CRANBROOK SCHOOL

Thursday September 7, 2006

Year 11 Extension 1 Mathematics

2006 Yearly Preliminary Examination

Time Allowed: 2 hours

General Instructions:

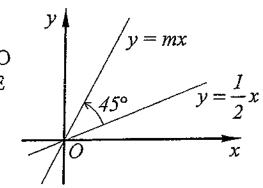
- There are 7 questions, each question is worth 12 marks
- Attempt all questions
- Begin a new booklet at the beginning of each question.
- Show all necessary working to obtain maximum marks.
- Marks will be deducted for poor and illegible work
- Only board approved calculators are permitted.

Question 1 [12 Marks] **Marked by HRK**

a) A and B are the points $(-5, 4)$ and $(2, -2)$ respectively. Find P, the point which divides AB externally in the ratio $2 : 3$ **3**

b) Find the obtuse angle between the lines $2x - y + 3 = 0$ and $y = 3x + 1$, giving the answer correct to the nearest minute. **3**

NOT TO SCALE



c) The angle between the lines $y = mx$ and $y = \frac{1}{2}x$ is $45°$ as shown in the diagram.

Find the exact value of m. **3**

d) Solve for x: $\frac{4}{5-x} \geq 1$ **3**

Question 2 [12 Marks] **Begin a new booklet** **Marked by HRK**

a) .

(i) Verify that $x = \frac{1}{3}$ and $x = 2$ satisfy the equation $7 - 3x = \frac{2}{x}$ **1**

(ii) On the same set of axes, sketch the graphs of $y = 7 - 3x$ and $y = \frac{2}{x}$ **2**

(iii) Using part (ii), or otherwise, write down all values of x for which $7 - 3x < \frac{2}{x}$ **2**

b) A point $P(x, y)$ moves in the XY-plane such that it is equidistant from the points $(2, 5)$ and $(5, -2)$. Find the equation of the path of P. **2**

c) The point $P(x, y)$ moves in XY-plane such that its distance from a fixed point $(-1, 1)$ is equal to its distance from the line $x = 1$. Prove that the locus of P is a parabola. Find its focus, directrix, vertex, axis of symmetry and focal length. Sketch the locus and clearly label these features on the graph. **5**

Question 3 [12 Marks] **Begin a new booklet** **Marked by HRK**

a) A parabola has equation $(x - 3)^2 = 8(y + 1)$

(i) What are the co-ordinates of its focus? **2**

(ii) Show that the line $x - 2y + 7 = 0$ is not a focal chord of this parabola. **2**

b) A parabola whose equation is of the form $y = Bx^2$ (where B is a constant), has the line $20x - y + 20 = 0$ as a tangent.

(i) Prove using simultaneous equations that $B = -5$ **3**

(ii) Sketch the parabola and the tangent line, showing the co-ordinates of the point of contact. **3**

(iii) Find the co-ordinates of the focus and the equation of the directrix of the parabola. **2**

Question 4 [12 Marks] **Begin a new booklet** **Marked by JJA**

a) Simplify $2\cos(90° - A) \times \sin(90° - A)$. **2**

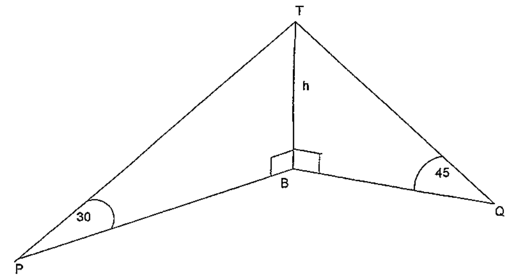
b) Given that $\cos 2\theta = 1 - 2\sin^2\theta$, show that : **3**

$$\frac{\cos x - \cos(x + 2\theta)}{2\sin\theta} = \sin(x + \theta)$$

c) If $\tan A$ and $\tan B$ are the roots of the equation $3x^2 - 5x - 1 = 0$, find the value of $\tan(A + B)$. **3**

Question 4 contd.

d) The angle of elevation from a boat at P to a point T at the top of a vertical cliff is measured to be 30^0. The boat sails 1km to a second point Q, from which the angle of elevation to T is measured to be 45^0. Let B be the point at the base of the cliff directly below T and let $h = BT$ be the height of the cliff in metres. The bearings of B from P and Q are 50^0 and 310^0 respectively.



