CRANBROOK
SCHOOL

Year 12 Mathematics

Term 1 Examination

Tuesday April 8, 2008

Instructions

- There are three (3) questions, each worth 20 marks
- Attempt all questions
- Answer each question in a new booklet
- Show all necessary working
- Calculators are allowed in all sections

Time Allowed: 90 minutes   Total Marks: 60

Question 1 (20 Marks) **START A NEW BOOKLET** **Marked by HRK**

(a) Dominic borrows \$10 000 at 15% p.a. over 5 years, repaid in monthly instalments. He does not have to make any repayments for the first two months.

(i) Write an expression for A_1, the amount owing after 2 months. **1**

(ii) Write an expression for A_2, the amount owing after 3 months. **1**

(iii) Write an expression for A_3, the amount owing after 4 months. **1**

(iv) Write an expression for the amount owing after the 5 years. **1**

(v) What is the amount of each repayment? **3**

(b) Find an expression for $f(x)$ if $f''(x) = 48x - 14$, $f'(-1) = 40$ and $f(-2) = -100$ **3**

(c) Find the primitive function of $y = x\sqrt{x} + 1$ **2**

(d) Evaluate the definite integral $\int_0^4 \sqrt{x+3}\, dx$, rounding your answer to 2 d.p. **2**

(e) If $\int_w^{w+3} (2x-4)dx = -13$, find the value of w **3**

(f) Find the area enclosed the curve $y = (x-3)^2$ and the line $y = 3 - x$. Leave your answer in exact form. **3**

EXAMINATION CONTINUES OVER THE PAGE

Question 2 (20 Marks) **START A NEW BOOKLET** **Marked by BMM**

(a) Consider the function $f(x) = x^2 - 5x + 6$.

(i) Sketch the graph of $f'(x)$ **2**

(ii) State the values of x for which:

1. $f'(x) < 0$ **1**

2. $f'(x) = 0$ **1**

3. $f'(x) > 0$ **1**

(b) A company manufactures items at \$2 per item and sells them for \$$x$ per item. If the number of items sold is $\frac{800}{x^2}$ per month, find the value of the selling price of each item for which the company could expect to maximise its monthly profit. **4**

(c) Consider the function $f(x) = -x^3 + 3x^2 - 3x$.

(i) Show that $f(x)$ only has one x intercept **2**

(ii) Determine the coordinates and nature of the stationary point on $f(x)$ **3**

(iii) Sketch $f(x)$ **2**

(iv) State the absolute minimum and absolute maximum for $f(x)$ in the domain $-1 \le x \le 3$ **2**

(d) If $f(x) = ax^3 + 12x^2 + 7$ has a point of inflexion at $\left(-\frac{2}{3}, \frac{95}{9}\right)$ find the value of a. **2**

EXAMINATION CONTINUES OVER THE PAGE

Question 3 (20 Marks) **START A NEW BOOKLET** **Marked by CJL**

(a) Evaluate $5^{x+1} = 9$ to 3 significant figures. **2**

(b) Simplify $\log_4 6 + \log_4 32 - \log_4 3$ **2**

(c) Solve $2\ln x = \ln(7x - 12)$ **3**

(d) Find the following indefinite integrals:

(i) $\int \frac{e^{5x}}{e^{5x} + 2}\, dx$ **2**

(ii) $\int \frac{2}{3x} - 9x\, dx$ **2**

(e) Determine the exact volume created by rotating the region below the curve $y = e^{3x}$ and between the lines $x = 0$ and $x = 4$ about the x-axis. **2**

(f) Find the equation of the tangent to the curve $y = 5 + 3\ln x$ at the point where $x = 1$. **4**

(g) Use Simpson's Rule with 5 ordinates to find an approximation for $\int_0^{12} \frac{dx}{2x+3}$ **3**

END OF EXAMINATION

YEAR 12 2U SOLUTIONS 2008

Question 1 (20 Marks) *Solutions*

(a) Dominic borrows \$10 000 at 15% p.a. over 5 years, repaid in monthly instalments. He does not have to make any repayments for the first two months.

