

CRANBROOK SCHOOL

YEAR 12 EXT1-11ACC MATHEMATICS – TEST

14th March, 2006

Circle teacher: SKB JJA CJL

- Geometrical Applications of Calculus
- Approximation to roots of $P(x) = 0$
- Parametrics
- Integration

Time: 50mins

All necessary working should be shown in every question.
Full marks may not be awarded if work is careless or badly arranged.
Approved silent calculators may be used.
Begin each question on a new page.

1. (12 marks) (Begin a new page) SKB

(a) A closed cylindrical can contains 2156 cm^3 of air.

(i) Show that the surface area can be expressed as:

$$SA = \frac{4312}{r} + 2\pi r^2 \quad 2$$

(ii) Hence calculate its minimum surface area to the nearest cm^2 . 3

(b) (i) Use long division to show that the curve $y = \frac{2x^2}{x-1}$ can be expressed as

$$y = 2x + 2 + \frac{2}{x-1} \quad 1$$

(ii) Hence or otherwise determine the oblique asymptote, any other asymptotes, any stationary points and any points of inflexion. 5

(iii) Hence sketch the curve $y = \frac{2x^2}{x-1}$, showing these features. 1

2. (12 marks) (Begin a new page) JJA

(a) By using the 'halving the interval' method twice find an approximation to the root of $x^3 + 2x - 8 = 0$ in the interval $1 < x < 2$. 3

(b) If $P(x) = 2x^3 - 3x^2 + 1.1$ has a root near $x = 1$. By using $z_1 = 0.4$ as a first approximation find a closer approximation to this root by using Newton's Method once leaving your answer correct to 3 decimal places. 3

Why would Newton's Method have failed if $z_1 = 1$ had been chosen as the first approximation? 1

(c) (i) Show that tangents to the parabola $x^2 = 4ay$ at the points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ intersect at the point $T(a(p+q), apq)$. 2

(ii) If $p^2 + q^2 = 2$ find the equation of the locus of T . Show that this locus is a parabola. 3

3. (12 marks) (Begin a new page)

CJL

(a) Use Simpson's Rule with 6 subintervals to find an approximation to the area bounded by the curve $y = \frac{1}{x^3 - 1}$, the x -axis and lines $x = 2$ and $x = 5$ correct to 4 decimal places. 4

(b) Find the exact volume generated when the area between the line $y = x + 2$ and the curve $y = x^2$ is rotated about the x -axis. 4

(c) By using the substitution $u^2 = 3x + 4$ evaluate $\int_0^4 x\sqrt{3x+4} dx$. 4

1. (a) (i) $2156 = \pi r^2 h$ — (1)

$SA = 2\pi r h + 2\pi r^2$ — (2)

From (1) $h = \frac{2156}{\pi r^2}$ sub into (2)

$\therefore SA = 2\pi r \left(\frac{2156}{\pi r^2} \right) + 2\pi r^2$

$= \frac{4312}{r} + 2\pi r^2$ ✓

(ii) $\frac{dSA}{dr} = -\frac{4312}{r^2} + 4\pi r$

$\frac{d^2SA}{dr^2} = \frac{8624}{r^3} + 4\pi$ ✓

For a possible max/min $\frac{dSA}{dr} = 0$

$\therefore \frac{4312}{r^2} = 4\pi r$

$\therefore r^3 = \frac{4312}{4\pi} = \frac{1078}{\pi}$

$\therefore r = \sqrt[3]{\frac{1078}{\pi}}$ ✓

when $r = \sqrt[3]{\frac{1078}{\pi}}$ $\frac{d^2SA}{dr^2} > 0 \Rightarrow$ min. surface area when $r = \sqrt[3]{\frac{1078}{\pi}}$

This minimum surface area = 923.876...
= 924 cm²
(to nearest cm²) ✓

(b) (i) $y = \frac{2x^2}{x-1}$

$$\begin{array}{r} x-1 \overline{) 2x^2} \\ \underline{-(2x^2 - 2x)} \\ 2x \end{array}$$

$$\begin{array}{r} \underline{-(2x - 2)} \\ 2 \end{array}$$

$\therefore y = 2x + 2 + \frac{2}{x-1}$ ✓

(ii) As $x \rightarrow \pm\infty$ $y \rightarrow 2x + 2$
 \Rightarrow oblique asymptote at $y = 2x + 2$. ✓

y is undefined when $x = 1 \Rightarrow$ vertical asymptote at $x = 1$

when $x = 0$ $y = 0 \Rightarrow$ intercept at $(0, 0)$.

