CRANBROOK SCHOOL

YEAR 12 MATHEMATICS – EXTENSION 1

Term 1 2005 — Time : 1.5 h / SKB, CJL and HRK

All questions are of equal value.
All necessary working should be shown in every question.
Full marks may not be awarded if work is careless or badly arranged.
Approved silent calculators may be used.
Submit your work in four 4 Page Booklets.

1. (15marks) (Begin a 4 page booklet.) HRK

(a) Sketch $y = -x(x+1)^2(x-1)^3$, showing any intercepts. 2

(b) Show that $x-3$ is a factor of the polynomial $P(x) = x^3 - x^2 - 14x + 24$.
Hence, find the other factors and solve $x^3 - x^2 - 14x + 24 < 0$. 4

(c) (i) If $(x-2)$ and $(2x+3)$ are two of the factors of the polynomial $P(x) = ax^3 - 5x^2 + bx + 6$, find the values of a and b. 3

(ii) By using the values of a and b found in (i) and letting the roots of $P(x) = 0$ be α, β and γ find the value of $\alpha^2 + \beta^2 + \gamma^2$. 2

(d) (i) Use Newton's Method with two applications to estimate the root of $P(x) = 0$ correct to 4 decimal places by taking $x = 1.2$ as a first approximation if $P(x) = e^x - x - 2$. 3

(ii) What value of x could not be used here as a first approximation? 1

2. (15marks) (Begin a 4 page booklet.) HRK

(a) (i) Find the point of intersection R, of the normals at P $(2ap, ap^2)$ and Q $(2aq, aq^2)$ on the parabola $x^2 = 4ay$. 4

(ii) If PQ passes through the point $(0, 8a)$ such that $pq = -8$ find the equation of the locus of R in the form $X^2 = 4A(Y-k)$. 3

Hence find the focus of the locus of R. 1

(b) Find $\int \frac{3x^2}{\sqrt{5x^3 - 6}}\,dx$, by using the substitution $u = 5x^3 - 6$. 3

(c) Evaluate $\int_0^1 x\sqrt{2x+2}\,dx$ in exact form by using the substitution $u^2 = 2x + 2$. 4

3. (15marks) (Begin a 4 page booklet.) SKB

(a) Differentiate with respect to x:

(i) x^2e^{-2x} (leave your answer in factored form) 2

(ii) $\frac{6x^2}{\sqrt{e^x - 1}}$ (leave your answer in factored form) 3

(b) Find the following integral:

$\int e^{2x-3}\,dx$ 1

(c) By using the substitution $u = e^{x^3}$ evaluate $\int_0^1 x^2 e^{x^3}\,dx$, exactly. 3

(d) Find the equation of the tangent to the curve $y = e^{2x}$ at the point where $x = -2$. **3**

(e) Find the volume of the solid generated when the area bounded by the curve $y = e^{-3x}$, the x-axis and lines $x = -1$ and $x = 1$ is rotated 360^0 about the x-axis.(Leave your answer in exact form) **3**

4. (15marks) (Begin a 4 page booklet.) **CJL**

(a) Sketch the curve $y = \dfrac{x^2+4}{x-2}$ showing clearly any intercepts, asymptotes and turning points. **8**

(b) Sketch the curve $y = 2xe^{-2x}$ showing clearly any intercepts, asymptotes, turning points and points of inflexion. **7**

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

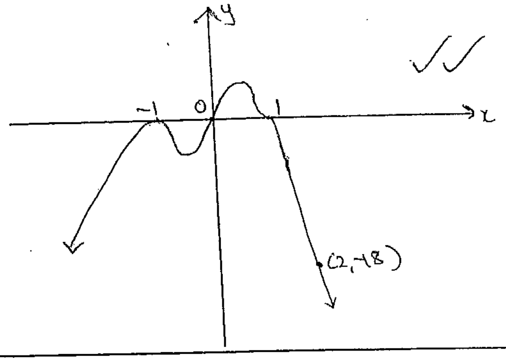
$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

(a) $y=-x(x+1)^2(x-1)^3$

Roots at $x=0$ (single), $x=-1$ (double) and $x=1$ (triple)

when $x=2$ $y=-18$



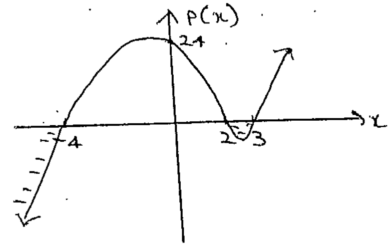
(b) $P(x)=x^3-x^2-14x+24$

$P(3)=27-9-42+24$
$=0$

$\Rightarrow x-3$ is a factor of $P(x)$.

