Centre Number | 1 | 2 | 5 |

Student Number | | | | | | | | |

CRANBROOK SCHOOL

2015

Half Yearly Examination

Assessment Task 2

Mathematics Extension 2

Reading time	5 minutes
Writing time	90 Minutes
Total Marks	55
Task weighting	30%

General Instructions

- Write using blue or black pen
- Diagrams drawn using pencil
- A Board-approved calculator may be used
- A table of standard integrals can be found on page 16 of this paper
- Use the Multiple-Choice Answer Sheet provided
- All relevant working should be shown for each question

Additional Materials Needed

- Multiple Choice Answer Sheet
- 3 writing booklets

Structure & Suggested Time Spent

Section I

Multiple Choice Questions

- Answer Q1 – 10 on the multiple choice answer sheet
- Allow 16 minutes for this section

Section II

Extended response Questions

- Attempt all questions in this section in a separate writing booklet
- Allow about 74 minutes for this section

This paper must not be removed from the examination room

Disclaimer

The content and format of this paper does not necessarily reflect the content and format of the HSC examination paper.

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0,\ \text{if } n < 0.$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0.$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0.$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a},\ a > 0, -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left\{x+\sqrt{(x^2-a^2)}\right\},\ |x| > |a|$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left\{x+\sqrt{(x^2+a^2)}\right\}$$

NOTE: $\ln x = \log_e x,\ x > 0.$

Section I

10 Marks

Allow about 16 minutes for this section

Use the multiple choice answer sheet for Questions 1 to 10.

Question 1

Which of the following is equivalent to $(2-2i)^3$?

(A) $16+16i$

(B) $-16-16i$

(C) $6+6i$

(D) $-6-6i$

Question 2

What is $-2\sqrt{2}-2\sqrt{2}i$ in modulus argument form?

(A) $4\left(\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\right)$

(B) $4\left(\cos\frac{3\pi}{4}+\sin\frac{3\pi}{4}\right)$

(C) $4\left(\cos\frac{\pi}{4}-\sin\frac{\pi}{4}\right)$

(D) $4\left(\cos\frac{3\pi}{4}-\sin\frac{3\pi}{4}\right)$

Question 3

What is the equation of an ellipse with the origin as the centre, a focus at $(0,3)$ and a corresponding directrix at $y=5$?

(A) $\frac{x^2}{6}+\frac{y^2}{15}=1$

(B) $\frac{x^2}{15}+\frac{y^2}{6}=1$

(C) $\frac{x^2}{\sqrt{6}}+\frac{y^2}{\sqrt{15}}=1$

(D) $\frac{x^2}{\sqrt{15}}+\frac{y^2}{\sqrt{6}}=1$

Question 4

The complex numbers 0, z_1 and z_2 where $Arg(z_1)<Arg(z_2)$, are three vertices of a parallelogram where the 4th point is the complex number z_1+z_2. All the vertices lie in the first quadrant.

Which statement is true if the parallelogram is a rhombus?

(A) $z_1=iz_2$ or $iz_1=z_2$

(B) $|z_1+z_2|=|z_1-z_2|$

(C) $|z_1+z_2|=|z_1|=|z_2|$

(D) $Arg(z_1+z_2)=Arg[i(z_1-z_2)]$

Question 5

Which of the following is the factorised form of the polynomial $P(z)=z^3-2z^2-\dfrac{27x}{4}+5$?

(A) $P(z)=(z+4)\left(z+1-\dfrac{i}{2}\right)\left(z+1+\dfrac{i}{2}\right)$

(B) $P(z)=(z-4)\left(z+1-\dfrac{i}{2}\right)\left(z+1+\dfrac{i}{2}\right)$

(C) $P(z)=(z-4)\left(z-1-\dfrac{i}{2}\right)\left(z-1+\dfrac{i}{2}\right)$

(D) $P(z)=(z+4)\left(z-1-\dfrac{i}{2}\right)\left(z-1+\dfrac{i}{2}\right)$

Question 6

Let the polynomial $P(x)=3x^3-2x^2+5x-7$ have roots α, β and γ.

Which of the following polynomials has roots $(1-\alpha)$, $(1-\beta)$ and $(1-\gamma)$?

