CRANBROOK SCHOOL

# Year 11 Mathematics Extension 1

## Preliminary Final Exam 2010

**General Instructions**

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

**Total marks – 72**

- Attempt Questions 1–6
- All questions are of equal value
- Six 4 – page Exam booklets should be used

**Question 1** — **Marks**

a) The equation $4x^2-3x+8=0$ has roots $\alpha$ and $\beta$.

Without finding the value of $\alpha$ and $\beta$, find

i) $\alpha^2+\beta^2$ — 1

ii) $\dfrac{\beta}{\alpha}+\dfrac{\alpha}{\beta}$ — 2

b) Solve the inequality $\dfrac{2x+3}{x-1}<3$ — 3

c) i) Find in exact form the tan of the acute angle between the lines — 3

$\sqrt{3}x+y-2=0$ and $y=x+1$

ii) Use the fact that $m=\tan\theta$ to show that the exact value of $\tan 75^0$ is — 3

$2+\sqrt{3}$

**End of Question 1**

**Question 2** *Start a new booklet* **Marks**

a) Given $y=(3x-2)^3(1-x)$

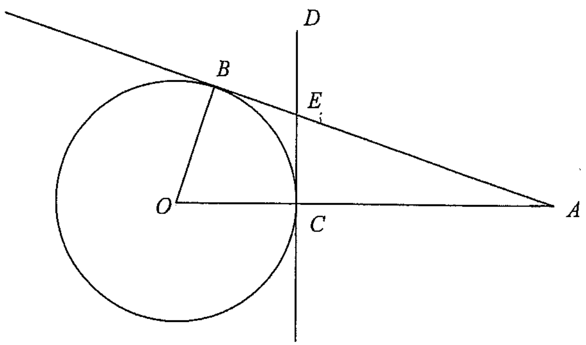
i) Find the derivative, $\frac{dy}{dx}$ in fully factorised form. **3**

ii) Find the equation of the normal at the point where $x=0$ **2**

b) i) Express $3\sin x-\sqrt{3}\cos x$ in the form $A\sin(x-\alpha)$ where $A>0$ and $\alpha$ is acute. **2**

ii) Hence or otherwise, solve the equation $3\sin x-\sqrt{3}\cos x=1$ for $0\le x\le 360^0$ **2**

c) $AB$ and $CD$ are tangents to a circle, centre $O$. The tangents intersect at a point $E$ and the line $AO$ passes through $C$ as illustrated.




i) Show that $\Delta AOB\ |||\ \Delta AEC$ **2**

ii) What kind of quadrilateral is $OBEC$? Justify your answer **1**

**End of Question 2**

**Question 3** *Start a new booklet* **Marks**

a) The point $P$ divides the interval $AB$ joining $A(-2,-3)$ and $B(1,4)$ internally in the ratio of 3:2. Find the coordinates of $P$ **2**

b) A function is defined by the equations $x=1-t$ and $y=\frac{t^2}{2}$

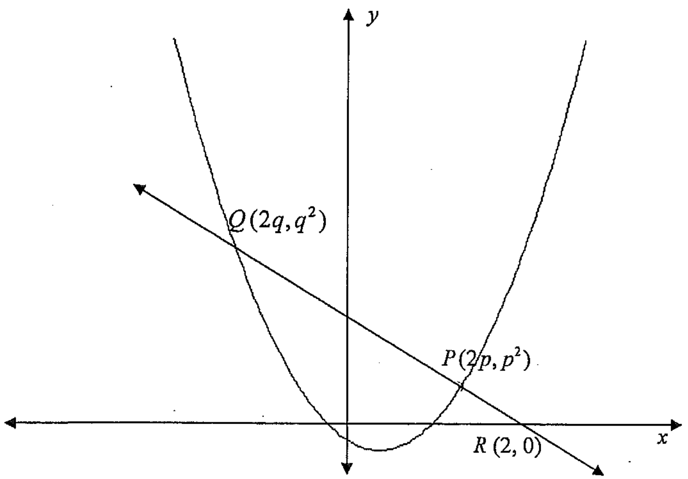
i) Find the value(s) of $x$ when $y=8$. **1**

ii) Express $y$ as a function of $x$. **2**

iii) Draw a neat sketch of the function. **2**

**Question 3 continues on the next page**

c) $P(2p, p^2)$ and $Q(2q, q^2)$ are points on the parabola $x^2 = 4y$. $R$ is the point $(2, 0)$ and $PQR$ is a straight line, as shown.




