CRANBROOK
SCHOOL

Term 3, 2009

Year 12 Mathematics

Trial HSC Examination

Friday July 31, 2009

Time Allowed: 3 hours, plus 5 minutes reading time

Total Marks: 120

There are 10 questions, all of equal value.
Submit your work in ten 4 Page booklets.
All necessary working should be shown in every question.
Full marks may not be awarded if work is careless or badly arranged.
Board of Studies approved calculators may be used.
A list of standard integrals is attached to the back of this paper.

STANDARD INTEGRALS

$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, n \neq -1;\ x \neq 0,$ if $n < 0$

$\int \frac{1}{x}\,dx = \ln x,\ x > 0$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$

$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$

$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$

$\int \sec^2 ax\,dx, = \frac{1}{a}\tan ax,\ a \neq 0$

$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$

$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln(x+\sqrt{x^2-a^2}),\ x > a > 0$

$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln(x+\sqrt{x^2+a^2})$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1 (12 marks) Use a SEPARATE writing booklet: **Marks**

(a) The points P and Q have coordinates (3, 8) and (9, 2) respectively. Find the gradient of PQ. **1**

(b) Evaluate : $e^{-2 \cdot 5}$ correct to 3 significant figures. **2**

(c) Solve: $|3x-2| \leq 7$ **2**

(d) Differentiate with respect to x: $\frac{x^2}{2} - \frac{1}{x}$ **2**

(e) Find the primitive: $\frac{1}{x-1}$ **1**

(f) Given $z = \frac{x+y}{xy}$, change the subject of the formula to y. **2**

(g) Write the exact value of $\cos 510^0$. **2**

Question 2 (12 marks) Use a SEPARATE writing booklet: **Marks**

(a) Find the equation of the tangent to the curve $y = \log_e\left(\frac{x}{2}\right)$ at the point (2, 0). **2**

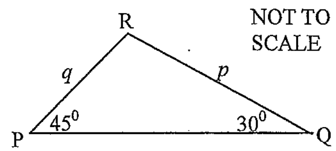
(b) Differentiate with respect to x:

(i) $x \cos x$ **2**

(ii) $\frac{\log_e x}{x^3}$ **2**

(c)

In the diagram, PQR is a triangle where angle RPQ $= 45^0$ and angle RQP $= 30^0$.



Find the exact value of $\frac{p}{q}$. **2**

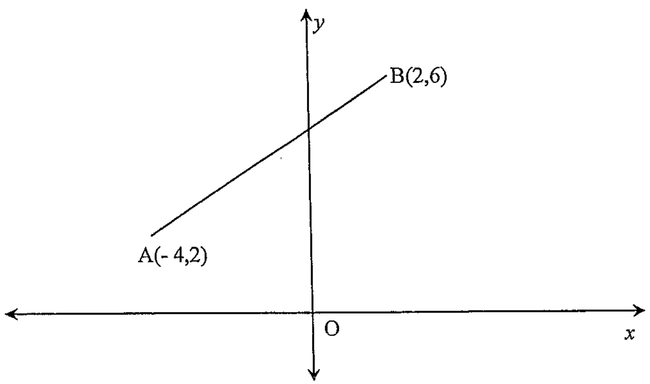
(d) Evaluate:

(i) $\int_0^1 \frac{dx}{4+3x}$ **2**

(ii) $\int_1^2 \left(\frac{1}{x^3} + x^2\right) dx$ **2**

Question 3 (12 marks) Use a SEPARATE writing booklet: **Marks**

(a)



The diagram shows two points A(−4, 2) and B(2, 6) on the number plane.
Copy the diagram.

(i) Find the coordinates of the mid point M of AB. **1**

(ii) Show that the equation of the perpendicular bisector of AB is $3x+2y-5=0$. **2**

(iii) Find the coordinates of C that lies on the X- axis and is equidistant from A and B. **1**

(iv) The point D lies on the intersection of the line $y=1$ and $3x+2y-5=0$. Find the coordinates of D and mark the position of D on your diagram. **2**

(v) Find size of $\angle ABC$ to the nearest degree. **2**

(b) Find the largest four digit number to be found in the sequence 1, 4, 7, 10, ... **2**

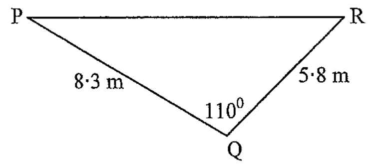
(c) Given that $\dfrac{d^2x}{dt^2}=e^{-t}+10$ and $\dfrac{dx}{dt}=1$ at t =0 and $x=0$ at t =0.
Find an expression for x. **2**

Question 4 (12 marks) Use a SEPARATE writing booklet: **Marks**

(a) Solve: $2^{2x}-9\times2^{x}+8=0$ **2**

(b) Find all the values of θ, where $0^0 \le \theta \le 360^0$, that satisfy the equation $\sin\dfrac{\theta}{2}-\dfrac{\sqrt{3}}{2}=0$ **2**

(c)



NOT TO SCALE

In the diagram PQR is a triangle where PQ = 8·3 m, QR=5·8 m and $\angle$PQR= 110^0.

