12 Extension 1 Mathematics	**Calculus, Approximation Methods & Parameters**
Term 1, 2011 | Week 6	*Time Allowed*: 50 mins *Marks*: 35

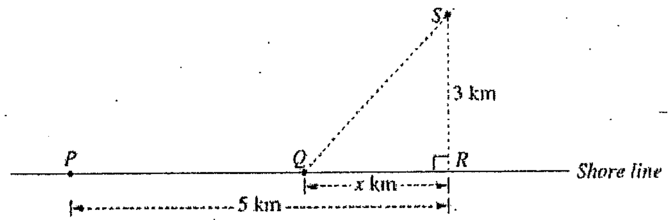
Show all working to gain maximum marks

Marks will be deducted for poorly presented or illegible work

X SOLNS

Question 1 (13 marks) **Start a new booklet** **Marked by GHW**

a) An oil rig, S, is 3km offshore. A power station, P, is on the shore. A cable is to be laid from P to S. It costs \$1000/km to lay the cable along the shore and \$2600/km to lay the cable underwater from the shore to S.
The point R is the point on the shore closest to S, and the distance PR is 5km.
The point Q is on the shore, at a distance of x km from R, as shown in the diagram.



i) Find the total cost of laying the cable in a straight line from P to R and then in a straight line from R to S **1**

ii) Find the cost of laying the cable in a straight line from P to S **1**

iii) Let \$C be the total cost of laying the cable in a straight line from P to Q, and then in a straight line from Q to S

Show that $C = 1000(5 - x + 2.6\sqrt{x^2+9})$ **2**

iv) Find the minimal cost of laying the cable **4**

b) Find the following integrals using the given substitution:

i) $\int x(x^2+2)^2\,dx$, *when* $u = x^2+2$ **2**

ii) $\int_3^{18} \frac{x}{\sqrt{x-2}}\,dx$, *when* $u = \sqrt{x-2}$ **3**

START A NEW BOOKLET

Question 2 (12 marks) **Marked by HRK**

a) Consider the equation $x^3 - 3x + 1 = 0$

i) Show that this equation has a root between 0 and 1. **2**

ii) Taking $x=0$ as a first approximation to this root, use Newton's Method to obtain a better approximation. **3**

iii) Explain why could you not have used $x=1$ as your first approximation. **1**

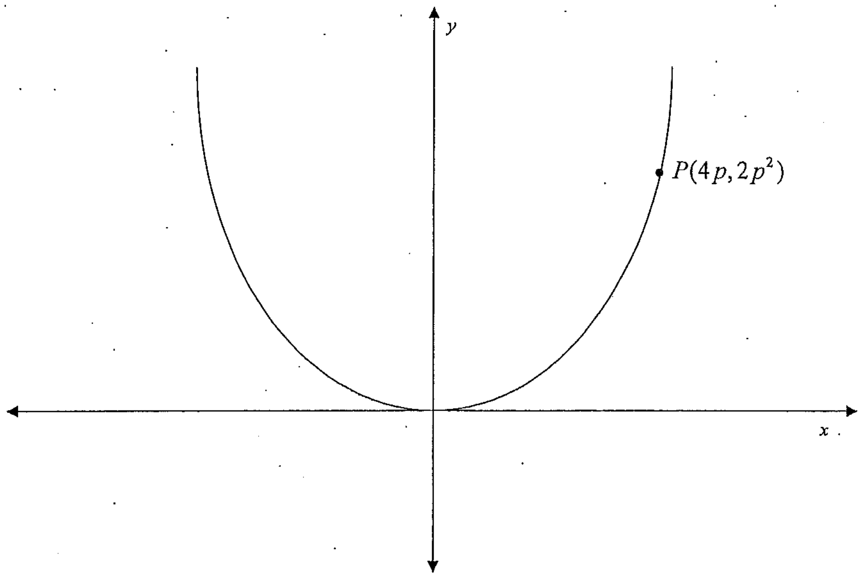
b) Prove by Mathematical induction for all positive integral n

$$1(2)+2(3)+3(4)+\dots\dots\dots+n(n+1)=\frac{n(n+1)(n+2)}{3}$$ **6**

START A NEW BOOKLET

Question 3 (10 marks) **Marked by RDS**

Consider the parabola $x^2 = 8y$ and the point, $P(4p, 2p^2)$, which lies on that parabola.



Copy this diagram into your work books

a) Show that the equation of the tangent at P is $y - px + 2p^2 = 0$. **2**

b) Find the points R and T, the x and y intercepts of the tangent. **2**

c) Show that the ratio of $RT{:}TP$ is independent of p. **2**

d) The normal at P has a y intercept at S. Show that **4**

$ST = 2a(1 + p^2)$

YEAR 12 EXT 1 2011

q.1 Solutions TEST MIDTERM

a) $PR + RS = 5\times 1000 + 3\times 2600$

$= \$12800$ (1)

ii) $PS = \sqrt{3^2+5^2} = \sqrt{34}$

$\therefore$ cost $= \sqrt{34}\times 2600$

$= \$15160.47$ (1)

iii) $\text{Cost} = \underset{PQ}{1000(5-x)} + \underset{QS}{2600\sqrt{(x^2+9)}}$ (1)

$\therefore \quad C = 1000(5-x+2.6\sqrt{x^2+9})$ (1)

iv) $\text{Cost} = 5000 - 1000x + 2600(x^2+9)^{\frac{1}{2}}$

$C' = -1000 + \frac{1}{\cancel{2}}\times \cancel{2}x \times 2600(x^2+9)^{-\frac{1}{2}}$