i) Give that the chord $PQ$ has the equation $(p+q)x - 2y - 2apq$ show that $p + q = pq$ **2**

ii) The the equation of the tangent at $P$. **1**

iii) Find the locus of the point $T$, the point where the tangents at $P$ and $Q$ intersect. **2**

**End of Question 3**

**Question 4** ***Start a new booklet*** **Marks**

a) A concave down parabola has a focal length of $\frac{1}{3}$ and a focus of (2, 2)

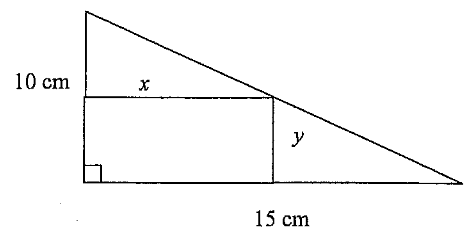
i) Find the vertex of the parabola **1**

ii) Find the equation of the parabola **2**

iii) Sketch the parabola showing the directrix, the focus and the vertex. **2**

b) Find the equation of the tangent to the curve $y = \dfrac{x^2}{x-1}$ at the point where $x = -1$ **3**

c) The diagram below shows a right angles triangle with sides 15 cm and 10 cm. A rectangle is inscribed inside the triangle.




i) Show that the area of the triangle is can be given by

$$A = xy + \frac{(15-x)y}{2} + \frac{(10-y)x}{2}$$ **2**

ii) Hence show that $y = \dfrac{30-2x}{3}$ **2**

**End of Question 4**

**Question 5** ***Start a new booklet*** **Marks**

a) Solve $|2x+1| > 4-x$ , graphically or otherwise. **3**

b) Solve $\cos 2\theta - \sin\theta = 0$ for $0 \le \theta \le 360^0$ **4**

c) $P$, $A$ and $B$ are the points $(x, y)$, $(4, 2)$ and $(1, -1)$ respectively.

i) Find the locus of the point $P$ if it moves so that $PA = 2PB$. **3**

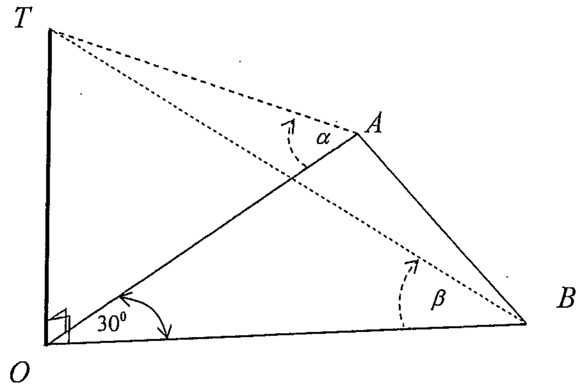
ii) Find the centre and radius of the locus. **2**

**End of Question 5**

**Question 6** ***Start a new booklet*** **Marks**

a) The parabola $y = x^2 + x + 1$ has a tangent with equation $y = -(mx + 2m)$. Find the possible values of $m$. Leave your answer in exact form. **2**

b) A tree's height is being measured by two observers. The first observer ($A$) records an angle of elevation to the top of the tree of $\alpha^0$. A second observer ($B$) measures the angle of elevation to be exactly $\beta^0$.