(A) $P(x)=-3x^3+7x^2-10x-1$

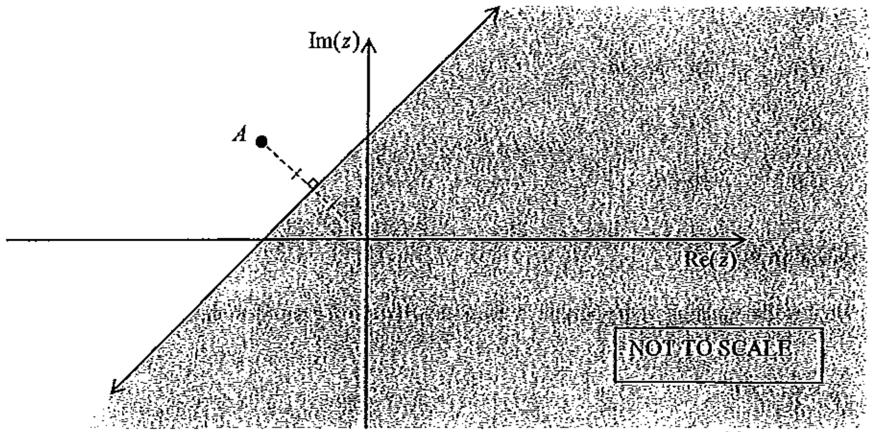
(B) $P(x)=3x^3-7x^2+10x-1$

(C) $P(x)=-3x^3-7x^2-10x-1$

(D) $P(x)=3x^3+7x^2+10x-1$

Question 7

The following is a sketch in the Argand Diagram. The point A is the complex number $-\pi+i\pi$.



Which of the following could represent the region shaded?

(A) $-\pi \le Arg(z) \le \pi$

(B) $Arg(z) \ge \pi, Arg(z) \le -\pi$

(C) $|z| \le |z+\pi-i\pi|$

(D) $|z| \ge |z+\pi-i\pi|$

Question 8

Let $P(x)=x^4-2px^2+3qx-m$ have roots α, β, γ and σ. If $\alpha^2+\beta^2+\gamma^2+\sigma^2=1$ what is the value of $\alpha^4+\beta^4+\gamma^4+\sigma^4$?

(A) 1

(B) $2p-3q+m$

(C) $2p+4m$

(D) $2p+m$

Question 9

What is the equation of the normal at the point $P(a\sec\theta, b\tan\theta)$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$?

(A) $\frac{ax}{\sec\theta}-\frac{by}{\tan\theta}=a^2+b^2$

(B) $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$

(C) $\frac{\sec\theta x}{a}-\frac{\tan\theta y}{b}=1$

(D) $\frac{\sec\theta x}{a}+\frac{\tan\theta y}{b}=1$

Question 10

A complex number, z, lies on a circle of radius $\frac{1}{2}$ with centre at the origin in the Argand diagram. w lies on another circle with radius $\frac{3}{2}$ and centre at the origin. If $\text{Im}(z)<0$, $\text{Re}(z)\neq 0$ and $Arg\left(\frac{w}{z}\right)=\frac{\pi}{2}$ then which statement is TRUE in regards to the complex number zw?

(A) zw lies on the unit circle.

(B) If $Arg(z)+\pi<\frac{\pi}{4}$ then $\text{Re}(zw)<0$.

(C) If $Arg(z)+\pi<\frac{\pi}{4}$ then $\text{Im}(zw)>0$.

(D) zw will be in the first quadrant.

END OF SECTION I

Section II

45 Marks

Allow about 74 minutes for this section

Answer questions 11 to 13 in separate booklets.

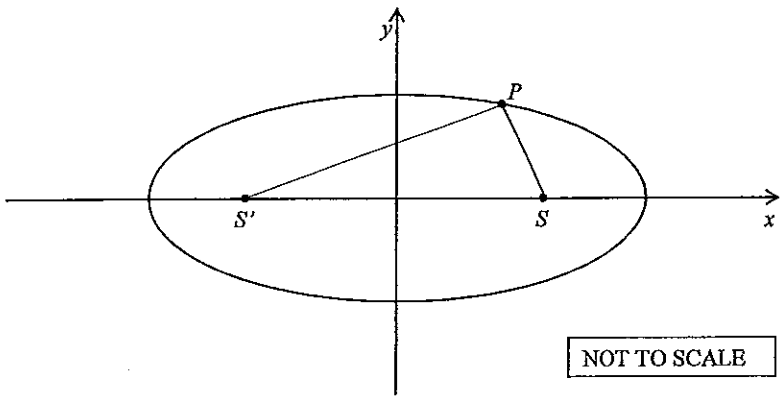
Question 11 **Begin a new booklet** **15 Marks**

a) Let $|z-1-i| \le 1$ and $\text{Re}(z) > 1$ define a region in the Argand plane.

i. Sketch the region showing points of intersection. **2**

ii. Find the value of z in the region where $Arg(z)$ is a maximum. **1**

b) Let P be a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $a > b$. Let S and S' be the foci of the ellipse.