$\therefore P(x)=(x-3)(x^2+2x-8)$ [by inspection]
$=(x-3)(x+4)(x-2)$

$\therefore$ other factors are $(x+4)$ and $(x-2)$.



$\therefore$ If $P(x)<0$

$\therefore x<-4$ or $2<x<3$.

(c) (i) $P(x)=ax^3-5x^2+bx+6$

$P(2)=0$ $\therefore 0=8a-20+2b+6$
$\therefore 14=8a+2b$ —— (1)

$P(-1\tfrac{1}{2})=0$ $\therefore 0=\frac{-27a}{8}-\frac{45}{4}-\frac{3b}{2}+6$

$\therefore 42=-27a-12b$ —— (2)

(1)×6: $84=48a+12b$ —— (1A)

(2)+(1A): $126=21a$ $\therefore a=6$

sub. $a=6$ into (1) $\therefore 14=48+2b$
$\therefore b=-17$

$\therefore (a,b)=(6,-17)$

(ii) $P(x)=6x^3-5x^2-17x+6$

Now $\alpha+\beta+\gamma=\frac{5}{6}$

$\alpha\beta+\alpha\gamma+\beta\gamma=\frac{-17}{6}$

$\therefore \alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$
$=\left(\frac{5}{6}\right)^2-2\left(\frac{-17}{6}\right)$
$=\frac{25}{36}+\frac{34}{6}$
$=6\frac{13}{36}$

(d) (i) $P(x)=e^x-x-2$

$P'(x)=e^x-1$

By Newton's Method $z_2=z_1-\frac{P(z_1)}{P'(z_1)}$

$\therefore$ if $z_1=1.2$ $z_2=1.2-\frac{P(1.2)}{P'(1.2)}$
$=1.2-\frac{0.120116922}{2.320116923}$
$=1.148228074\ldots$

$\therefore z_3=z_2-\frac{P(z_2)}{P'(z_2)}$
$=1.14822\ldots-\frac{P(1.148\ldots)}{P'(1.148\ldots)}$
$=1.14822\ldots-\frac{0.00437370}{2.15260178}$
$=1.146196251\ldots$
$=1.1462$ (4 d.p.)

(ii) Newton's Method cannot be used when $P'(x)=0$

Now $P'(x)=0$ if $x=0$

2 (a) (i) $x^2=4ay$

$\therefore y=\frac{x^2}{4a}$

$\frac{dy}{dx}=\frac{2x}{4a}=\frac{x}{2a}$

At $P(2ap,ap^2)$ $\frac{dy}{dx}=\frac{2ap}{2a}=p=m_{\text{tangent}}$

$\therefore m_{\text{normal}}=\frac{-1}{p}$

$\therefore$ Eq'n of normal to curve at P is:

$y-ap^2=\frac{-1}{p}(x-2ap)$

$\therefore py-ap^3=-x+2ap$

$\therefore x+py=ap^3+2ap$ —— (1)

Similarly the eq'n of the normal at Q is:

$x+qy=aq^3+2aq$ —— (2)

Solving (1) and (2) to find R:

(1)−(2): $y(p-q)=a(p^3-q^3)+2a(p-q)$

$\therefore y=\frac{a(p-q)(p^2+pq+q^2)+2a(p-q)}{(p-q)}$

$\therefore y=a(p^2+pq+q^2)+2a$

$\therefore y=a(p^2+pq+q^2+2)$ sub into (1)

$\therefore x+ap(p^2+pq+q^2+2)=ap^3+2ap$

$\therefore x+\cancel{ap^3}+ap^2q+apq^2+\cancel{2ap}=\cancel{ap^3}+\cancel{2ap}$

$\therefore x=-apq(p+q)$

$\Rightarrow R=(-apq(p+q),\ a(p^2+pq+q^2+2))$

(ii) If $pq=-8$ $\therefore$ R becomes:

$R=(8a(p+q),\ a(p^2+q^2-6))$

$\therefore x=8a(p+q)$ $\Rightarrow p+q=\frac{x}{8a}$ —— (1)

$y=a(p^2+q^2-6)$ $\Rightarrow p^2+q^2=\frac{y}{a}+6$ —— (2)

But $p^2+q^2=(p+q)^2-2pq$
$=(p+q)^2+16$

$\therefore \frac{y}{a}+6=\left(\frac{x}{8a}\right)^2+16$ (sub (1) and (2))

$\therefore \frac{y}{a}-10=\frac{x^2}{64a^2}$

$\therefore x^2=64a^2\left(\frac{y}{a}-10\right)$

$\therefore x^2=64ay-640a^2$

$\therefore x^2=64a(y-10a)$

in the form $X^2=4A(y-k)$.