If $\angle AOB = 30^0$ and $AB$ is 10 m.

i) Use $\Delta AOB$ and the Cosine rule show that **2**

$$\sqrt{3}AO.BO + 100 = AO^2 + BO^2$$

ii) Show that if $BO = 3AO$ then **2**

$$AO = \frac{10}{\sqrt{10 - 3\sqrt{3}}}$$

**Question 6 continues on the next page**

iii) Using $\Delta TOA$ and $\Delta TOB$ show that **2**

$$\frac{\tan\alpha}{\tan\beta} = 3$$

iv) If the angle of elevation at $A$ is twice the size of the angle of elevation at $B$ ($\alpha = 2\beta$) then show that the $\beta = 30^0$. **3**

iv) Hence or otherwise find the height of the tree to one decimal place. **1**

**-- End of Exam --**

Solutions Yr 11 Prelim 2010

Q1 a) $4x^2 - 3x + 8 = 0$

$\alpha + \beta = \frac{3}{4}$

$\alpha\beta = \frac{8}{4} = 2$

i) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= \left(\frac{3}{4}\right)^2 - 2.2$

$= \frac{9}{16} - 4$

$= -\frac{55}{16}$ ①

ii) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

$\frac{\alpha^2 + \beta^2}{\alpha\beta}$ ①

$= \frac{-55}{16} / 2$

$= \frac{-55}{32}$ ①

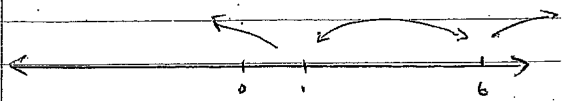
b) $\frac{2x+3}{x-1} < 3$

$x \neq 1$

Solve $\frac{2x+3}{x-1} = 3$

$2x+3 = 3x-3$

$x = 6.$



test $x=0$

$\frac{3}{-1} < 3$ true

$\therefore x < 1 \quad x > 6.$

c) $\sqrt{3}\,x + y - 2 = 0$

$y = -\sqrt{3}\,x + 2$

$m_1 = -\sqrt{3}$ ①

$y = x + 1$

$m_2 = 1$ ①

$\therefore \tan\theta = \left|\frac{1-\sqrt{3}}{1-\sqrt{3}}\right|$

$= \frac{1+\sqrt{3}}{1-\sqrt{3}}$ ①

ii) $m_1$ $\tan\theta = -\sqrt{3}$

$\theta = 120°$

$m_2$ $\tan\theta = 1$ ①

$\therefore \theta = 45°$

$\therefore$ Angle between lines is $75°$ ①.

$\therefore \tan 75° = \frac{1+\sqrt{3}}{1-\sqrt{3}}$

$\tan 75° = \left|\frac{1+\sqrt{3}}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}}\right|$

$= \left|\frac{1+2\sqrt{3}+3}{1-3}\right|$

$= \left|\frac{4+2\sqrt{3}}{-2}\right|$

$= \left|-(2+\sqrt{3})\right|$ ①

As required.

QUESTION 2

a)

$y = (3x-2)^3(1-x)$

$u = (3x-2)^3 \quad u' = 9(3x-2)^2$

$v = (1-x) \quad v' = -1$ (1)

$y' = vu' + uv'$ (1)

$= 9(3x-2)^2(1-x) + -1(3x-2)^3$

$= (3x-2)^2(9(1-x) - (3x-2))$

$= (3x-2)^2(9 - 9x - 3x + 2)$

$= (3x-2)^2(11 - 12x)$

when $x = 0$

$y = (0-2)^3(1-0)$

$= -8$

when $x = 0$

$y' = (-2)^2(11)$

$= 44$

$m_1 m_2 = -1$

$\therefore m_2 = -\frac{1}{44}$ (1)

$y - y_1 = m(x - x_1)$

$y + 8 = -\frac{1}{44}(x - 0)$

$44y + 352 = -x$

$x - 44y - 352 = 0$ (1)

b) (i)

$3\sin x - \sqrt{3}\cos x = r\sin(x - \alpha)$

$r = \sqrt{3^2 + (\sqrt{3})^2}$

$= \sqrt{12}$

$= 2\sqrt{3}$

$\tan\alpha = \frac{\sqrt{3}}{3}$

$\alpha = 30^\circ$ (1)

hence $3\sin x - \sqrt{3}\cos x$

$= 2\sqrt{3}\sin(x - 30^\circ)$ (1)