Prove that $PS + PS' = 2a$ **2**

c) The points $M\left(2m, \frac{2}{m}\right)$ and $N\left(2n, \frac{2}{n}\right)$ lie on opposite branches of the rectangular hyperbola $xy = 4$.

i. Find the equation of the chord MN **2**

ii. MN goes through the y axis at 1. Write the coordinates of the point N using the parameter m. **2**

d) Sketch $|z-3+i| = \text{Re}(z)-1$ showing all intercepts. **2**

e) The polynomial $P(z) = 3z^4 + (14-6i)z^3 - (28i+8)z^2 + (10i-14)z + 5$ has a double root at $z = i$.

Solve $P(z) = 0$ over the complex field. **4**

Question 12 **Begin a new booklet** **15 Marks**

a) Let $P(x)$ be a polynomial of degree n with a double root at $x=\alpha$.

i. Show that $P'(\alpha)=0$ **1**

ii. Hence find the values of a and b given the polynomial $Q(x)=ax^3+bx+5i$ has a double root at $x=4i$ **2**

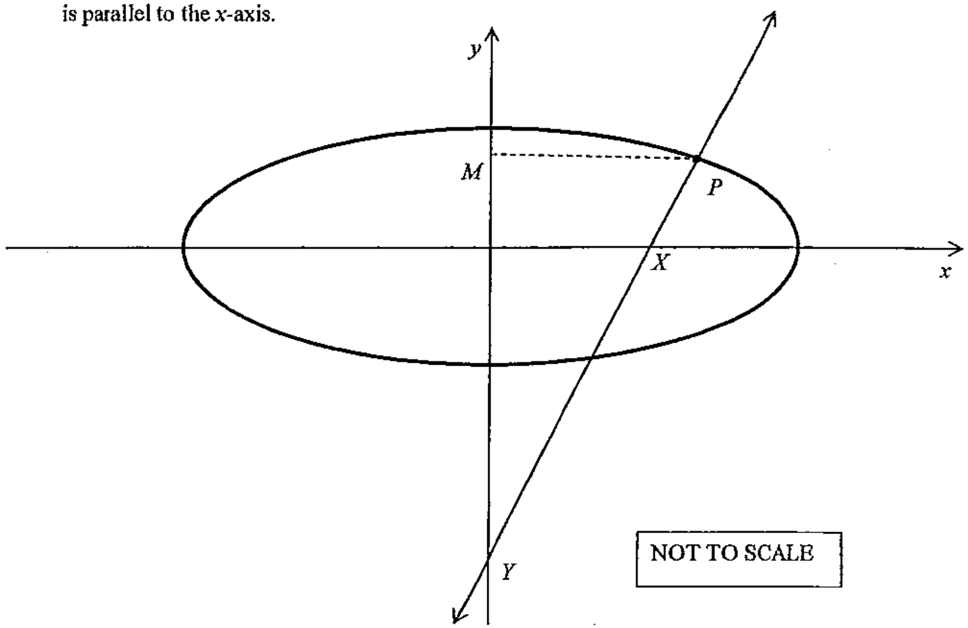
b) Let $P(x)=x^3-3px^2+q$ have roots have roots α, β and γ.

i. Form a polynomial whose roots are $\dfrac{1}{\alpha}$, $\dfrac{1}{\beta}$ and $\dfrac{1}{\gamma}$. **2**

ii. Hence or otherwise find a polynomial whose roots are $\dfrac{1+\alpha}{\alpha}$, $\dfrac{1+\beta}{\beta}$ and $\dfrac{1+\gamma}{\gamma}$. **2**

c) Sketch $\dfrac{x^2}{2}-\dfrac{y^2}{8}=1$ showing all intercepts, asymptotes, foci and directrices. **2**

d) The point $P(x_p, y_p)$ lies on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. The normal at the point P cuts the x- and y-axis at X and Y. The point M is the y intercept of the line drawn through P that is parallel to the x-axis.