(i) Show that $\angle PBQ = 100^0$. **1**

(ii) Find expressions for PB and QB in terms of h. **1**

(iii) Hence show that $h^2 = \dfrac{1000^2}{\cot^2 30^0 + \cot^2 45^0 - 2\cot 30^0 \cot 45^0 \cos 100^0}$ **1**

(iv) Use a calculator to find the height of the cliff. **1**

Question 5 [12 Marks] **Begin a new booklet** **Marked by JJA**

a) Using $t = \tan\dfrac{\theta}{2}$ write the results for **6**

(i) $\sin\theta$

(ii) $\cos\theta$

(iii) By substituting $t = \tan\dfrac{x}{2}$, find the solutions to the equation: $3\sin x + 4\cos x = 5$ for $0^0 \le x \le 360^0$, giving your answer to the nearest degree.

Question 5 contd.

b) 6

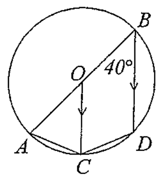
(i) Write the equation $\sqrt{3}\,\cos x - \sin x = 1$ in the form $R\cos(x+\alpha) = 1$.

(ii) Solve the equation for $0 \le x \le 360°$

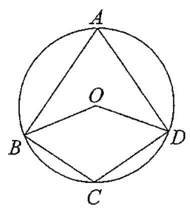
(iii) What is the general solution of the equation?

Question 6 [12 Marks] **Begin a new booklet** **Marked by BMM**

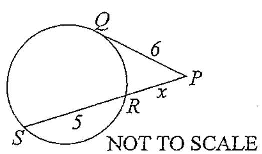
a) AB is the diameter of the circle with centre O. BD and OC are parallel, and $\angle\, OBD = 40°$. If C and D are joined, find the size of $\angle\, OCD$, giving reasons. 3



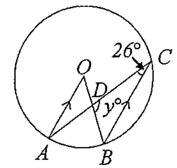
b) In the diagram A, B, C, and D are points on a circle with centre O. $\angle BAD = x°$ and $\angle BOD = \angle BCD$. Find the value of x 3



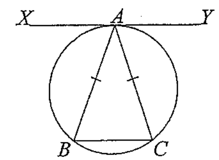
c) PQ is a tangent to a circle QRS, while PRS is a secant intersecting the circle in R and S, as in the diagram. Given that $PQ = 6$, $RS = 5$, $PR = x$, find x 2



d) The points A, B and C lie on a circle with centre O. The lines AO and BC are parallel, and OB and AC intersect at D. Also, $\angle\, ACB = 26°$ and $\angle\, BDC = y°$, as shown in the diagram. Find y. Justify your answer 2



e) Given that $AB = AC$ and XY is tangent to circle ABC at A, prove that XY is parallel to BC 2



Question 7 [12 Marks] **Begin a new booklet** **Marked by CAB**

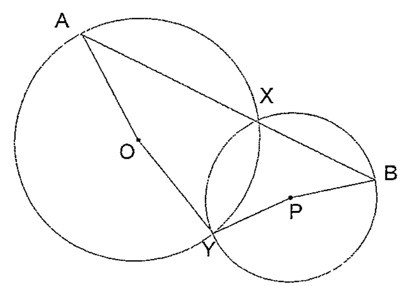
a) Two points A and B are taken on a circle, and C is the other end of the diameter through A. AE is the line from A perpendicular to the tangent at B. 4.