(ii) $2\sqrt{3}\sin(x - 30^\circ) = 1$

$\sin(x - 30^\circ) = \frac{1}{2\sqrt{3}}$

$\sin^{-1}\left(\frac{1}{2\sqrt{3}}\right) = x - 30$

$\therefore x - 30 = 16^\circ 47'$ or

$x - 30 = 180^\circ - 16^\circ 47'$ (1)

$x = 46^\circ 47'$ or $193^\circ 13'$

(1)

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Question 3

Common issue was the use of an incorrectly remembered formula.

a) $A(-2,-3)$ $B(1,4)$

3 : 2

$x = \frac{2 \times -2 + 3 \times 1}{3+2}$ $y = \frac{2 \times -3 + 3 \times 4}{3+2}$ (1)

$= \frac{-1}{5}$ $= \frac{6}{5}$

$\left(-\frac{1}{5}, \frac{6}{5}\right)$ (1)

b) i) $x = 1 - t$ $y = \frac{t^2}{2}$

$y = 8$

$8 = \frac{t^2}{2}$

$t^2 = 16$

$t = \pm 4$

must remember all square roots are + and −

$\therefore$ when $t = 4$ $x = -3$ (1)

$t = -4$ $x = 5$

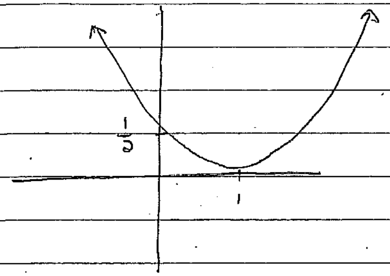
* only got the mark if both the t values were subbed in to find x values.

ii) $t = 1 - x$ (1)

$\therefore y = \frac{(1-x)^2}{2}$ (1)

$y = \frac{x^2 - 2x + 1}{2}$

* students who rearranged the $y = \frac{t^2}{2}$ were unable to express as y in terms of x

iii) 


2 marks
(1) correct placing
(1) marking the intercepts

c i) $(p+q)x - 2y - 2apq = 0$ where $a = 1$ as $x^2 = 4y$ (1)
and $x = 2$, $y = 0$ as $R(2,0)$

$(p+q)2 - 2(0) - 2\times pq = 0$

$2(p+q) = 2pq$ (1)

$p+q = pq$ as required

ii) $y = \frac{x^2}{4}$

$y' = \frac{x}{2}$

@ $x = 2p$

$y' = \frac{2p}{2} = p$

$y - p^2 = p(x - 2p)$

$y - p^2 = px - 2p^2$

$y = px - p^2$ (1)

iii) T occurs when $y = px - p^2$ intersects with $y = qx - q^2$

$\therefore px - p^2 = qx - q^2$

$px - qx = p^2 - q^2$

$\cancel{(p-q)}x = \cancel{(p-q)}(p+q)$

$x = p+q$

sub $x = p+q$ into

$y = px - p^2$

$= p(p+q) - p^2$

$= pq$ (1)

Needed to get this for to get 1 mark!

C iii) As $p+q = pq$

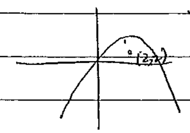
then $y = p+q$

$\therefore y = x$

$\therefore$ locus is the line $y = x$

Question 4

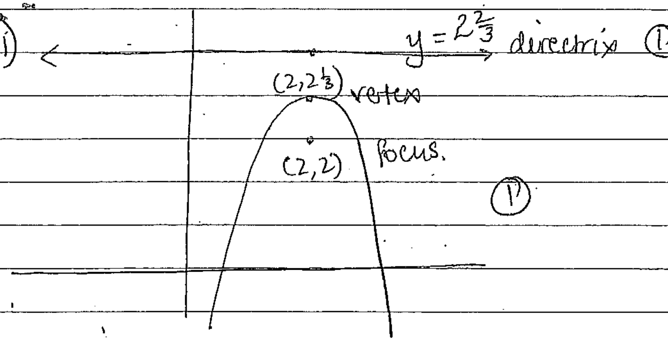
a) i) $a = \frac{1}{3}$ focus $(2,2)$ concave down