$y = \frac{2x^2}{x-1}$

$\therefore y' = \frac{(x-1) \cdot 4x - 2x^2 \cdot 1}{(x-1)^2}$

$= \frac{4x^2 - 4x - 2x^2}{(x-1)^2}$

$= \frac{2x^2 - 4x}{(x-1)^2}$

$= \frac{2x(x-2)}{(x-1)^2}$ ✓

$y'' = \frac{(x-1)^2 \cdot (4x-4) - (2x^2-4x) \cdot 2(x-1)}{(x-1)^4}$

$= \frac{2(x-1)[(x-1)(2x-2) - (2x^2-4x)]}{(x-1)^4}$

$= \frac{2[2x^2 - 4x + 2 - 2x^2 + 4x]}{(x-1)^3}$

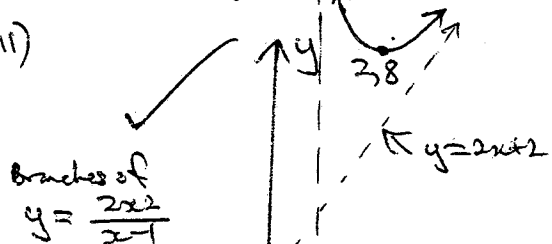
$= \frac{4}{(x-1)^3}$ ✓

For a stat. pt $y' = 0 \therefore x = 0$ or 2

when $x = 0$ $y'' < 0 \Rightarrow$ max. tempt at $(0, 0)$ ✓

when $x = 2$ $y'' > 0 \Rightarrow$ min. tempt at $(2, 8)$

(iii)



No points of inflexion as $y'' \neq 0$ for all x . ✓

2 (a) Let $P(x) = x^3 + 2x - 8$
 $P(1) = -5 < 0$
 $P(2) = 4 > 0$ } as $P(1)$ and $P(2)$ have opposite signs and $P(x)$ is cts for all x .

\Rightarrow there is at least 1 real root in the interval $1 < x < 2$.

Let $x_1 = \frac{1+2}{2} = 1.5$

$P(1.5) = -1.625 < 0$

\therefore as $P(1.5)$ and $P(2)$ have opposite signs \Rightarrow at least 1 real root in interval $1.5 < x < 2$

$\therefore x_2 = \frac{1.5+2}{2} = 1.75$

\therefore Approx to root of $P(x) = 0$ is $x = 1.75$ after using the 'halving the interval' method twice.

(b) $P(x) = 2x^3 - 3x^2 + 1.1$

$P'(x) = 6x^2 - 6x$

$z_1 = 0.4$

By Newton's method $z_2 = z_1 - \frac{P(z_1)}{P'(z_1)}$
 $\therefore z_2 = 0.4 - \frac{P(0.4)}{P'(0.4)}$
 $= 0.4 - \frac{0.748}{-1.44}$
 $= 0.919$ (3dp)

At $z_1 = 1$ $P'(1) = 0$

\Rightarrow there is a stationary point on the curve at $x=1$. A tangent drawn to the curve at this point will be parallel to the x -axis and \therefore not give a 2nd approx to the root. \therefore Newton's method will fail (z_2 will be undefined).

(c) (i) $x^2 = 4ay \therefore y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}$

At $P(2ap, ap^2)$ $\frac{dy}{dx} = p = \text{slope of tangent}$
 \therefore Eqn of tangent at P is:

$y - ap^2 = p(x - 2ap)$

$\therefore y - ap^2 = px - 2ap^2$

$\therefore y = px - ap^2$ — (1)

Similarly, the eqn of the tangent at Q is:

$y = qx - aq^2$ — (2)

(1) - (2): $0 = x(p - q) - a(p^2 - q^2)$

$\therefore x = \frac{a(p - q)(p + q)}{(p - q)} = a(p + q)$

sub $x = a(p + q)$ into (1)

$\therefore y = p a(p + q) - ap^2$

$\therefore y = ap^2 + apq - ap^2 = apq$

\Rightarrow point of intersection T is: $(a(p + q), apq)$

(ii) Now from T : $x = a(p + q)$

and $y = apq$

$\therefore p + q = \frac{x}{a}$ — (1)

$pq = \frac{y}{a}$ — (2)