(i) Draw a diagram showing this information

(ii) Prove that AB bisects $\angle\, CAE$

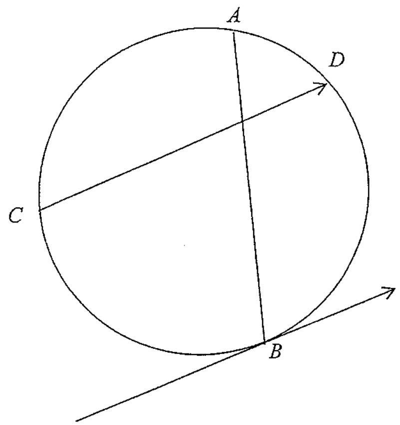
b) O and P are the centres of the circles: AXB is a straight line.

Prove that *obtuse* $\angle AOY = obtuse \angle BPY$ 4



Question 7 contd.

c) *AB* and *CD* are two intersecting chords of a circle and *CD* is parallel to the tangent to the circle at *B*. **4**



(i) Draw a neat sketch of the diagram in your examination booklet.

(ii) Prove that *AB* bisects $\angle CAD$

End of Examination

Q1 a) LEARN FORMULA !!

$P = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$

$m = 2 \quad n = -3$ Since EXTERNAL
$x_1 = -5 \quad y_1 = 4$
$x_2 = 2 \quad y_2 = -2$

$= \left(\frac{2(2) + (-3)(-5)}{2-3}, \frac{2(-2) + -3(4)}{2-3}\right)$

$= (-19, 16)$ ✓

b) $m_1 = 2 \quad m_2 = 3$ ✓

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \frac{2-1}{1+6}$ ✓

$\therefore$ acute $\theta = 8°8'$
Then Obtuse $\angle = 171°52'$ ✓

c) $\tan 45° = \left|\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m}\right|$ ✓

$1 = \left|\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m}\right|$ ✓

(Recall $|x - 3| = 1$ is solved by $x - 3 = 1$ as well as $x - 3 = -1$)

EITHER
$\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m} = 1$ ✓

$m - \frac{1}{2} = 1 + \frac{1}{2}m$

$\frac{m}{2} = \frac{3}{2} \quad \therefore m = 3$ //

$\left|\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m}\right| = -1$

$m - \frac{1}{2} = -1 - \frac{1}{2}m$

$\frac{3m}{2} = -\frac{1}{2}$

$m = -\frac{1}{3}$ ✓

d) $\frac{4}{5-x} \geq 1$ NOTE $x \neq 5$

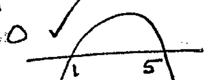
$\frac{4}{(5-x)} \times (5-x)^2 \geq 1 \times (5-x)^2$ ✓

$4(5-x) \geq (5-x)^2$

$4(5-x) - (5-x)^2 \geq 0$

$(5-x)[4 - (5-x)] \geq 0$

$(5-x)[-1 + x] \geq 0$ ✓



$1 \leq x < 5$ ✓ Since $x \neq 5$ (see 1st line)

2 a)(i) SUBSTITUTE BOTH

$x = \frac{1}{3}$
LHS $= 7 - 3x$
$= 7 - 3(\frac{1}{3})$
$= 6$

RHS $= \frac{2}{x}$
$= \frac{2}{\frac{1}{3}}$
$= 6$
$=$ LHS //

$x = 2$
LHS $= 7 - 3x$
$= 7 - 3(2)$
$= 1$

RHS $= \frac{2}{x}$
$= \frac{2}{2}$
$= 1 =$ LHS

$\therefore$ BOTH SATISFY // ✓✓

(Note 2 marks given for (i))

Q3 b(iii)

$y = Bx^2$

$\frac{dy}{dx} = m_T = 2Bx$

Now tangent has $m = 20$

$\therefore 2Bx = 20$

$x = \frac{10}{B}$ ④

Subst ④ into ③

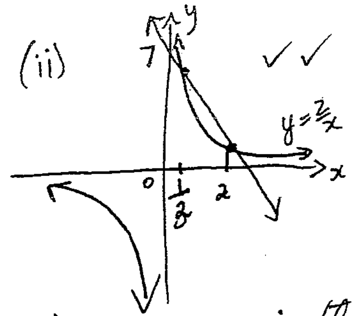
$B\frac{100}{B^2} - 20\frac{10}{B} - 20 = 0$

