CRANBROOK SCHOOL

YEAR 12 MATHEMATICS – 2 UNIT

Term 1 2005 **Time : 1.5 h / HRK, GC and SKB**

All questions are of equal value.
All necessary working should be shown in every question.
Full marks may not be awarded if work is careless or badly arranged.
Approved silent calculators may be used.
Submit your work in four 4 Page Booklets.

1. (15marks) (Begin a 4 page booklet.) **HRK**

(a) Find the amount of money there will be in an account after 5 years if initially I have $8 000 and the account earns interest at 10% p.a. with interest paid quarterly. (Leave your answer to the nearest cent.) 3

(b) John pays $7 000 at the end of each year into a superannuation fund. If interest is paid at 7.5% p.a. how much superannuation will John have in the fund at the end of 30 years? (Leave your answer to the nearest cent.) 4

(c) A family buys a house for $920 000 by taking out a loan at 8% p.a. interest, charged monthly over 20 years. If no repayments need to be paid for the first six months,

find : (i) the amount of each monthly repayment (to the nearest cent). 5

(ii) the amount that would still be owing on the loan after 10 years (to the nearest cent). 3

2. (15marks) (Begin a 4 page booklet.) **SKB**

(a) Determine the values of x for which the curve $y = -2x^3 - 6x^2$ is: 4

(i) increasing (ii) decreasing

(iii) concave down (iv) concave up

(b) Consider the curve given by $y = 2x^3 + 5x^2 - 4x$

(i) Find the coordinates of any stationary points and determine their nature. 2

(ii) Find the coordinates of the point of inflexion proving its existence. 2

(iii) Hence sketch the curve $y = 2x^3 + 5x^2 - 4x$ including any intercepts. 2

(c) A cylindrical silo is required to hold $9000\,m^3$ of wheat.

(i) Show that the surface area, S, of the silo is given by $S = 2\pi r^2 + \dfrac{18000}{r}$ 2

(ii) Hence find the least surface area required to hold $9000\,m^3$ of wheat, to the nearest m^2. 3

3. (15marks) (Begin a 4 page booklet.) **GC**

(a) The 2^{nd} derivative of a function is given by $f''(x) = 15x^2 - 6x + 5$. If $f'(0) = 5$ and $f(-1) = 6$ find the function $f(x)$ and the value of $f(2)$. 5

(b) Using the trapezoidal rule, find the area bounded by the curve $y = \sqrt{x(x+1)}$, the $x-$axis and lines $x = 0$ and $x = 4$, with 5 function values correct to 2 decimal places. 5

(c) An irregular coal seam has a length of 60 metres and a width of 25 metres. The height of the coal seam at 10 metre intervals is given by 15m, 17m, 18m, 18.5m, 17m, 16m and 14.5m respectively. Find the volume of the coal seam, in m^3, using Simpson's Rule, correct to 3 significant figures. 5

4. **(15marks) (Begin a 4 page booklet.)** GC

(a) Evaluate: $\int_{-3}^{3} x^4 + 5\ dx$ **2**

(b) Find: $\int \frac{5x^5 - 4x - 6}{x^3}\ dx$ **2**

(c) Find: $\int \sqrt[3]{7x-2}\ dx$ **2**

(d) Find the area enclosed by the curve $x = y^2 - 5y + 6$ and the y-axis. **3**

(e) Find the area between the curve $y = x^2 - 2$ and the line $x + y = 10$. **3**

(f) Find the volume of the solid formed when the area bounded by the curve $x = y^2 + 1$, the y-axis and lines $y = -3$ and $y = 3$ is rotated about the y-axis. Leave your answer in terms of π. **3**

Year 12 2UMT MINI 2005

1 (a) $A_n = P\left(1+\frac{r}{100}\right)^n$

$P = 8000$, $r = \frac{10}{4} = 2.5$

$n = 5\times 4 = 20$.

$\therefore A_{20} = 8000\left(1+\frac{2.5}{100}\right)^{20}$

$= 8000(1.025)^{20}$

$\therefore$ Amount $= \$13108.93$ (to nearest cent)

(b)

The 1st \$7000 accumulates to $7000(1.075)^{29}$

the 2nd \$7000 " " $7000(1.075)^{28}$

⋮

the 30th \$7000 " " $7000(1.075)^{0}$

$\therefore$ Total amount in fund at end of 30 years

$= 7000 + 7000(1.075)^1 + \cdots + 7000(1.075)^{29}$

$= 7000\underbrace{\left[1 + 1.075 + \cdots + 1.075^{29}\right]}$

GP $a=1$, $r=1.075$, $n=30$

$= 7000\left[1\left[\frac{1.075^{30}-1}{1.075-1}\right]\right]$

$= \$723\,795.82$ (to nearest cent)