(i)
$$A_1 = [\$10000(1.0125)](1.0125)$$
$$= \$10000(1.0125^2) \quad ✓$$

(ii)
$$A_2 = A_1(1.0125) - M$$
$$= [\$10000(1.0125^2)](1.0125) - M$$
$$= \$10000(1.0125^3) - M \quad ✓$$

(iii)
$$A_3 = A_2(1.0125) - M$$
$$= [\$10000(1.0125^3) - M](1.0125) - M$$
$$= \$10000(1.0125^4) - M(1.0125) - M$$
$$= \$10000(1.0125^4) - M(1.0125 + 1) \quad ✓$$

(iv)
$$A_{59} = \$10000(1.0125^{60}) - M(1.0125^{57} + .. + 1) \quad ✓$$

(v) After 60 months, $A_{60} = 0$
$$0 = \$10000(1.0125^{60}) - M(1.0125^{57} + .. + 1)$$
$$\$10000(1.0125^{60}) = M(1.0125^{57} + .. + 1)$$
$$M = \frac{\$10000(1.0125^{60})}{(1.0125^{57} + .. + 1)} \quad ✓$$

The denominator is a geometric series where a = 1, r = 1.0125, n = 58

$$S_n = \frac{a(r^n - 1)}{r - 1}$$
$$= \frac{1(1.0125^{58} - 1)}{0.0125} \quad ✓$$

$$M = \frac{\$10000(1.1025^{60})(0.0125)}{1.0125^{58} - 1}$$
$$= \$249.55 \quad ✓$$

(b)
$$f''(x) = 48x - 14$$
$$f'(x) = 24x^2 - 14x + c_1 \quad ✓$$
$$f'(-1) = 24 + 14 + c_1$$
$$40 = 38 + c_1$$
$$c_1 = 2$$
$$f(x) = 8x^3 - 7x^2 + 2x + c_2 \quad ✓$$
$$f(-2) = -64 - 28 - 4 + c_2$$
$$-100 = -96 + c_2$$
$$c_2 = -4$$
$$f(x) = 8x^3 - 7x^2 + 2x - 4 \quad ✓$$

(c)
$$y = x\sqrt{x} + 1$$
$$= x^1 \times x^{\frac{1}{2}} + 1$$
$$= x^{\frac{3}{2}} + 1 \quad ✓$$

$$\int x^{\frac{3}{2}} + 1 \, dx = \frac{2x^{\frac{5}{2}}}{5} + x + c \quad ✓$$

(d) 2
$$\int_0^4 \sqrt{x+3} \, dx = \int_0^4 (x+3)^{\frac{3}{2}} \, dx$$
$$= \left[\frac{2(x+3)^{\frac{3}{2}}}{3}\right]_0^4 \quad ✓$$
$$= \left(\frac{2}{3} \times 7^{\frac{3}{2}}\right) - \left(\frac{2}{3} \times 3^{\frac{3}{2}}\right)$$
$$= 8.88 \quad ✓$$

(e)
$$\int_w^{w+3} (2x - 4) \, dx = \left[x^2 - 4x\right]_w^{w+3}$$
$$= \left((w+3)^2 - 4(w+3)\right) - (w^2 - 4w)$$
$$= w^2 + 6w + 9 - 4w - 12 - w^2 + 4w$$
$$-13 = 6w - 3 \quad ✓$$
$$-10 = 6w$$
$$w = \frac{-10}{6} = \frac{-5}{3} = -1\frac{2}{3} \quad ✓$$

(f) Where are the points of intersection?

$$(x-3)^2 = 3-x$$
$$x^2-6x+9 = 3-x$$
$$x^2-5x+6=0$$
$$(x-3)(x-2)=0$$

At $x=3$ and $x=2$ ✓

$$\int_2^3 3-x-(x-3)^2\,dx$$
$$=\int_2^3 -x^2+5x-6\,dx \quad ✓$$
$$=\left[-\frac{x^3}{3}+\frac{5x^2}{2}-6x\right]_2^3$$
$$=\left(\frac{-27}{3}+\frac{45}{2}-18\right)-\left(\frac{-8}{3}+10-12\right)$$
$$=-4\frac{1}{2}-\left(-4\frac{2}{3}\right)$$
$$=\frac{1}{6}\text{ units}^2 \quad ✓$$

Q1 a) Markers Comments

More care needed with reading question and numbering A_1 A_2 .. etc accurately but general concept known well

b) More care needed with substitution and carrying c_1 forward into 2nd integration

c) NOT a product though if used carefully product rule will work but waste time. Rather use $x\sqrt{x} = x^{\frac{3}{2}}$ and remember to integrate the 1

d) Well done except for calculator work.