$\therefore$ vertex $(2, 2+\frac{1}{3}) = (2, 2\frac{1}{3})$
or
$= (2, \frac{7}{3})$

ii) $(x-2)^2 = -4\cdot\frac{1}{3}(y - \frac{7}{3})$

$(x-2)^2 = -\frac{4}{3}(y - \frac{7}{3})$ (1)
(1)

iii) 


(1)

b) $y = \dfrac{x^2}{x-1}$ $\quad \dfrac{u}{v}$

$u = x^2 \quad v = x-1$

$u' = 2x \quad v' = 1$

$v^2 = (x-1)^2$

$y' = \dfrac{(x-1)2x - 1.x^2}{(x-1)^2}$ (1)

$= \dfrac{2x^2-2x-x^2}{(x-1)^2}$

$= \dfrac{x^2-2x}{(x-1)^2}$

~~when x = 1~~

when $x = -1$

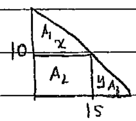
$y = \dfrac{(-1)^2}{-2} = -\dfrac{1}{2}$

$y' = \dfrac{(-1)(-1-2)}{(-1-1)^2} = \dfrac{3}{4}$ (1)

$\dfrac{3}{4} = \dfrac{y+\frac{1}{2}}{x+1}$

$4y + 2 = 3x+3$

$3x - 4y + 1 = 0$ (1).

C i) 



$A_1 = \frac{1}{2}.x.(10-y)$ $\qquad A_2 = xy$

$A_3 = \frac{1}{2}.y.(15-x)$

$\therefore A = \dfrac{(10-y)x}{2} + xy + \dfrac{(15-x)y}{2}$ as required.

ii) $A = \dfrac{10 \times 15}{2} = 75$

$\therefore 75 = \dfrac{(10-y)}{2}x + xy + \dfrac{(15-x)y}{2}$

$150 = 10 - xy + 2xy + 15y - xy$

$150 = 10x + 15y$

$\therefore y = \dfrac{30-2x}{3}$ as required.

QUESTIONS

a) $|2x+1| > 4-x$ $\quad$ OR// $\quad 2x+1 > 4-x$ or $\quad 2x+1 < -4+x$

$(2x+1)^2 > (4-x)^2$ (1)

$4x^2+4x+1 > 16-8x+x^2$

$3x^2+12x-15 > 0$

$x^2+4x-5 > 0$

$(x+5)(x-1) > 0$

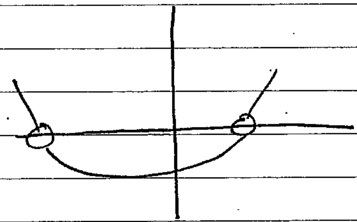
$3x > 3$

$x > 1$

test: $x = 2$

$|4+1| > 4-2$

$5 > 2$ ✓



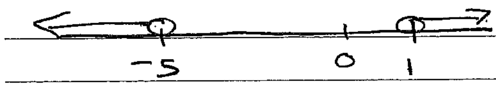
$x < -5$

test: $x = -6$.

$|-12+1| > 4 - -6$

$11 > 10$ ✓

$x < -5, \; x > 1$ (1)





(1) for test or graph.

b) $\cos 2\theta - \sin\theta = 0$

$1 - 2\sin^2\theta - \sin\theta = 0$ (1)

$2\sin^2\theta + \sin\theta - 1 = 0$

$(2\sin\theta - 1)(\sin\theta + 1) = 0$

$\sin\theta = \frac{1}{2}$ or $\sin\theta = -1$ (1)

$\theta = \underbrace{30°, 150°}_{(1)}, 270°$ (1)