(c) Let amount owing after n months $= A_n$

(i) Amount owing after 6 months,

$A_6 = 920000\left(1+\frac{\frac{8}{12}}{100}\right)^6$

$= 920000(1.00\dot{6})^6$

Amount owing after 7 months, $A_7 = A_6(1.00\dot{6}) - M$

[1 repayment of \$M]

$\therefore A_7 = \left(920000(1.00\dot{6})^6\right)1.00\dot{6} - M$

$= 920000(1.00\dot{6})^7 - M$

Amount owing after 8 months, $A_8 = A_7(1.00\dot{6}) - M$

$\therefore A_8 = \left(920000(1.00\dot{6})^7 - M\right)1.00\dot{6} - M$

$= 920000(1.00\dot{6})^8 - M(1 + 1.00\dot{6})$

$\therefore$ continuing this pattern, the amount owing after 20 years is 240 months is:

$A_{240} = 920000(1.00\dot{6})^{240} - M(1 + 1.00\dot{6} + \cdots + 1.00\dot{6}^{2\ldots})$

But after 240 months the amount owing is zero.

$\therefore 0 = 920000(1.00\dot{6})^{240} - M(1 + 1.00\dot{6} + \cdots + 1.00\dot{6}^{23\ldots})$

$\therefore M = \dfrac{920000(1.00\dot{6})^{240}}{\underbrace{1 + 1.00\dot{6} + \cdots + 1.00\dot{6}^{233}}}$

GP $a=1$, $r=1.00\dot{6}$, $n=234$

$\therefore$ Amount owing, $M = \dfrac{920\,000(1.00\dot{6})^{240}}{1\left[\frac{1.00\dot{6}^{234}-1}{1.00\dot{6}-1}\right]}$

$= \$8092.05$ (to nearest cent)

(ii) Amount owing after 10 years $= A_{10\times 12}$

$= A_{120}$

$\therefore A_{120} = 920000(1.00\dot{6})^{120} - 8092.05\left[1 + 1.00\dot{6} + \cdots + 1.00\dot{6}^{11\ldots}\right]$

$\therefore A_{120} = 920\,000(1.00\dot{6})^{120} - 8092.05\left[1\left[\frac{1.00\dot{6}^{114}-1}{1.00\dot{6}-1}\right]\right]$

$= \$666\,958.63$ (to nearest cent)

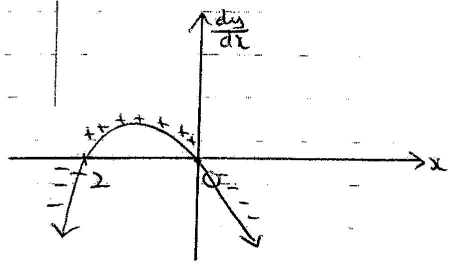
2 (a) $y=-2x^3-6x^2$

$\frac{dy}{dx} = -6x^2-12x$

$= -6x(x+2)$

$\frac{d^2y}{dx^2} = -12x-12$

$= -12(x+1)$

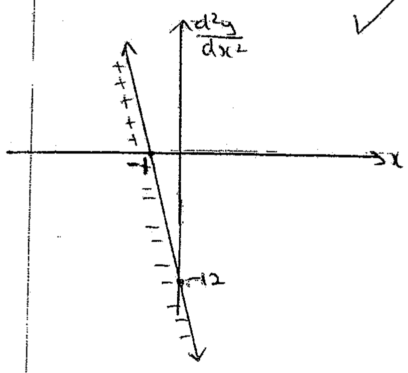


(i) For an increasing curve $\frac{dy}{dx}>0$

∴ from graph: $-2<x<0$

(ii) For a decreasing curve $\frac{dy}{dx}<0$

∴ from graph $x<-2$ or $x>0$



(iii) For concave down $\frac{d^2y}{dx^2}<0$

∴ from graph $x>-1$

(iv) For concave up $\frac{d^2y}{dx^2}>0$

∴ from graph $x<-1$.