$100 - 200 = 20B$

$-100 = 20B$

$\therefore B = -5$ //

(ii) ✓✓



(iii) ie where is the straight line BELOW the hyperbola ?
READ OFF DIAGRAM using (i) to give x values of Points of intersection

FOR $0 < x < \frac{1}{3}$ and $x > 2$

$7 - 3x < \frac{2}{x}$ // ✓ NOTE only 1 mark for this.

b) $PA = PB$

$\sqrt{(x-2)^2 + (y-5)^2} = \sqrt{(x-5)^2 + (y+2)^2}$ ✓

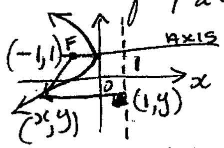
$x^2 - 4x + 4 + y^2 - 10y + 25 = x^2 - 10x + 25 + y^2 + 4y + 4$

$6x - 14y = 0$ ✓

SIMPLIFY !!

ie $3x - 7y = 0$ is equation of Path of P ✓✓✓

c)



$a = 1$

$\sqrt{(x+1)^2 + (y-1)^2} = \sqrt{(x-1)^2 (y-y)^2}$ ✓

$\cancel{x^2} + 2x + \cancel{1} + (y-1)^2 = \cancel{x^2} - 2x + \cancel{1}$

$(y-1)^2 = -4x$ ✓

OR

$y^2 - 2y + 4x + 1 = 0$

NOTE: MUST PROVE NOT JUST WRITE IN FORMULA

Focus $(-1,1)$ focal length $= 1$
Vertex $(0,1)$ axis of symmetry is $y = 1$
Directrix $x = 1$

3(a)(i) VERTEX is $(3,-1)$
$a = 2$ ∪ ✓
$\therefore$ Focus is $(3,1)$ ✓

(ii) Show $(3,1)$ does NOT satisfy

LHS $= x - 2y + 7$ RHS $= 0$
$= 3 - 2(1) + 7$ ✓
$= 8$
$\neq 0$
$\therefore$ LHS $\neq$ RHS

Since line does not pass through focus it is NOT a focal chord. ✓

b) $y = Bx^2$ ①

$y = 20x + 20$ ②

$\therefore Bx^2 = 20x + 20$

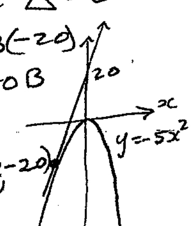
$Bx^2 - 20x - 20 = 0$ ③

Since ② is a tangent there is only 1 point of intersection ie $\Delta = 0$ in ③

$\Delta = (-20)^2 - 4B(-20)$

$0 = 400 + 80B$

$\therefore B = -5$



ie $x^2 = -\frac{1}{5}y$

$4a = \frac{1}{5}$

$\therefore a = \frac{1}{20}$

focus $(0, -\frac{1}{20})$

DIRECTRIX $y = \frac{1}{20}$ //

(Some students used calculus successfully).
See previous page.

③

4\. A) $2\cos(90-A)\sin(90-A)$

$= 2\sin A\cos A$

$= \sin 2A$

Must learn (90 − A) Rules

B) $\dfrac{\cos x - \cos(x+2\theta)}{2\sin\theta} = \sin(x+\theta)$

20 expansions

$LHS = \dfrac{\cos x - (\cos x\cos 2\theta - \sin x\sin 2\theta)}{2\sin\theta}$

Use info given!

$= \dfrac{\cos x - (\cos x(1-2\sin^2\theta) - \sin x \times 2\sin\theta\cos\theta)}{2\sin\theta}$

$= \dfrac{\cancel{\cos x} - \cancel{\cos x} + 2\sin^2\theta\cos x + 2\sin\theta\cos\theta\sin x}{2\sin\theta}$

$= \dfrac{\sin\theta\cos x + \cos\theta\sin x}{1}$

$= \sin(\theta + x)$

$= \sin(x+\theta)$

c) $3x^2 - 5x - 1 = 0$ 2 Methods

Method 1: Find $\tan A$ and $\tan B$.