i. Show that the equation of the normal at P is **2**

$$\frac{a^2x}{x_p}-\frac{b^2y}{y_p}=a^2-b^2$$

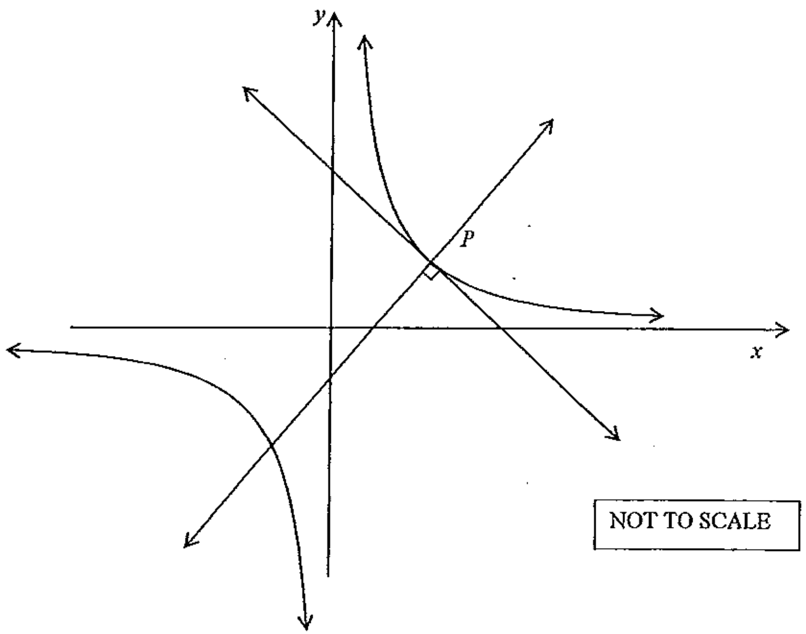
ii. Find the ratio of the areas of ΔMPY and ΔOXY. **2**

e) The line $x+q^2y-2cq=0$ is a tangent to the rectangular hyperbola $xy=c^2$ at the point $Q\left(cq,\frac{c}{q}\right)$. It cuts the asymptotes at A and B.

Show that the distance AB is equal to twice the distance of Q from the origin. **2**

Question 13 **Begin a new booklet** **15 Marks**

a) The point $P\left(cp,\frac{c}{p}\right)$ lies on the rectangular hyperbola $xy=c^2$ in the first quadrant.



i. Show that the equation of the normal at P is $p^3x-py+c-cp^4=0$ **1**

ii. The point W is chosen so that the y intercept of the normal is the midpoint of PW.

Find the coordinates of W. **2**

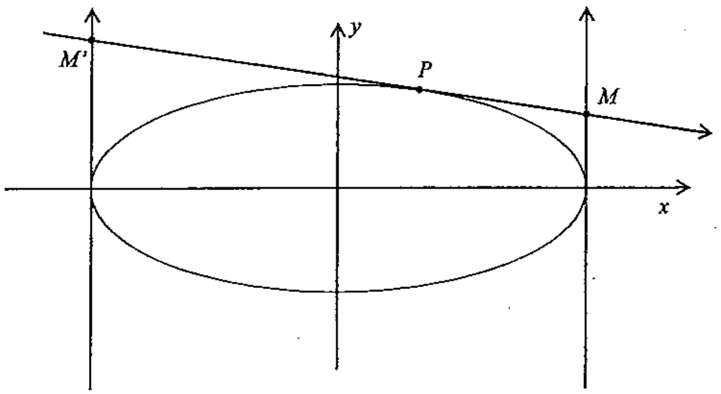
iii. Hence show that the locus of W as P varies is $c^2xy=2x^4-c^4$ and explain why the domain of that locus is $x<0$. **2**

b) Let $z^5+1=0$ have roots α, β, γ, δ and ε where α has the smallest positive argument.

i. Find the values of all the roots in modulus argument form and identify the conjugate pairs. **3**

ii. By first expressing each root as a power of α find the monic Quadratic equation whose roots are $\alpha-\alpha^2$ and $\alpha^3-\alpha^4$. **3**

c) A point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $a>b$. The tangent through P meets the tangents through $x=\pm a$ at M and M' respectively.