(i) Find the length of PR correct to one decimal place. **2**

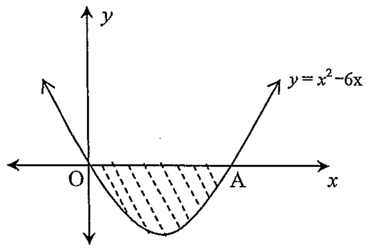
(ii) Find the size of the smallest angle of the triangle.
Give your answer to the nearest degree. **2**

(d) Jasmin invests $5 000 in a bank that pays interest at 5·25% p.a. compounded annually. What will be the value of her investment at the end of 15 years? **2**

(e) In a colony of bacteria each divides into two, every hour. How many bacteria will be produced from a single bacterium if the rate of division continues for 20 hours? **2**

Question 5 (12 marks) Use a SEPARATE writing booklet:

(a) **Marks**



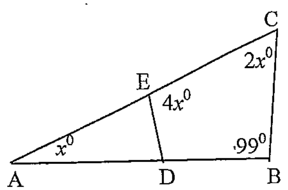
The above diagram shows the graph of the function $y=x^2-6x$.

(i) Find the coordinates of the point A where the curve crosses the x- axis. **1**

(ii) Find the area of the shaded region contained by the curve and the x- axis. **2**

(iii) Write a pair of inequalities that specify the shaded region. **1**

(b)



Use the information in the diagram to

(i) find the value of x. Give reasons. **2**

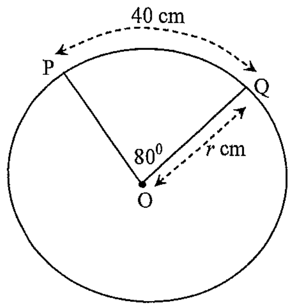
(ii) find the size of $\angle$BDE. Give reasons. **1**

(c) Consider the parabola $x^2+2x+12y-23=0$,

(i) Find the coordinates of its vertex. **2**

(ii) Find the coordinates of its focus. **1**

(d)



In the given circle on the left, the length of the arc PQ which subtends 80^0 at the centre of the circle is 40 cm.
Find the length of the radius (r) correct to one decimal place. **2**

Question 6 (12 marks) Use a SEPARATE writing booklet: **Marks**

(a) Consider the function $y=\sqrt{4-x^2}$.

(i) State its domain. **1**

(ii) Sketch the graph **1**

(b) The gradient function of a curve is given by $f'(x)=(3x-4)(x-4)$ and the curve $y=f(x)$ passes through (1,9).

(i) Find the equation of the curve $y=f(x)$. **2**

(ii) Find any stationary points and their nature **2**

(iii) Sketch the curve $y=f(x)$ clearly labelling turning points. **2**

(c) Consider the geometric series: $1+(5-\sqrt{a})+(5-\sqrt{a})^2+(5-\sqrt{a})^3+----$

(i) Find the values of a for which this geometric series has a limiting sum. **2**

(ii) Find the limiting sum of the series given that a is 20. Write your answer with a rational denominator. **2**

Question 7 (12 marks) Use a SEPARATE writing booklet: **Marks**

(a) A rain water tank which is full is drained so that at time 't' minutes, the volume of water V in litres is given by

$V=500\left(1-\dfrac{t}{60}\right)^2$ for $0\le t\le 60$.

(i) How much water was initially in the tank? **1**

(ii) After how many minutes was the tank half full? **2**

(iii) At what rate was the water draining when the time is 58 min. **2**

(b) Given that $x^2-(2+k)x+3k=0$, find k if:

(i) The sum of the roots is 5 **2**

(ii) The product of the roots is 4 times the sum of the roots **2**

(c) The following table gives values of $f(x)=x\log_e x$

x	1	2	3	4	5
$f(x)$	0	1·39	3·30	5·55	8·05

Use Simpson's Rule using these five functional values to find an approximate value of $\int_1^5 x\log_e x\,dx$. **3**

Question 8 (12 marks) Use a SEPARATE writing booklet: **Marks**

(a) A particle moves in a straight line such that at time t seconds its distance x metres from a fixed point O on the line is given by $x = 1 + \cos 2t$

(i) Sketch the graph of x as a function of t for $0 \le t \le 2\pi$. **2**

(ii) Using your graph or otherwise, find times and positions when the particle is at rest between $0 \le t \le \pi$. **2**

(iii) Describe the motion completely. **2**

(b) A condenser discharges at a rate proportional to the charge present. i.e. $\frac{dC}{dt} = -kC$, where C is the charge at time t seconds. When $t = 0$, $C = 90$. The charge reduces from 90 to 20 in 10 seconds,

