n=k$ and prove it is true for $n=k+2$.

i.e. $\dfrac{h(k)}{8}=M$ (where M is an integer)

$\therefore h(k)=8M$ ....(1)

Now from part (i) $h(k+2)-h(k)=8(k+1)(k^2+2k+5)$

Furthermore, if k is odd then $k+1$ is even.

Let $k+1=2J$ (where J is an integer).

$\therefore h(k+2)-h(k)=8(2J)(k^2+2k+5)$

$\therefore h(k+2)=16J(k^2+2k+5)+h(k)$

$\therefore h(k+2)=16J(k^2+2k+5)+8M$ (substituting (1))

$\therefore h(k+2)=8[2J(k^2+2k+5)+M]$, which is divisible by 8.

$\therefore$ if it is true for $n=k$ so it is true for $n=k+2$. **[2 marks]**

Step 3: It is true for $n=1$ and so it is true for $n=1+2=3$.

It is true for $n=3$ and so it is true for $n=3+2=5$

and so on for all positive odd integral values of n. **[1 mark]**