Now $(p + q)^2 = p^2 + q^2 + 2pq$

$\therefore \left(\frac{x}{a}\right)^2 = 2 + 2\left(\frac{y}{a}\right)$

$\therefore \frac{x^2}{a^2} = 2\left(1 + \frac{y}{a}\right)$

$\therefore x^2 = 2a^2\left(1 + \frac{y}{a}\right)$

$x^2 = 2a^2 + 2ay$

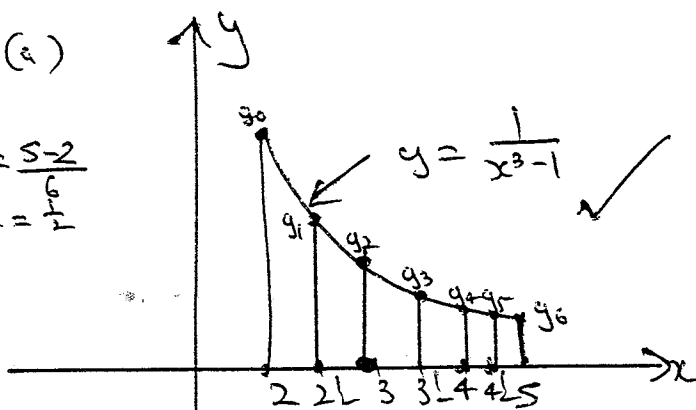
$\therefore x^2 = 2a(y + a)$ is the basis!

which is of the form $x^2 = 4a(y - k)$
 \therefore locus is a parabola.

3(a)

$$h = \frac{5-2}{6}$$

$$\therefore h = \frac{1}{2}$$



By Simpson's Rule,

$$\text{Area} = \frac{h}{3} [y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$\therefore \text{Area} = \frac{1}{6} \left[\frac{1}{7} + \frac{1}{124} + 4 \left(\frac{1}{14.625} + \frac{1}{41.875} + \frac{1}{90.125} \right) + 2 \left(\frac{1}{28} + \frac{1}{63} \right) \right]$$

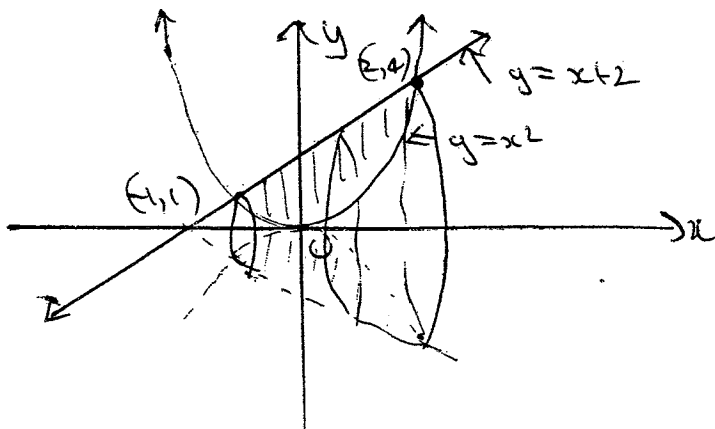
$$= 0.111250954 \dots$$

$$= 0.1113 \text{ units}^2 \text{ (4 dp)}$$

(b) $y = x + 2$ — (1)
 $y = x^2$ — (2)

(2) - (1): $0 = x^2 - x - 2$
 $\therefore 0 = (x-2)(x+1)$
 $\therefore x = 2 \text{ or } -1$

when $x = 2$, $y = 4$, when $x = -1$, $y = 1$
 \Rightarrow pts of int at $(2, 4)$ and $(-1, 1)$



$$\text{Volume} = \pi \int_{-1}^2 (x+2)^2 - (x^2)^2 dx$$

$$= \pi \int_{-1}^2 (x^2 + 4x + 4 - x^4) dx$$

$$= \pi \left[\frac{x^3}{3} + 2x^2 + 4x - \frac{x^5}{5} \right]_{-1}^2$$

$$= \pi \left[\left(\frac{8}{3} + 8 + 8 - \frac{32}{5} \right) - \left(-\frac{1}{3} + 2 - 4 + \frac{1}{5} \right) \right]$$

$$= \frac{72\pi}{5} \text{ units}^3$$

(c) $I = \int_0^4 x \sqrt{3x+4} dx$

let $u^2 = 3x+4$, $x = \frac{u^2-4}{3}$
 $\therefore 2u \frac{du}{dx} = 3$ when $x=0$, $u=2$
 $\therefore \frac{2u}{3} du = dx$ when $x=4$, $u=4$

$$\therefore I = \int_2^4 \left(\frac{u^2-4}{3} \right) u \cdot \frac{2u}{3} du$$

$$= \frac{2}{9} \int_2^4 (u^4 - 4u^2) du$$

$$= \frac{2}{9} \left[\frac{u^5}{5} - \frac{4u^3}{3} \right]_2^4$$

$$= \frac{2}{9} \left[\left(\frac{1024}{5} - \frac{286}{3} \right) - \left(\frac{32}{5} - \frac{32}{3} \right) \right]$$

$$= 27 \frac{67}{135}$$