(i) Show that $C = Ae^{-kt}$ satisfies the equation $\frac{dC}{dt} = -kC$ **1**

(ii) Find k **2**

(iii) What is the charge after 5 seconds? **1**

(iv) At what time does the charge reach 60? **2**

Question 9 (12 marks) Use a SEPARATE writing booklet: **Marks**

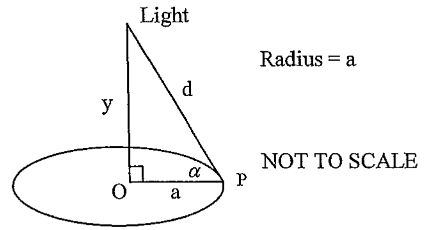
(a) John and Helen are two farmers. They borrow \$400 000 from a bank. They make monthly repayments and the interest is 6% p.a. compounded monthly. The loan is for 20 years. Because of a drought the bank allow them to begin repaying the loan at the end of the fourth month. Let A_n be the amount owing at the end of n months and m the monthly repayments.

(i) Show that $A_5 = 400000(1.005)^5 - 2.005m$. **2**

(ii) Find the monthly repayment m. **3**

(b) Solve the equation $2\log x = \log(2x + 8)$ **2**

(c) A light is to be placed over the centre of a circle. The intensity (I) of the light varies as the sine of the angle (α) at which the rays strike the illuminated surface, divided by the square of the distance (d) from the light i.e. $I = \frac{k \sin \alpha}{d^2}$ where k is a constant.



(i) Show that $I = \frac{ky}{(y^2 + a^2)^{\frac{3}{2}}}$ **2**

(ii) Find the best height for a light to be placed over the centre of a circle in order to provide maximum illumination to the circumference. **3**

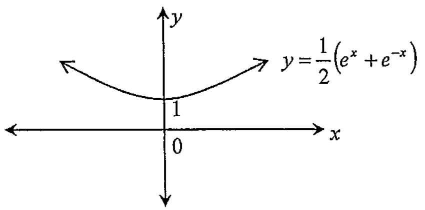
Question 10 (12 marks) Use a SEPARATE writing booklet: **Marks**

(a) Find the value(s) of k for which $kx^2 - 6x + 2$ is positive definite **2**

(b) For what values of p does the equation $\sin x = px$ have a solution in the domain $0 \le x \le \pi$ **2**

(c) The sketch of the curve catenary $y = \frac{1}{2}\left(e^x - e^{-x}\right)$ is given below.

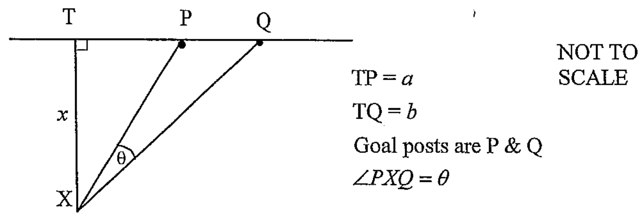
The catenary is the shape obtained when a chain or rope is strung between two points.



(i) Calculate the area enclosed between the x – axis and the ordinates $x = -3$ to 3. **2**

(ii) Calculate the volume of the solid generated when the curve $y = \frac{1}{2}\left(e^x + e^{-x}\right)$ is rotated about the x- axis between the ordinates $x =$ -3 and $x = 3$. **3**

(d) A rugby league try is scored outside the posts at point T. The conversion attempt will be kicked from point X at a distance x metres back from the goal line.



Using the formula, $\tan(\beta - \alpha) = \dfrac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha}$ or otherwise

show that $\tan\theta = \dfrac{x(b-a)}{x^2 + ab}$ where a and b are the distances from the try position T to each post. **3**

2U Cranbrook 2009
Trial Solutions.

Question 1. (BMM)

(a) $M_{PQ} = \frac{2-8}{9-3} = -1$ ✓

(b) $e^{-2.5} = 0.082084998$ ✓
$= 0.0821$ (3 sig fig) ✓

(c) $|3x-2| \le 7$

$3x-2 \le 7$ $\quad$ $-3x+2 \le 7$

$3x \le 9$ $\quad$ $-3x \le 5$

$x \le 3$ ✓ $\quad$ $x \ge -\frac{5}{3}$ ✓

(d) $\frac{x^2}{2} - \frac{1}{x} = \frac{1}{2}x^2 - x^{-1}$ ✓

$\frac{d}{dx} = x + \frac{1}{x^2}$ ✓

(e) $\int \frac{1}{x-1}\,dx$

$= \ln(x-1) + c$ ✓ $\quad$ * marks lost if no (+c)

(f) $z = \frac{x+y}{xy}$

$zxy = x+y$ ✓

$zxy - y = x$

$y(zx-1) = x$

$y = \frac{x}{zx-1}$ ✓

(g) $\cos 510°$

- in Q2
- cos -ve in Q2
- $360 + (180-\theta) = 510$

$\theta = 30°$

$\therefore \cos 510° = -\cos 30°$ ✓

$= -\frac{\sqrt{3}}{2}$ ✓

Question 2. (BMM)