$x = \dfrac{5 \pm \sqrt{25+12}}{6}$

$\tan A = \dfrac{5+\sqrt{37}}{6}$

$\tan B = \dfrac{5-\sqrt{37}}{6}$

④

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

$= \dfrac{\frac{5+\sqrt{37}}{6} + \frac{5-\sqrt{37}}{6}}{1 - \frac{5+\sqrt{37}}{6} \times \frac{5-\sqrt{37}}{6}}$

$= \dfrac{\frac{10}{6}}{1 - \frac{(25-37)}{36}}$

$= \dfrac{\frac{10}{6}}{1 + \frac{12}{36}}$

$= \dfrac{\frac{10}{6}}{\frac{48}{36}}$

$= \dfrac{10}{6} \times \dfrac{36}{48}$

$= \dfrac{5}{4}$

Method 2: Sums and products.

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

$\tan A + \tan B = \dfrac{-b}{a} = \dfrac{5}{3}$

Sum of roots

$\tan A\tan B = \dfrac{c}{a}$

$= \dfrac{-1}{3}$

$\Rightarrow \tan(A+B) = \dfrac{\frac{5}{3}}{1+\frac{1}{3}}$

$\Rightarrow \dfrac{5}{4}$

5.a) i) $\sin\theta = \frac{2t}{1+t^2}$ ✓ ⟸ Easy Marks

ii) $\cos\theta = \frac{1-t^2}{1+t^2}$ ✓

iii) $t = \tan\frac{x}{2}$

$3\sin x + 4\cos x = 5$

$3\left(\frac{2t}{1+t^2}\right) + 4\frac{(1-t^2)}{1+t^2} = 5$ ✓

$\frac{6t + 4 - 4t^2}{1+t^2} = 5$

$6t + 4 - 4t^2 = 5 + 5t^2$

$9t^2 - 6t + 1 = 0$ ✓

$(3t-1)(3t-1) = 0$

$t = \frac{1}{3}$ ✓

$\tan\frac{x}{2} = \frac{1}{3}$

$\frac{x}{2} = 18.43\ldots$

$x = 36.86\ldots$

$x = 37^\circ$ ✓

This does $\boxed{\text{not!}}$ $\Rightarrow \tan x = \frac{2}{3}$!!

b) $\sqrt{3}\cos x - \sin x = 1$ ← This we means using $R\cos(x+\alpha)$

$R = \sqrt{3+1}$

$R = 2$ ✓

$\tan\alpha = \frac{1}{\sqrt{3}}$ ← Take $\tan\alpha = \left|\frac{b}{a}\right|$

$\alpha = 30^\circ$ ✓

$2\cos(x+30) = 1$ ✓

ii)

$\cos(x+30) = \frac{1}{2}$

$(x+30) = 60^\circ$

$x = 30, 270^\circ$

iii) For general solution, take from $\cos(x+3) = \frac{1}{2}$

$\Rightarrow (x+30) = 360 \times n \pm 60^\circ$

$x = 360n \pm 60 - 30^\circ$

Important

EXT 1 ⑥

Qn 6 - BMM

a) $\angle AOC = 40°$ (corresponding $\angle$'s on $\parallel$ lines)
$\angle OAC = \angle OCA = 70°$ (Base $\angle$'s of isosc Δ)

$\angle ABO + \angle ACO = 180°$ (opp $\angle$'s of cyclic quad supp.)
$40° + (70° + \angle OCO) = 180°$

$110° + \angle OCO = 180°$
$\angle OCO = 70°$

2 mks - logical mathematical order
1 mk - correct reasons.

b) $\angle BAD = x$
$\therefore \angle BOO = 2x$ (angle @ centre is twice angle @ circumf)
$\therefore \angle BCO = 2x$ (data)

$\angle BAO + \angle BCO = 180°$ (opp $\angle$'s of cyclic quad supp.)
$x + 2x = 180°$
$3x = 180°$
$x = 60°$

2 mks - correct and logical mathematical order
1 mk - correct reasons
1 mk - if only answer given.

c) $QP^2 = SP \times RP$
$6^2 = (5+x) \times x$
$6^2 = 5x + x^2$
$x^2 + 5x - 36 = 0$
$(x+9)(x-4) = 0$
$x = -9, 4$
$x > 0$
$\therefore x = 4$

1 mk - correct rule
1 mk - answer
1 mk - answer followed incorrect substitution
0 mk - if rule not stated and incorrect.