(b) $y=2x^3+5x^2-4x$

$\frac{dy}{dx} = 6x^2+10x-4$

$= 2(3x^2+5x-2)$

$= 2(3x-1)(x+2)$

$\frac{d^2y}{dx^2} = 12x+10$

(i) For a stationary point $\frac{dy}{dx}=0$

∴ $x=\frac{1}{3}$ or -2

when $x=\frac{1}{3}$ $\frac{d^2y}{dx^2}>0$ ⇒ min. turn. pt at $\left(\frac{1}{3}, -\frac{19}{27}\right)$

when $x=-2$ $\frac{d^2y}{dx^2}<0$ ⇒ max. turn. pt at $(-2,12)$

(ii) For a possible pt of inflexion $\frac{d^2y}{dx^2}=0$

∴ $x=-\frac{5}{6}$

x	$\left(-\frac{5}{6}\right)^-$	$-\frac{5}{6}$	$\left(-\frac{5}{6}\right)^+$
$\frac{d^2y}{dx^2}$	$-$	0	$+$

concavity change

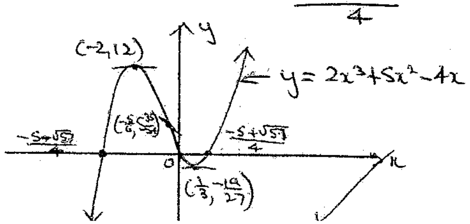
⇒ pt. of inflexion at $\left(-\frac{5}{6}, 5\frac{35}{54}\right)$

(iii) For x-intercepts: $y=x(2x^2+5x-4)$

∴ $0=x(2x^2+5x-4)$

∴ $x=0$ or $x=\frac{-5\pm\sqrt{25-4\cdot 2\cdot -4}}{4}$

$= \frac{-5\pm\sqrt{57}}{4}$



(c)(i) $V=\pi r^2h$

∴ $9000=\pi r^2 h$ —(1)

$S = 2\pi r^2 + 2\pi rh$ —(2)

from (1) $h=\frac{9000}{\pi r^2}$ sub into (2)

∴ $S = 2\pi r^2 + 2\pi r\left(\frac{9000}{\pi r^2}\right)$

$= 2\pi r^2 + \frac{18000}{r}$

(ii) $S = 2\pi r^2 + 18000\,r^{-1}$

∴ $\frac{dS}{dr} = 4\pi r - 18000\,r^{-2}$

$\frac{d^2S}{dr^2} = 4\pi + 36000\,r^{-3}$

For a possible max/min $\frac{dS}{dr}=0$

∴ $4\pi r = \frac{18000}{r^2}$

∴ $r^3 = \frac{18000}{4\pi}$

∴ $r = \sqrt[3]{\frac{9000}{2\pi}}$

when $r=\sqrt[3]{\frac{9000}{2\pi}}$, $\frac{d^2S}{dr^2}>0$ ⇒ min. occurs when $r=\sqrt[3]{\frac{9000}{2\pi}}$

∴ Least surface area required

$= 2\pi\left(\frac{9000}{2\pi}\right)^{2/3} + \frac{18000}{\left(\frac{9000}{2\pi}\right)^{\frac{1}{3}}}$

$= 2395\ m^2$ (to nearest m^2)

3 (a) $f''(x) = 15x^2-6x+5$

∴ $f'(x) = \frac{15x^3}{3} - \frac{6x^2}{2} + 5x + c_1$

$= 5x^3-3x^2+5x+c_1$

As $f'(0)=5$

∴ $5=c_1$

∴ $f'(x) = 5x^3-3x^2+5x+5$

∴ $f(x) = \frac{5x^4}{4} - \frac{3x^3}{3} + \frac{5x^2}{2} + 5x + c$

As $f(-1)=6$

∴ $6 = \frac{5}{4} + 1 + \frac{5}{2} - 5 + c_2$

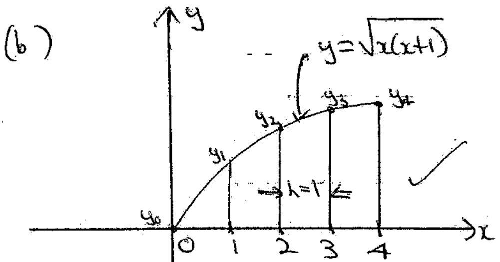
∴ $c = \frac{25}{4}$

∴ $f(x) = \frac{5x^4}{4} - x^3 + \frac{5x^2}{2} + 5x + \frac{25}{4}$

$f(2) = \frac{5}{4}(2)^4 - 2^3 + \frac{5}{2}(2)^2 + 5(2) + \frac{25}{4}$

$= 38\frac{1}{4}$

(b)



$h = \frac{b-a}{n} = \frac{4-0}{4} = 1$ [Graph or equivalent table of values

By the Trapezoidal Rule,

Area $= \int_0^4 \sqrt{x(x+1)}\,dx \doteqdot \frac{h}{2}\left[y_0+y_4+2(y_1\right.$

∴ Area $= \frac{1}{2}\left[0+\sqrt{20}+2(\sqrt{2}+\sqrt{6}+\sqrt{12}\right.$

$= 9.56$ units^2 (2 d.p.)