(a) $y = \log_e\left(\frac{x}{2}\right)$

$y = \ln x - \ln 2$

$y' = \frac{1}{x}$

$m = \frac{1}{2}$ ✓

$y - 0 = \frac{1}{2}(x-2)$

$2y = x - 2$

$x - 2y - 2 = 0$ ✓

(b)(i) $x\cos x$

$\frac{d}{dx} = \cos x - x\sin x$ ✓✓

(ii) $\frac{\ln x}{x^3}$

$\frac{d}{dx} = \frac{x^2 - 3x^2\ln x}{x^6}$ ✓

$= \frac{x^2(1-3\ln x)}{x^6}$

$= \frac{1-3\ln x}{x^4}$ ✓

(c) $\frac{p}{\sin 45°} = \frac{q}{\sin 30°}$

$\frac{p}{q} = \frac{\sin 45°}{\sin 30°}$ ✓

$= \frac{1}{\sqrt{2}} \div \frac{1}{2}$

$= \frac{2}{\sqrt{2}}$

$= \sqrt{2}$ ✓

(d)(i) $\int_0^1 \frac{1}{4+3x}\,dx$

$= \frac{1}{3}\left[\ln(4+3x)\right]_0^1$ ✓

$= \frac{1}{3}(\ln 7 - \ln 4)$

$= \frac{1}{3}\ln\left(\frac{7}{4}\right)$ ✓

or: 0.18653...

(ii) $\int_1^2 \frac{1}{x^3} + x^2\,dx$

$= \int_1^2 x^{-3} + x^2\,dx$

$= \left(-\frac{1}{2x^2} + \frac{x^3}{3}\right)_1^2$

$= 2\frac{17}{24}$

or: $\frac{65}{24}$ $\quad$ or: 2.708

Question 3 (BMM)

(a)(i) $M_{AB} = \left(\frac{-4+2}{2}, \frac{2+6}{2}\right)$

$= (-1, 4)$ ✓

(ii) $m_{AB} = \frac{2}{3}$

$\therefore m_{\perp} = \frac{-3}{2}$ ✓

$y - 4 = \frac{-3}{2}(x+1)$

$3x + 2y - 5 = 0$ ✓

(iii) lies on $3x+2y-5=0$

$\therefore$ sub $y=0$.

$3x + 2(0) - 5 = 0$

$3x = 5$

$x = \frac{5}{3}$ ✓

$\therefore C\left(\frac{5}{3}, 0\right)$

(iv) $3x + 2(1) - 5 = 0$

$3x = 3$

$x = 1$ ✓

$\therefore D(1, 1)$ ✓

(v) $m_{MC} = \frac{-3}{2}$ → $\tan\theta = \frac{-3}{2}$

$\theta = 124°$ ✓

$m_{BC} = 18$ → $\tan\theta = 18$

$\theta = 87°$

$\therefore \angle ACB = 2(124 - 87) = 74°$

Δ is isoc. $\therefore \angle ABC = (180 - 74) \div 2$

$= 53°$. ✓

(b) $a = 1$ $d = 3$

$a + (n-1)d < 10000$

$1 + 3n - 3 < 10000$

$3n < 10002$

$n < 3334$

$\therefore n = 3333$ ✓

$T_{3333} = 9997$ ✓

(c) $\frac{dx}{dt} = \int e^{-t} + 10 \, dt$

$= -e^{-t} + 10t + C$

$1 = -e^{0} + 10(0) + C$

$1 = -1 + C$

$C = 2$

$\therefore \frac{dx}{dt} = -e^{-t} + 10t + 2$ ✓

$x = \int -e^{-t} + 10t + 2 \, dt$

$x = e^{-t} + 5t^2 + 2t + C$

$0 = e^{0} + 5(0)^2 + 2(0) + C$

$-1 = C$

$\therefore x = e^{-t} + 5t^2 + 2t - 1$ ✓

ANSWERS QUESTION 4

Question 4 (a)

Criteria	Marks
• One for forming a quadratic equation and one for simplification	2

Answer:

If $2^x = p$ then $2^{2x} - 9 \cdot 2^x + 8 = 0$

becomes $p^2 - 9p + 8 = 0$ ✓

i.e. $(p-8)(p-1) = 0 \therefore p = 8$ or $p = 1$

$\therefore 2^x = 8$ *or* $2^x = 1$ i.e. $x = 3$ or 0 ✓

Question 4 (b)

Criteria	Marks
• One mark for each answer	2

Answer:

$\sin\frac{\theta}{2} = \frac{\sqrt{3}}{2} \quad \therefore \frac{\theta}{2} = 60^0, 120^0 \therefore \theta = 120^0$ ✓ *or* 240^0 ✓

Question 4 (c) (i)