⑦

d) $\angle AOB° = 2 \times 26°$ (angle @ centre is twice angle at circumf on same arc)

$\therefore \angle AOB = 52°$

$\angle OBC = 52°$ (alt. $\angle$'s on $\parallel$ lines)

In ΔOBC, $26° + 52° + y = 180°$ (angle sum of Δ)
$y = 102°$

1 mk - correct working + answer.
1 mk - reasons correct.

e) let $\angle YAC = x$

$\angle YAC = \angle ABC = x$ (angle in alt. segment)

ΔABC is isosceles, $\therefore \angle ACB = x$ (base $\angle$'s of isosc Δ)

$\angle ACB$ is alternate to $\angle YAC$ and equal,

$\therefore XY \parallel BC$.

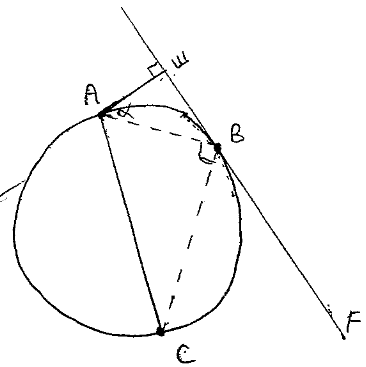
1 mk - correct, logical order + reasons
1 mk - stating correctly why $XY \parallel BC$

1 mk only - if use of 'angle in alt segment' twice and no reference to alternate $\angle$'s.

8

QUESTION 7

a) (i)



ii) PROVE AB BISECTS $\angle CAE$

let $\angle BAE = \alpha$

$\therefore \angle ABE = 90 - \alpha$ ($\angle$ sum of $\triangle AEB = 180°$)

$\therefore \angle BCA = 90° - \alpha$ ($\angle$ in the alt. seg)

$\therefore \angle BAC = \alpha$ ($90° - (90 - \alpha)$ $\angle$ sum of $\triangle ABC = 180°$)

$\therefore$ AB bisects $\angle CAE$

b) let $\underset{\text{(obtuse)}}{\angle AOY} = \alpha$

$\therefore \underset{\text{(reflex)}}{\angle AOY} = 360° - \alpha$

$\therefore \angle AXY = \dfrac{360° - \alpha}{2} = 180° - \dfrac{\alpha}{2}$

(angle at centre is twice angle at circumference)

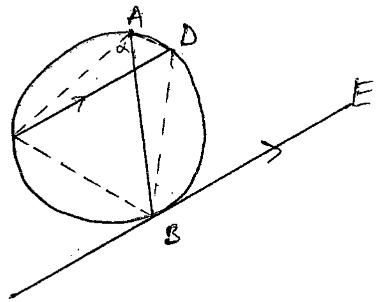
$\angle YXB = 180° - \left[180° - \dfrac{\alpha}{2}\right]$

$\therefore \angle YPB = 2 \times \dfrac{\alpha}{2}$

$= \alpha$ (angle at centre equal twice angle at circumference)

$\therefore \angle AOY = \angle BPY$

c) i)



ii) PROVE AB bisects $\angle CAD$

let $\angle CAB = \alpha$

$\therefore \angle CDB = \alpha$ (angles subtended by equal arcs)

$\therefore \angle DBE = \alpha$ (alt angles are equal on parallel lines)

$\angle BCD = \alpha$ (angle in the alternate segment)

$\therefore \angle DAB = \alpha$ (angles subtended by equal chords are equal)

$\therefore$ AB bisects $\angle CAD$