c)(i) $PA = 2PB$

$$\sqrt{(x-4)^2+(y-2)^2} = 2\sqrt{(x-1)^2+(y+1)^2}$$

$$(x-4)^2+(y-2)^2 = 4\left[(x-1)^2+(y+1)^2\right]$$

$$x^2-8x+16+y^2-4y+4 = 4(x^2-2x+1+y^2+2y+1$$

$$x^2-8x+y^2-4y+20 = 4x^2-8x+4+4y^2+8y+4$$

$$0 = 3x^2+3y^2+12y-12$$

$$0 = x^2+y^2+4y-4$$

$$x^2+(y+2)^2 = 4+4$$

$$x^2+(y+2)^2 = 8 \quad (1)$$

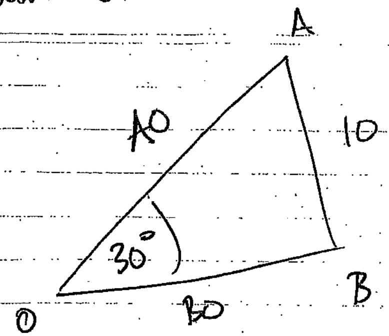
(ii) Centre $(0,-2)$ (1)

radius $2\sqrt{2}$ (1)

Question 6 Ext 1 prelim Solutions 2010

b)

i)




$$\cos 30^\circ = \frac{AO^2+BO^2-10^2}{2.AO.BO} \quad (1)$$

$$\frac{\sqrt{3}}{\cancel{2}} = \frac{AO^2+BO^2-100}{\cancel{2}AO.BO}$$

$$\sqrt{3}\,AO.BO + 100 = AO^2+BO^2 \quad (1)$$

As required.

ii) $BO = 3AO$

$$\sqrt{3}.\,AO.\,3AO + 100 = AO^2+9AO^2$$

$$3\sqrt{3}\,AO^2+100 = 10AO^2 \quad (1)$$

$$100 = 10AO^2 - 3\sqrt{3}\,AO^2$$

$$\frac{100}{(10-3\sqrt{3})} = AO^2 \quad (1)$$

$$\therefore AO = \frac{10}{\sqrt{10-3\sqrt{3}}}$$ As required.

iii) $\frac{OT}{OA} = \tan\alpha$ (1)

$\frac{OT}{OB} = \tan\beta$

$\therefore \quad OA \tan\alpha = OB \tan\beta$

$OB = 3OA$ (1)

$\therefore \quad \cancel{OA} \tan\alpha = 3\cancel{OA} \tan\beta$

$$\frac{\tan\alpha}{\tan\beta} = 3$$

iv) $\alpha = 2\beta$

$\frac{\tan 2\beta}{\tan\beta} = 3$ let $t = \tan\beta$

$\tan 2\beta = \frac{2t}{1-t^2}$

$\therefore \quad \frac{2\cancel{t}}{1-t^2} \cdot \frac{1}{\cancel{t}} = 3$

$$\frac{2}{1-t^2} = 3$$

$2 = 3 - 3t^2$

$3t^2 = 1$

$t^2 = \frac{1}{3}$

$t = \pm \frac{1}{\sqrt{3}}$ only +ve as $\beta$ is acute.

$\therefore \tan\beta = \frac{1}{\sqrt{3}}$ $\beta = 30°$

v) Using $AO = \frac{10}{\sqrt{10-3\sqrt{3}}}$ & $\alpha = 2\beta = 60°$

$\frac{OT}{OA} = \tan\alpha$

$\therefore \quad OT = \frac{10}{\sqrt{10-3\sqrt{3}}} \tan 60°$

$= \frac{10\sqrt{3}}{\sqrt{10-3\sqrt{3}}}$

$\therefore \quad OT = 7.9$ metres (1dp)

6. a) $y = x^2 + x + 1$

Tangent eqⁿ: $y = -(mx + 2m)$

~~tet~~ tangent intersects at one point

sub $y = -(mx + 2m)$ into $y = x^2 + x + 1$
for intersection

$-mx - 2m = x^2 + x + 1$

$x^2 + (m+1)x + (1+2m) = 0$

only one root for tangent $\therefore \Delta = 0$

$\Delta = (m+1)^2 - 4(1+2m) = 0$ ✓

$= m^2 + 2m + 1 - 4 - 8m = 0$

$m^2 - 6m - 3 = 0$

$m = \dfrac{6 \pm \sqrt{36 + 12}}{2}$ ✓

$= \dfrac{6 \pm 4\sqrt{3}}{2}$

$= \underline{\underline{3 \pm 2\sqrt{3}}}$ ✓