(c)

x	0	10	20	30	40	50	60
y_i	15	17	18	18.5	17	16	14.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's Rule $\quad h = \frac{60-0}{6} = 10$

$$\int_0^{60} f(x)\,dx \doteqdot \frac{h}{3}\left[y_0+y_6+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{10}{3}\left[15+14.5+4(17+18.5+16)+2(18+17)\right]$$

$$= 1018\tfrac{1}{3}$$

$\therefore$ Volume of coal seam $= 1018\frac{1}{3} \times 25$

$= 25458.\dot{3}\ m^3$

$= 25500\ m^3$ (to 3 sig. figs)

4 (a) $I = \int_{-3}^{3} x^4+5\,dx$

$= 2\left[\frac{x^5}{5}+5x\right]_0^3$ (x^4+5 is an even function)

$= 2\left[\left(\frac{243}{5}+15\right)-0\right]$

$= 127\frac{1}{5}$

(b) $I = \int \frac{5x^5-4x-6}{x^3}\,dx$

$= \int 5x^2-4x^{-2}-6x^{-3}\,dx$

$= \frac{5x^3}{3} - \frac{4x^{-1}}{-1} - \frac{6x^{-2}}{-2} + c$

$= \frac{5x^3}{3} + \frac{4}{x} + \frac{3}{x^2} + c$

(c) $I = \int \sqrt[3]{7x-2}\,dx$

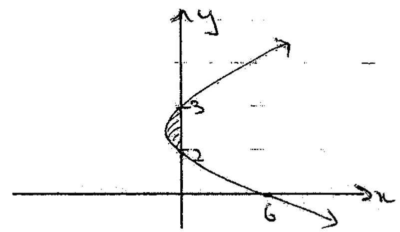
$= \int (7x-2)^{\frac{1}{3}}\,dx$

$\therefore I = \frac{(7x-2)^{4/3}}{4/3 \cdot 7} + c$

$= \frac{3(7x-2)^{4/3}}{28} + c$

(d) $x = y^2-5y+6$

$\therefore x = (y-3)(y-2)$



Area $= \left|\int_2^3 x\,dy\right|$

$= \left|\int_2^3 y^2-5y+6\,dy\right|$

$= \left|\left[\frac{y^3}{3}-\frac{5y^2}{2}+6y\right]_2^3\right|$

$= \left|\left[9-\frac{45}{2}+18\right]-\left[\frac{8}{3}-10+12\right]\right|$

$= \frac{1}{6}$ units^2

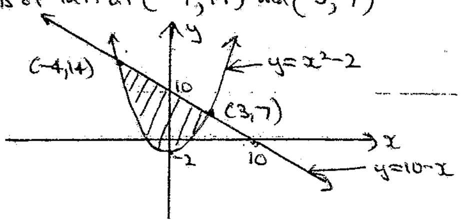
(e) $y = x^2-2$ — ①

$y = 10-x$ — ②

①−②: $0 = x^2+x-12$

$0 = (x+4)(x-3)$

$\Rightarrow$ pts of int. at $(-4,14)$ and $(3,7)$



Area $= \int_{-4}^{3} y_2-y_1\,dx$

$= \int_{-4}^{3} (10-x)-(x^2-2)\,dx$

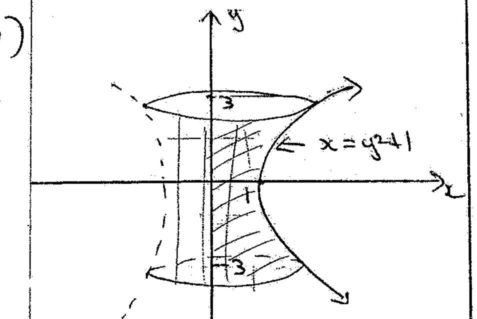
$= \int_{-4}^{3} 12-x-x^2\,dx$

$= \left[12x-\frac{x^2}{2}-\frac{x^3}{3}\right]_{-4}^{3}$

$= \left[\left(36-\frac{9}{2}-9\right)-\left(-48-8+\frac{64}{3}\right)\right]$

$= 57\frac{1}{6}$ units^2

(f)



Req'd Volume $= \pi\int_{-3}^{3} x^2\,dy$

$= \pi\int_{-3}^{3} (y^2+1)^2\,dy$

$= \pi\int_{-3}^{3} y^4+2y^2+1\,dy$

$= 2\pi\left[\frac{y^5}{5}+\frac{2y^3}{3}+y\right]_0^3$

$= 2\pi\left[\left(\frac{243}{5}+\frac{54}{3}+3\right)-0\right]$

$= \frac{696\pi}{5}$ units^3