Criteria	Marks
• One mark each for substituting into the cosine formula and one for simplification	2

Answer:

$PR^2 = PQ^2 + QR^2 - 2PQ \cdot QR \cos\angle PQR$

$= 8 \cdot 3^2 + 5 \cdot 8^2 - 2 \times 8 \cdot 3 \times 5 \cdot 8 \times \cos 110^0$ ✓

$PR = 11.638.. = 11.6$ (1 dec.pl) ✓

Question 4 (c) (ii)

Criteria	Marks
• One mark for finding $\cos\angle QPR$ and one for simplification	2

Answer:

$\cos\angle QPR = \frac{8.3^2 + 11.6^2 - 5.8^2}{2 \times 8.3 \times 11.6} = 0.881855.. \therefore \angle QPR = 28^0$ *(nearest degree)* ✓ ✓

Question 4 (d))

Criteria	Marks
• One mark for $5000(1.0525)^{15}$ and one for simplification	2

Answer:

Jasmin's investment after 15 years

$=5000(1.0525)^{15} = \$10772 \cdot 13$ *(to the nearest cent)* ✓

Question 4 (e)

Criteria	Marks
• One mark for using the geometric sequence formula and one for simplification	2

Answer:

The division of bacteria corresponds to the geometric sequence 1, 2, 4, 8, 16 , ------------

where $a=1$ and $r=2$

$\therefore Tn = ar^{n-1}$ ✓

$T_{20} = 1 \times 2^{19} = 524\,288$ ✓

ANSWERS QUESTION 5

Question 5 (a) (i)

Criteria	Marks
• One mark for the correct answer	1

Answer:

To find where the curve cuts x- axis, put

$x^2 - 6x = 0 \Rightarrow x(x-6) = 0$

i.e $x = 0$ *or* $x = 6 \therefore A$ *is* $(6,0)$ ✓

Question 5 (a) (ii)

Criteria	Marks
• One mark for $\left\|\int_0^6 (x^2 - 6x)dx\right\|$ and one for simplification	2

Answer:

shaded area $= \left|\int_0^6 (x^2 - 6x)dx\right|$ ✓

$= \left|\left[\frac{x^3}{3} - 3x^2\right]_0^6\right| = |-36| = 36\,unit^2$ ✓

Question 5 (a) (iii)

Criteria	Marks
• One mark for correct answer	1

Answer:

$y \geq x^2 - 6x$ *and* $y \leq 0$ ✓

Question 5 (b) (i)

Criteria	Marks
• One mark for correct answer and one for the reason	2

Answer:

(a) *In* $\triangle ABC$, $x + 2x + 99 = 180$ *(Angle sum of a triangle)* ✓

$x = \frac{180-99}{3} = 27$ ✓

Question 5 b) (ii)

Criteria	Marks
• One mark for correct answer	1

Answer:

In quad. DBCE, $\angle BDE + 99^0 + 2x^0 + 4x^0 = 360^0$ *(Angle sum of a quadrilateral)*	$\angle BDE + 99^0 + 2\times27^0 + 4\times27^0 = 360^0$ $\therefore \angle BDE = 360^0 - 261^0 = 99^0$ ✓

Question 5 (c) (i)

• One mark for completing the square and one for simplification	2

Answer:

$x^2 + 2x = -12y + 23$

$x^2 + 2x + 1 = -12y + 23 + 1$

$(x+1)^2 = -12y + 24$

$(x+1)^2 = 12(2-y)$ ✓

$\therefore$ vertex $= (-1, 2)$ ✓

5 (c) (ii) One mark for correct answer.

Focus $= (-1, -1)$. ✓

5 (d) One mark for $40 = r \times \frac{\pi}{180} \times 80$

One mark for simplification

$l = r\theta \quad \therefore 40 = r \times \pi/180 \times 80$ ✓

$\therefore r = 28.64 = 28.6$ (1dp) ✓

ANSWERS QUESTION 6

Question 6 (a) (i)

Criteria	Marks
• One mark for the correct answer	1

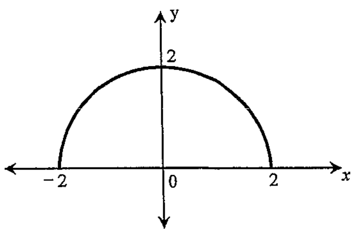
Answer:

$y=\sqrt{4-x^2}$ Here $4-x^2=(2-x)(2+x)\geq 0$ $\therefore$ *Domain*: $-2\leq x\leq 2$ ✓

Question 6 (a) (ii)

Criteria	Marks
• One mark for the correct graph	1

Answer:



Question 6 (b) (i)

Criteria	Marks
• One mark for integration and one for simplification	2

Answer:

$f'(x)=3x^2-16x+16$

$\therefore\ f(x)=\int(3x^2-16x+16)dx$

$=x^3-8x^2+16x+C$ ✓

(1,9) *is a point on* $y=f(x)$

i.e. $9=1-8+16+C$

$\therefore\ C=0$

Equation of the curve is

$y=x^3-8x^2+16x=x(x-4)^2$ ✓

Question 6 (b) (ii)