Now $4A=64a$ $\therefore A=16a$

and as vertex $=(0,10a)$

$\Rightarrow$ focus $=(0,26a)$.

(b) $I=\int\frac{3x^2}{\sqrt{5x^3-6}}\,dx$

let $u=5x^3-6$

$\therefore \frac{du}{dx}=15x^2$

$\therefore \frac{du}{5}=3x^2\,dx$

$\therefore I=\int\frac{\frac{du}{5}}{\sqrt{u}}$
$=\frac{1}{5}\int u^{-\frac{1}{2}}\,du$
$=\frac{1}{5}\cdot 2u^{\frac{1}{2}}+c$
$=\frac{2}{5}\sqrt{5x^3-6}+c$

(c) $I=\int_0^1 x\sqrt{2x+2}\,dx$

$u^2=2x+2$ | when $x=0$ $u=\sqrt{2}$
$\therefore 2u\frac{du}{dx}=2$ | $x=1$ $u=2$
$\therefore u\,du=dx$ | $x=\frac{u^2-2}{2}$

$\therefore I=\int_{\sqrt{2}}^{2}\frac{u^2-2}{2}\cdot u\cdot u\,du$
$=\frac{1}{2}\int_{\sqrt{2}}^{2}u^4-2u^2\,du$
$=\frac{1}{2}\left[\frac{u^5}{5}-\frac{2u^3}{3}\right]_{\sqrt{2}}^{2}$
$=\frac{1}{2}\left[\left(\frac{32}{5}-\frac{16}{3}\right)-\left(\frac{4\sqrt{2}}{5}-\frac{4\sqrt{2}}{3}\right)\right]$
$=\frac{1}{2}\left[\frac{16}{15}-\left(\frac{-8\sqrt{2}}{15}\right)\right]$
$=\frac{1}{2}\left[\frac{16+8\sqrt{2}}{15}\right]$
$=\frac{8+4\sqrt{2}}{}$

(a) (i) let $y = x^2 e^{-2x}$

$\therefore \frac{dy}{dx} = x^2 \cdot -2e^{-2x} + e^{-2x} \cdot 2x$

$= 2xe^{-2x}[1-x]$

(ii) $y = \frac{6x^2}{\sqrt{e^x-1}}$

$\therefore \frac{dy}{dx} = \frac{(e^x-1)^{\frac{1}{2}} \cdot 12x - 6x^2 \cdot \frac{1}{2}(e^x-1)^{-\frac{1}{2}} \cdot e^x}{e^x-1}$

$= \frac{3x(e^x-1)^{-\frac{1}{2}}[4(e^x-1) - xe^x]}{e^x-1}$

$= \frac{3x(4e^x-4-xe^x)}{(e^x-1)^{3/2}}$

(b) $I = \int e^{2x-3}\,dx$

$= \frac{1}{2}e^{2x-3} + c$

(c) $I = \int_0^1 x^2 e^{x^3}\,dx$

let $u = e^{x^3}$ — when $x=0$ $u=1$, $x=1$ $u=e$

$\therefore \frac{du}{dx} = 3x^2 e^{x^3}$

$\therefore \frac{du}{3} = x^2 e^{x^3}\,dx$

$\therefore I = \int_1^e \frac{du}{3}$

$= \frac{1}{3}[u]_1^e$

$= \frac{1}{3}[e-1]$

(d) $y = e^{2x}$

$\frac{dy}{dx} = 2e^{2x}$

At $x=-2$ $\frac{dy}{dx} = 2e^{-4}$ = m tangent

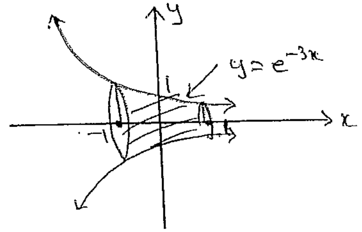
when $x=-2$ $y = e^{-4}$

∴ Eqn of req'd tangent is:

$y - e^{-4} = 2e^{-4}(x - -2)$

$\therefore 2e^{-4}x - y + 5e^{-4} = 0$

(e)