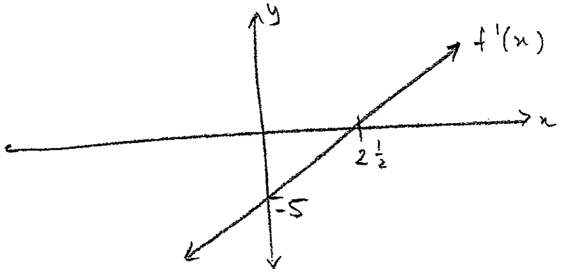
e) An unusual question handled well by many. However, again careless substitution and equation solving let many down

f) Those who drew careful sketches found this straightforward. More practice with basic curve sketching will benefit others.

Question 2 Solns   12 (2u) Mini 2008

a) $f(x) = x^2 - 5x + 6$

(i) $f'(x) = 2x - 5$



✓✓

(ii) 1. $f'(x) < 0$ for $x < 2\frac{1}{2}$ ✓

2. $f'(x) = 0$ for $x = 2\frac{1}{2}$ ✓

3. $f'(x) > 0$ for $x > 2\frac{1}{2}$ ✓

b) Profit per item $= \$(x-2)$

Profit per month if $\frac{800}{x^2}$ items sold

$\therefore$ is $P = \frac{800}{x^2}(x-2)$ ✓

Now find x such that P is a max.

$$P = \frac{800x}{x^2} - \frac{1600}{x^2}$$

$$P = \frac{800}{x} - \frac{1600}{x^2}$$

$$P = 800x^{-1} - 1600x^{-2}$$

$$P' = -800x^{-2} + 3200x^{-3}$$

$$P' = -\frac{800}{x^2} + \frac{3200}{x^3}$$

Let $P' = 0$

$$\frac{800}{x^2} = \frac{3200}{x^3}$$

$$800x^3 = 3200x^2$$

$$800x^3 - 3200x^2 = 0 \quad ✓$$

$$800x^2(x-4)$$

$x = 0 \qquad x = 4$ ✓

Check if $x = 4$ gives max Profit.

$$P'' = 1600x^{-3} - 9600x^{-4}$$

$$P'' = \frac{1600}{x^3} - \frac{9600}{x^4}$$

When $x = 4$ $P'' < 0$

$\therefore$ max ✓

$\therefore$ selling price/item must be \$4.

c) $f(x) = -x^3 + 3x^2 - 3x$

(i) $f(x) = 0$

$x^3 - 3x^2 + 3x = 0$

$x(x^2 - 3x + 3) = 0$ ✓

$x = 0$ or $b^2 - 4ac = 9 - 4(1)(3)$
$= -3$
$\therefore$ no roots ✓

$\therefore x = 0$ is the only intercept

(ii) $f'(x) = -3x^2 + 6x - 3$

$f'(x) = 0$

$3x^2 - 6x + 3 = 0$

$x^2 - 2x + 1 = 0$

$(x-1)(x-1) = 0$

$x = 1$

$\therefore$ Stat pt @ $(1, -1)$ ✓

Nature: $f''(x) = -6x + 6$

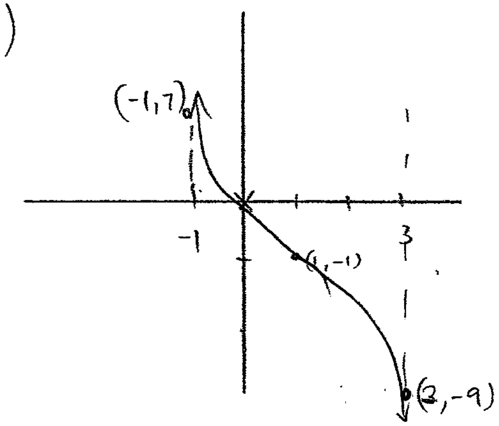
$f''(1) = -6 + 6$
$= 0$ ✓

$\therefore$ possible pt of inflexion → test concavity

x	-1	1	2
$f''(x)$	>0	X	<0

$\therefore$ concavity changes ✓
$\therefore$ horizontal pt of inflexion.

iii)