Criteria	Marks
• One mark for each stationary point and their nature	2

Answer:

$y=x^3-8x^2+16x$

$f'(x)=3x^2-16x+16=(3x-4)(x-4)$

At a stationary point $f'(x)=0$

$\therefore$ $(3x-4)=0$ or $x-4=0$

i.e $x=4/3$ or 4

$f''(x)=6x-16$

$f''(4/3)<0$ *and* $f''(4)>0$

$\therefore$ At $x=4/3$, $f(x)$ *is maximum and* ✓

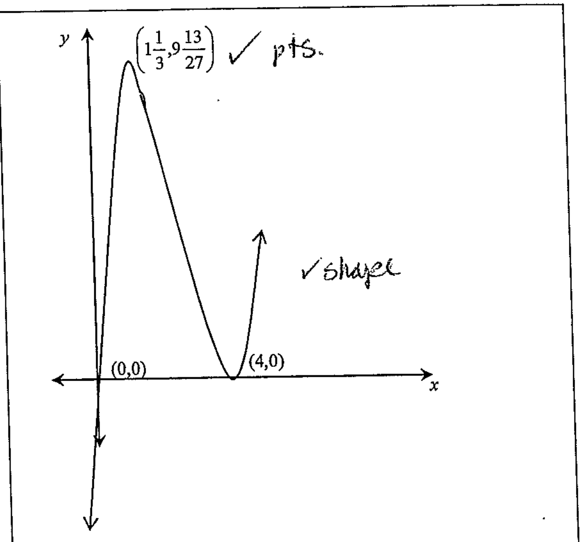
At $x=4$, $f(x)$ *is minimum* ✓

max $\left(\frac{4}{3},9\frac{13}{27}\right)$ *and min* $(4,0)$

Question 6 (b) (iii)

Criteria	Marks
• One mark for the shape and one for labelling the turning points	2

Answer:



Question 6 (c) (i)

Criteria	Marks
• One mark for method and one for noting a is not equal to 25	2

Answer:

For limiting sum $|r|<1$

i.e. $|5-\sqrt{a}|<1$ or $-1<5-\sqrt{a}<1$ ✓

or $-6<-\sqrt{a}<-4$ ✓

or $16<a<36$ *Except* $a\neq 25$ for $a=25$, *it will not be a series!*

Question 6 (c) (ii)

Criteria	Marks
• One mark limiting sum and one for rationalising	2

Answer:

If $r=5-\sqrt{a}$ *Limiting sum* $=\dfrac{1}{1-(5-\sqrt{20})}=\dfrac{1}{-4+2\sqrt{5}}$ ✓

$=\dfrac{2\sqrt{5}+4}{(2\sqrt{5}-4)(2\sqrt{5}+4)}=\dfrac{2\sqrt{5}+4}{4}=\dfrac{\sqrt{5}+2}{2}$

either ✓

7 (a) $V = 500\left(1 - \frac{t}{60}\right)^2$, $0 \le t \le 60$.

(i) When $t = 0$ $V = 500$

$\therefore$ there was initially 500 litres in the tank.

(ii) If half full $V = 250$

$\therefore 250 = 500\left(1 - \frac{t}{60}\right)^2$

$\therefore \frac{1}{2} = \left(1 - \frac{t}{60}\right)^2$ $\therefore \pm\sqrt{\frac{1}{2}} = 1 - \frac{t}{60}$

$\therefore \frac{t}{60} = 1 \mp \sqrt{\frac{1}{2}}$

$\therefore t = 60\left(1 \mp \frac{1}{\sqrt{2}}\right)$

$\therefore t = 17.5735\ldots$ or $102.426\ldots$

But $0 \le t \le 60$

$\therefore$ half full after 18 mins (or 17.5735... mins)

(iii) $\frac{dV}{dt} = 1000\left(1 - \frac{t}{60}\right) \cdot \frac{-1}{60}$

when $t = 58$

$\frac{dV}{dt} = 1000\left(1 - \frac{58}{60}\right) \cdot -\frac{1}{60}$

$= -\frac{5}{9}$

$\therefore$ tank is draining at $\frac{5}{9}$ Litre/minute

(b) $x^2 - (2+k)x + 3k = 0$

(i) $S = \frac{-b}{a}$

$\therefore S = \frac{2+k}{1}$ $\therefore k = 3$

(ii) $\frac{c}{a} = 4\left(-\frac{b}{a}\right)$

$\therefore 3k = 4(2+k)$

$\therefore 3k = 8 + 4k$

$\therefore k = -8$

some students carried on condition for part(i) to part(ii) when these were ...