i. Show that the equation of the tangent at P is given by $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$ **1**

ii. Prove that $MM'S'S$ is a circle with diameter MM'. **3**

END OF SECTION II

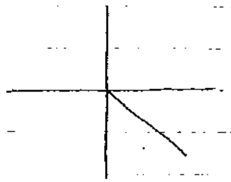
END OF EXAM

Year 12 Extension 2 Half Yearly Exam (1)

Solutions

Question 1

$z = (2 - 2i)$



let $\operatorname{cis}\theta$

$= \cos\theta + i\sin\theta$

$|z| = 2\sqrt{2}$

$\operatorname{Arg}(z) = \frac{-\pi}{4}$

$z^3 = (2\sqrt{2})^3 \operatorname{cis} 3 \times \frac{-\pi}{4}$

$= 16\sqrt{2} \operatorname{cis} -\frac{3\pi}{4}$

$= 16\sqrt{2}\left(\cos -\frac{3\pi}{4} + i\sin -\frac{3\pi}{4}\right)$

$= 16\sqrt{2}\left(-\frac{1}{\sqrt{2}} + i \cancel{\sin} -\frac{1}{\sqrt{2}}\right)$

$= -16 - 16i$

$\therefore$ B

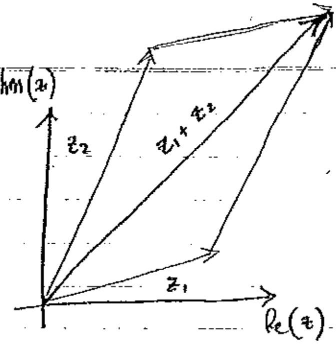
NB/ It would be quicker to simply think about the quadrant z^3 will end up in.

(3)

Question 4

Diagonal of a parallelogram are perpendicular if it is a Rhombus.

$\therefore$ D



Question 5

$P(z) = z^3 - 2z^2 - \frac{27z}{4} - 5$

$P(4) = 64 - 32 - \frac{27 \times 4}{4} - 5$

$= 0$

$\therefore P(z) = (z - 4)(az^2 + bz + c)$

Comparing coefficients

$a = 1$

$c = \frac{5}{4}$

$-4a + b = -2$

$b = 2$

$\therefore P(z) = (z - 4)\left(z^2 + 2z + \frac{5}{4}\right)$

Q7

This is the perpendicular bisector of AO.

The region is closer to the origin.

$\therefore$ C

Question 8

$P(\alpha) = \alpha^4 - 2p\alpha^2 + 3q\alpha - m = 0$. This is true for all roots

$\therefore \sum \alpha^4 - 2p\left(\sum \alpha^2\right) + 3q\sum \alpha - 4m = 0$

$\sum \alpha^4 = 2p\sum \alpha^2 - 3q\sum \alpha + 4m$

$= 2p + 4m$

$\therefore$ C

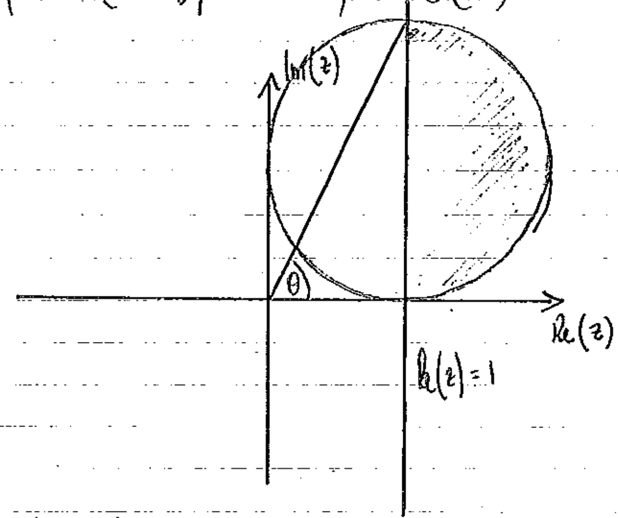
Question 9

B See text book for this.