Volume $= \pi \int_{-1}^{1} y^2\,dx$

$= \pi \int_{-1}^{1} e^{-6x}\,dx$

$= \pi \left[\frac{e^{-6x}}{-6}\right]_{-1}^{1}$

$= \frac{-\pi}{6}\left[e^{-6} - e^{6}\right]$

$= \frac{\pi}{6}(e^6 - e^{-6})$ units^3

4 (a) $y = \frac{x^2+4}{x-2}$

when $x=0$ $y=-2$ ∴ intercept at $(0,-2)$

y is undefined when $x=2$

⇒ vertical asymptote at $x=2$.

$$\begin{array}{r} x+2 \\ x-2\,\overline{)\,x^2 \qquad +4} \\ -(x^2-2x) \\ \hline 2x+4 \\ -(2x-4) \\ \hline 8 \end{array}$$

$\therefore y = x+2 + \frac{8}{x-2}$

As $x \to \pm\infty$ $y \to x+2$

∴ oblique asymptote at $y = x+2$

$\frac{dy}{dx} = \frac{(x-2)\cdot 2x - (x^2+4)\cdot 1}{(x-2)^2}$

$= \frac{2x^2-4x-x^2-4}{(x-2)^2}$

$= \frac{x^2-4x-4}{(x-2)^2}$

For a stationary point $\frac{dy}{dx} = 0$

$\therefore x^2 - 4x - 4 = 0$

$\therefore x = \frac{4 \pm \sqrt{16-4\cdot1\cdot-4}}{2}$

$= \frac{4 \pm \sqrt{32}}{2}$

$= \frac{4 \pm 4\sqrt{2}}{2}$

$= 2 \pm 2\sqrt{2}$

At $x = 2+2\sqrt{2}$

x	$(2+2\sqrt{2})^-$	$2+2\sqrt{2}$	$(2+2\sqrt{2})^+$
y'	$-$	0	$+$

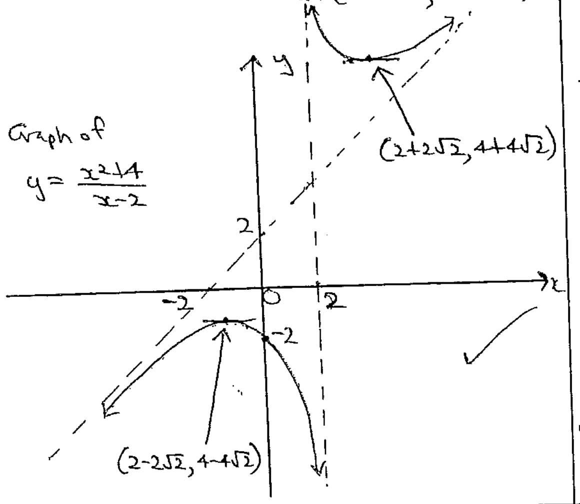
⇒ min. turn pt at $(2+2\sqrt{2}, 4+4\sqrt{2})$

At $x = 2-2\sqrt{2}$

x	$(2-2\sqrt{2})^-$	$2-2\sqrt{2}$	$(2-2\sqrt{2})^+$
y'	$+$	0	$-$

⇒ max. turn pt at $(2-2\sqrt{2}, 4-4\sqrt{2})$

Graph of $y = \frac{x^2+4}{x-2}$



(b) $y = 2xe^{-2x}$

$\frac{dy}{dx} = 2x \cdot -2e^{-2x} + e^{-2x} \cdot 2$

$= 2e^{-2x}(1-2x)$

$\frac{d^2y}{dx^2} = 2e^{-2x}(-2) + (1-2x)\cdot -4e^{-2x}$

$= -4e^{-2x}[1 + (1-2x)]$

For a stationary point $\frac{dy}{dx} = 0$

$\therefore x = \frac{1}{2}$

When $x = \frac{1}{2}$ $\frac{d^2y}{dx^2} < 0$

⇒ max. turn pt. at $(\frac{1}{2}, e^{-1})$

For a possible point of inflexion

$\frac{d^2y}{dx^2} = 0$ $\therefore x = 1$

At $x=1$

x	1^-	1	1^+
y''	$-$	0	$+$

concavity change

⇒ pt. of inflexion at $(1, 2e^{-2})$

When $x=0$ $y=0$ ∴ intercept at $(0,0)$

As $x \to \infty$ $y \to 0$

∴ Asymptote at $y=0$ for $x > 1$

As $x \to -\infty$ $y \to -\infty$.