(c) By Simpson's Rule:

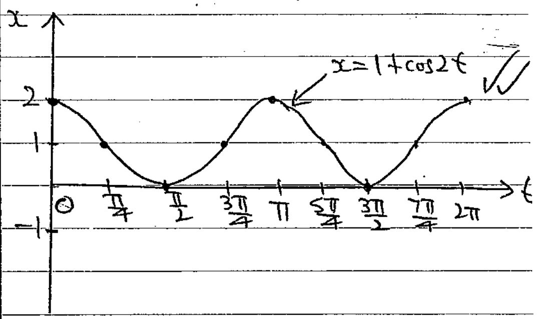
$\int_1^5 x \log_e x \, dx \doteq \frac{h}{3}\left[y_0 + y_4 + 4(y_1 + y_3) + 2y_2\right]$

$= \frac{1}{3}\left[0 + 8.05 + 4(1.39 + 5.55) + 2(3.30)\right]$

$= 14.13666\ldots$

8 (a) (i) $x = 1 + \cos 2t$

period $= \frac{2\pi}{2} = \pi$; sub-interval widths $\frac{\pi}{4}$



(ii) Particle is at rest when $t = 0$, $x = 2$; $t = \frac{\pi}{2}$, $x = 0$ and $t = \pi$, $x = 2$ for $0 \le t \le \pi$.

(iii) The particle oscillates on a straight line between $x = 0$ and $x = 2$ about a centre of motion of $x = 1$ every π seconds.

8 (b) (i) $\frac{dC}{dt} = -kC$ — (1)

$C = Ae^{-kt}$ — (2)

sub (2) into (1): LHS $= \frac{dC}{dt}$

$= \frac{d}{dt}\left(Ae^{-kt}\right)$

$\therefore$ LHS $= -kAe^{-kt}$

$= -kC$

$=$ RHS.

$\therefore C = Ae^{-kt}$ satisfies eqn.

(ii) when $t = 0$, $C = 90$

$\therefore 90 = Ae^0$ $\therefore A = 90$

$\therefore C = 90e^{-kt}$

when $t = 10$ $C = 20$

$\therefore 20 = 90e^{-10k}$

$\therefore k = -\frac{1}{10}\ln\frac{2}{9}$

(iii) when $t = 5$ $C = 90e^{\left(\frac{1}{10}\ln\frac{2}{9}\right)5}$

$= 90\left(\frac{2}{9}\right)^{\frac{1}{2}}$

$= 42.426\ldots$

$\therefore$ Charge is approx 42.4.

(iv) When $C = 60$, $t = ?$

$\therefore 60 = 90e^{\left(\frac{1}{10}\ln\frac{2}{9}\right)t}$

$\therefore t = \frac{\ln\frac{2}{3}}{\frac{1}{10}\ln\frac{2}{9}}$

$= 2.69577\ldots$

$\therefore$ time taken is 2.7 s (1dp)

9 (i) Amount owing after 1 month, $A_1 = P(1.005)$ where $P = 400\,000$

Similarly, amount owing after 2 months,

$A_2 = P(1.005)^2$

and $A_3 = P(1.005)^3$

and $A_4 = P(1.005)^4 - M$

and $A_5 = \left(P(1.005)^4 - M\right)1.005 - M$

$= P(1.005)^5 - M(1 + 1.005)$

$= P(1.005)^5 - 2.005M$

(ii) Now $A_6 = A_5(1.005) - M$

$= \left[P(1.005)^5 - M(1 + 1.005)\right]1.005 - M$

$\therefore A_6 = P(1.005)^6 - M(1 + 1.005 + 1.005^2)$

Now continuing this pattern $\therefore$ amount owing after 20 years ie 240 months

$A_{240} = P(1.005)^{240} - M(1 + 1.005 + \ldots + 1.005^{236})$

But as loan is repaid after 20 years

$A_{240} = 0$.

$\therefore M = \frac{P(1.005)^{240}}{1 + 1.005 + \ldots + 1.005^{236}}$

G.P. $a = 1$, $r = 1.005$, $n = 237$

$\therefore M = \frac{400\,000(1.005)^{240}}{1\left[\frac{1.005^{237} - 1}{0.005}\right]}$

$= 2928.03$ (2dp)

$\therefore$ monthly repayment, $M = \$2928.03$ (to nearest cent)

(b) $2\log x = \log(2x+8)$ $[x > 0 \;\&\; x > -4$ $\therefore x > 0]$

$\therefore \log x^2 = \log(2x+8)$

$\therefore x^2 = 2x + 8$

$\therefore x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

$\therefore x = -2$ or 4

But $x > 0$ $\therefore x = 4$ only.

(c) (i) $I = \frac{k \sin\alpha}{d^2}$

Now $\sin\alpha = \frac{y}{d}$ and $d^2 = y^2 + a^2$ $\therefore d = \sqrt{y^2 + a^2}$ $(d > 0)$

$\therefore I = \frac{k\frac{y}{d}}{y^2 + a^2}$

$= \frac{ky}{d(y^2 + a^2)}$

$= \frac{ky}{(y^2 + a^2)^{3/2}}$ (upon substitution of d ...)