$= -1000 + 2600x(x^2+9)^{-\frac{1}{2}}$

when $c' = 0$

$1000 = \dfrac{2600x}{\sqrt{x^2+9}}$

$1000\sqrt{x^2+9} = 2600x$

$\sqrt{x^2+9} = 2.6x$

$x^2+9 = 6.76x^2$

$9 = 5.76x^2$

$x = \pm 1.25$

Marks
(1) correct differentiation
(1) $c'=0$
(1) find $x=1.25$ + prove min
(1) final cost

x	1	1.25	2
$f'(x)$	$-$	0	$+$

$\therefore \quad x = 1.25$ is a minimum

$C = 1000(5-1.25+2.6\sqrt{1.25^2+9}) = \$12200$

b) i) $\int x(x^2+2)^2\,dx$

$u = x^2+2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$.

$= \int \frac{1}{2}u^2\,du$

$= \frac{1}{2}\int u^2\,du$

$= \frac{1}{2}\left(\frac{u^3}{3}\right) + c$

$= \frac{u^3}{6} + c$

$= \frac{(x^2+2)^3}{6} + c$

must remember to sub in the original expression at the end!!

ii) $\int_3^{18} \frac{x}{\sqrt{x-2}}\,dx$

$u = \sqrt{x-2}$

if $x=18 \quad u=\sqrt{16} = 4$.

$x = 3 \quad u = \sqrt{1} = 1$.

$x = u^2+2$.

$u^2 = x-2$

$2u\frac{du}{dx} = 1$

$2u\,du = dx$

$= \int_1^4 \frac{u^2+2}{\cancel{u}}\cdot 2\cancel{u}\,du$

$= 2\int_1^4 u^2+2\,du$

$= 2\left[\frac{u^3}{3} + 2u\right]_1^4$

$= 2\left(\left(\frac{64}{3}+8\right) - \left(\frac{1}{3}+2\right)\right)$

$= 54$.

Marks
(1) new bounds and diff
(1) correct integration
(1) final answer

Q2a i) $f(0)=1$

$f(1)=-1$

Since $f(0)>0$ and $f(1)<0$ and the function is continuous* a root exists between $x=0$ and $x=1$

*The fact that the function is continuous should be mentioned.

ii) $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 0 - \frac{1}{-3}$

$= \frac{1}{3}$

$f'(x) = 3x^2 - 3$

$f'(0) = -3$

(iii) Given $f'(1)=0$, x_2 would be undefined

If the gradient is zero, tangent is horizontal and will not cut the x-axis to give another approximation

b) i) Prove true for $n=1$

LHS $= n(n+1)$

$= 1(1+1)$

$= 2$

RHS $= \frac{n(n+1)(n+2)}{3}$

$= 1(2)(3)$

$= 2$

$=$ LHS $\therefore$ true for $n=1$

ii) Assume true for $n=k$

ie $1(2)+2(3)+3(4)+\dots+k(k+1) = \frac{k(k+1)(k+2)}{3}$

iii) Prove true for $n=k+1$

ie $1(2)+2(3)+3(4)+\dots+k(k+1)+(k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$

LHS $= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$ (using assumption

$= \frac{1}{3}\left[k(k+1)(k+2) + 3(k+1)(k+2)\right]$ ★ takes time to expand

$= \frac{1}{3}(k+1)(k+2)(k+3)$ (MUCH QUICKER TO factorise

$=$ RHS

$\therefore$ If true for $n=k$ then it is true for $n=k+1$

It has been shown true for $n=1$ hence by Mathematical Induction it is true for all positive integers

Ext 1 Solutions Term 1 Test 1

Question 3

a) $x^2 = 8y$

$y = \frac{x^2}{8}$

$\frac{dy}{dx} = \frac{2x}{8} = \frac{x}{4}$

At P, $m_{tangent} = \frac{4p}{4} = p$

$\therefore p = \frac{y-2p^2}{x-4p}$

$px - 4p^2 = y - 2p^2$

$y - px + 2p^2 = 0$ As required.

b) R occurs when $y=0$

$0 - px + 2p^2 = 0$

$x = 2p$

$\therefore R(2p, 0)$

T occurs when $x=0$

$y - 0 + 2p^2 = 0$

$y = -2p^2$

$T(0, -2p^2)$

c) $RT^2 = (2p)^2 + (-2p^2)^2$

$= 4p^2 + 4p^4$

$= 4p^2(1+p^2)$

$TP^2 = (4p)^2 + (2p^2+2p^2)^2$

$= 16p^2 + 16p^4$

$= 16p^2(1+p^2)$

$RT : TP = \dfrac{RT}{TP}$

$= \sqrt{\dfrac{4\cancel{p^2}\cancel{(1+p^2)}}{16\cancel{p^2}\cancel{(1+p^2)}}}$

$= \dfrac{1}{2}$ which is independent of p

d) Normal at P

$m_{Normal} = -\dfrac{1}{p}$

$-\dfrac{1}{p} = \dfrac{y-2p^2}{x-4p}$

$-x + 4p = py - 2p^3$

S occurs when $x = 0$

$4p = py - 2p^3$

$\cancel{p}y = 2\cancel{p}(2+p^2)$

$y = 2(2+p^2)$

As both points lie on the y axis
ST = difference in their y coordinates

$= 2(2+p^2) + 2p^2$

$= 4 + 4p^2$

$= 4(1+p^2)$

As $x^2 = 4ay$ and this equation is $x^2 = 8y$ then $a = 2$

$\therefore ST = 2a(1+p^2)$