✓✓

iv) Absolute Min $= -9$ ✓
Absolute Max $= 7$. ✓

d) $f(x) = ax^3 + 12x^2 + 7$

$f'(x) = 3ax^2 + 24x$

$f''(x) = 6ax + 24$

$0 = 6a\left(-\frac{2}{3}\right) + 24$

$0 = -\frac{12a}{3} + 24$ ✓

$0 = -4a + 24$

$4a = 24$

$a = 6$ ✓

12-2U MINI 2008

a) $5^{x+1} = 9$

$(x+1)\ln 5 = \ln 9$

$x+1 = \dfrac{\ln 9}{\ln 5}$ ✓

$x = 0.365$ ✓

b) (i) $\log_4 6 + \log_4 32 - \log_4 3$

$= \log_4 (6 \times 32 \div 3)$

$= \log_4 64$ ✓

$= 3$ ✓

(ii) $2\ln x = \ln(7x-12)$

$\ln x^2 = \ln(7x-12)$

$\therefore x^2 = 7x - 12$ ✓

$x^2 - 7x + 12 = 0$

$(x-4)(x-3) = 0$ ✓

$\therefore x = 4,\ 3$ ✓

c) $\displaystyle\int \frac{e^{5x}}{e^{5x}+2}\,dx$

(i)

$= \dfrac{1}{5}\displaystyle\int \frac{5e^{5x}}{e^{5x}+2}\,dx$

$= \dfrac{1}{5}\ln(e^{5x}+2) + c$ ✓ ✓

(ii) $\displaystyle\int \frac{2}{3x} - 9x\,dx$

$= \dfrac{2}{3}\ln x - \dfrac{9x^2}{2} + c$ ✓ ✓

d) $V = \pi\displaystyle\int y^2\,dx$

$= \pi\displaystyle\int_0^4 (e^{3x})^2\,dx$

$= \pi\displaystyle\int_0^4 e^{6x}\,dx$ ✓

$= \dfrac{\pi}{6}\left[e^{6x}\right]_0^4$

$= \dfrac{\pi}{6}(e^{24}-1)\ u^3$ ✓

e) $y = 5 + 3\ln x$

$y' = \dfrac{3}{x}$ ✓

at $x = 1$ $y' = 3 = m_T$ ✓

at $x = 1$ $y = 5$ $(1,5)$ ✓

$y - 5 = 3(x-1)$ ✓

$y - 5 = 3x - 3$

$y = 3x + 2$

f)

x	0	3	6	9	12
y	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{15}$	$\frac{1}{21}$	$\frac{1}{27}$

$y = \dfrac{1}{2x+3}$ $h = 3$

$\displaystyle\int_0^{12} \frac{dx}{2x+3} \doteqdot \frac{3}{3}\left(\frac{1}{3} + \frac{1}{27} + 4\left(\frac{1}{9}+\frac{1}{21}\right) + 2\left(\frac{1}{15}\right)\right)$

$= 1\dfrac{131}{945}$ ✓✓✓

$\doteqdot 1.14$ (to 2 d.p.)

MARKERS NOTES 2U

a) Many students did not know what 3 sig figs was and gave the answer as 0.37. This is only 2 s.f.

b) (i) Some students used change of base which is perfectly O.K. Probably better to use log laws. Many students stopped at $\log_4 64$ and didn't evaluate it.

(ii) Most students had no clue. They failed to recognise $2\ln x = \ln x^2$. At $\ln x^2 = \ln(7x-12)$ simply equate both sides to obtain $x^2 = 7x-12$ then solve as a quadratic equation. Logs do NOT EXPAND.
ie $\ln(7x-12) \neq \ln 7x - \ln 12$

c) Use $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + c$

(i) This question uses log integration

(ii) Integrate each part separately ie $\displaystyle\int \frac{2}{3x}$ by itself then $\displaystyle\int 9x$

If you try to integrate $\displaystyle\int \frac{2}{3}x^{-1}$ and get $\dfrac{2}{3}\dfrac{x^0}{0}$ you know the answer will be a log as you can't divide by zero

d) Common mistake was $\displaystyle\int e^{6x}\,dx = 6e^{6x}$. This is the differentiation! Some students forgot π, others forgot to square the function. $V = \pi\displaystyle\int y^2\,dx$
so $(e^{3x})^2 = e^{6x}$

(e) Many could not differentiate $3\ln x$ to get $3 \times \dfrac{1}{x} = \dfrac{3}{x}$

Some forgot to substitute $x = 1$ into y' for gradient

(f) Most students did not know what "h" is.
If you use $\dfrac{b-a}{n}$, know what is means. Many used 3 function values or 6 function values, not 5 as asked.
Those who used this version $\dfrac{b-a}{n}\left(f(a) + 4f\left(\dfrac{a+b}{2}\right) + f(b)\right)$ had no clue what it meant!
Easiest way is to use a table of values with $y = \dfrac{1}{2x+3}$ as the function