9 c (ii) $I = \frac{ky}{(y^2+a^2)^{3/2}}$

$\therefore \frac{dI}{dy} = \frac{(y^2+a^2)^{3/2} \cdot k - ky \cdot \frac{3}{2}(y^2+a^2)^{\frac{1}{2}} \cdot 2y}{(y^2+a^2)^3}$

$= \frac{k(y^2+a^2)^{\frac{1}{2}}\left[(y^2+a^2) - 3y^2\right]}{(y^2+a^2)^3}$

$= \frac{k(a^2-2y^2)}{(y^2+a^2)^{5/2}}$ ✓

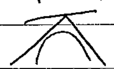
~~For a possible max/min~~ $\frac{dI}{dy} = 0$

$\therefore a^2 - 2y^2 = 0$ ✓

$\therefore y = \frac{a}{\sqrt{2}}$ $(y > 0)$

When $y = \frac{a}{\sqrt{2}}$

y	$\frac{a}{\sqrt{2}}^-$	$\frac{a}{\sqrt{2}}$	$\frac{a}{\sqrt{2}}^+$
$\frac{dI}{dy}$	+	0	−



⇒ max. illumination when $y = \frac{a}{\sqrt{2}}$ ✓

$\therefore$ best height is $\frac{a}{\sqrt{2}}$ above the centre of the circle.

Q10 2U Cranbrook 2009

Solutions & Markers Notes.

a) POSITIVE DEFINITE IF $k > 0$, $\Delta < 0$. $kx^2 - 6x + 2$

MARKS

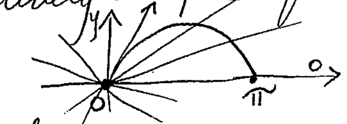
$\Delta = b^2 - 4ac$
$= 36 - 8k$

$36 - 8k < 0$
$-8k < -36$
$k > 4\frac{1}{2}$ and $k > 0$

$\therefore$ FOR $k > 4\frac{1}{2}$ $kx^2 - 6x + 2$ IS POSITIVE DEFINI[TE]

BOTH conditions should be stated and satisfied. TOO many lost these easy marks for not knowing the basic rules. ☹ learn them carefully! ☺.

b) MARKS a deceptively simple question if you consider the SKETCH



note: $\frac{d \sin x}{dx} = \cos x$ for gradient of tangent
at $x = 0$, $= \cos 0$
$= 1$

2 solutions for $0 \leq p < 1$
and 1 solution for all other P values
$\therefore$ true for all P

	Marks
c) i) • One mark for $A = 2 \times \frac{1}{2}\int_0^3 e^x + e^{-x}dx = \left[e^x - e^{-x}\right]_0^3$ and one for simplification	2

Answer:
Since the curve is symmetrical about the y axis

$\therefore A = 2 \times \frac{1}{2}\int_0^3 (e^x + e^{-x})dx = \left[e^x - e^{-x}\right]_0^3$

$= e^3 - e^{-3} - (1-1)$

AREA $= (e^3 - e^{-3})$ *unit*^{2}

Well done

	Marks
ii) • One mark for $V = \frac{1}{2}\pi\int_0^3 e^{2x} + 2 + e^{-2x}dx$, one for integration and one for simplification	3

Answer:

$V = 2 \times \frac{1}{4}\pi\int_0^3 \left(e^x + e^{-x}\right)^2$ *

$= \frac{1}{2}\pi\int_0^3 e^{2x} + 2 + e^{-2x}dx$

$\frac{1}{2}\pi\left[\frac{e^{2x}}{2} + 2x - \frac{e^{-2x}}{2}\right]_0^3 = \frac{1}{2}\pi\left[\frac{e^6}{2} + 6 - \frac{e^{-6}}{2}\right]$

VOLUME $= \pi\left[\frac{e^6}{4} + 3 - \frac{e^{-6}}{4}\right]u^3$

* More care needed with expansion

	Marks
d) One for $\tan\alpha = \frac{a}{x}$ *and* $\tan\beta = \frac{b}{x}$, one for $\frac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha} = \frac{\frac{b}{x} - \frac{a}{x}}{1 + \frac{b}{x}\cdot\frac{a}{x}}$ and one for simplification	3

Answer:

Let ∠TXP = α and ∠TXQ = β, then

$\tan\alpha = \frac{a}{x}$ *and* $\tan\beta = \frac{b}{x}$ $\therefore \theta = \beta - \alpha$

$\tan\theta = \tan(\beta - \alpha)$

$= \frac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha} = \frac{\frac{b}{x} - \frac{a}{x}}{1 + \frac{b}{x}\cdot\frac{a}{x}}$

$= \frac{x(b-a)}{x^2 + ab}$

Too many avoided this question ☹ You must learn to make a start and earn some marks