Question 11

a) $|z-(1+i)| \leq 1$, $\text{Re}(z) > 1$

i)



① correct shapes

① Correct region

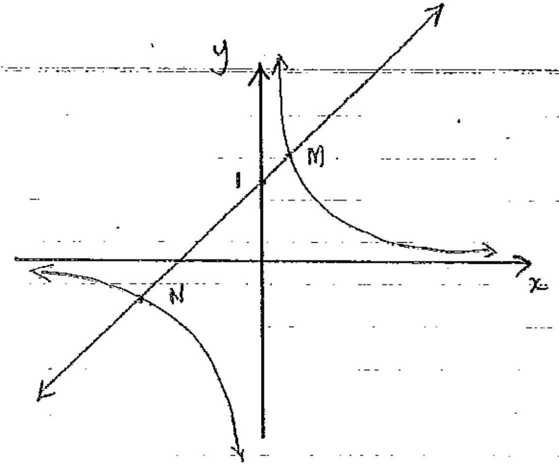
ii) See diagram.

let θ be $\text{Arg}\left(\frac{z}{z}\right)$ when it is a maximum.

$\tan\theta = \left(\frac{2}{1}\right)$ → Diameter of circle → on $\text{Re}(z) = 1$

$\theta = \tan^{-1}\left(\frac{2}{1}\right)$

c) (9)



i) $$m_{MN} = \frac{\frac{2}{m} - \frac{2}{n}}{2m - 2n}$$

$$= \frac{\frac{\cancel{2}\cancel{(n-m)}}{mn}}{-\cancel{2}\cancel{(n-m)}}$$

$$= -\frac{1}{mn} \quad (1)$$

$$-\frac{1}{mn} = \frac{y - \frac{2}{m}}{x - 2m}$$

$$-x + 2m = mny - 2n$$

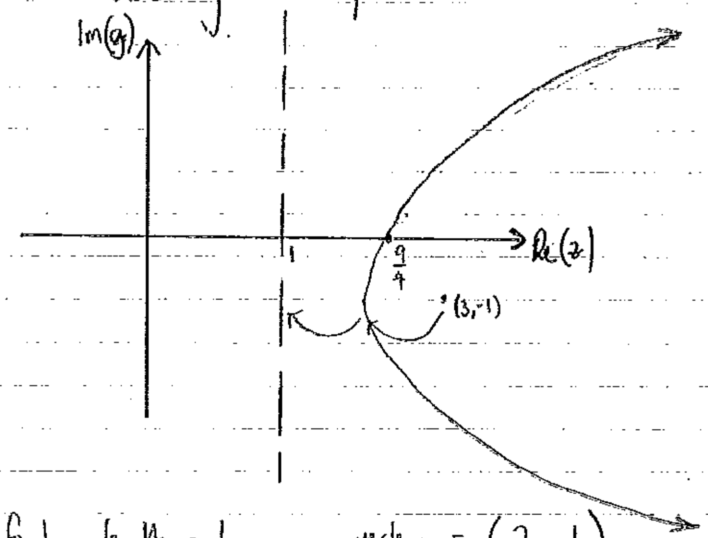
$$0 = x + mny - 2(m+n) \quad (1)$$

d) $|z - (3-i)| = Re(z) - 1$ (11)

2 options

(1) This is a point that moves so that is equidistant from the point $(3,-1)$ & the line $x = 1$

$\therefore$ 'sideways' parabola.



focal length = 1, vertex = $(2,-1)$

$\therefore \quad (y+1)^2 = 4(x-2)$

When $y = 0$, $4x - 8 = 1$

$4x = 9$

$x = \frac{9}{4}$

$P(z) = (z-i)^2(3z-1)(z+5)$ ① 13

$\therefore$ When $P(z)=0$ $z = i, \frac{1}{3}$ & -5 ①

$-192a + 4b = 0$

$-64a + 4b = 5$

$-128a = -5$

$a = -\frac{5}{128}$

$\therefore b = 48 \times -\frac{5}{128}$

$= -\frac{15}{8}$

"Otherwise" could be expanding and equating coefficients of like terms but this is cumbersome!

c) $\frac{x^2}{2} - \frac{y^2}{8} = 1$

Normal hyperbola - Major axis is x axis

$b^2 = a^2(e^2-1)$

$8 = 2(e^2-1)$

$e^2-1 = 4$

$e^2 = 5$

$e = \sqrt{5}$

① finding values.

foci $(\pm\sqrt{2}.\sqrt{5}, 0)$

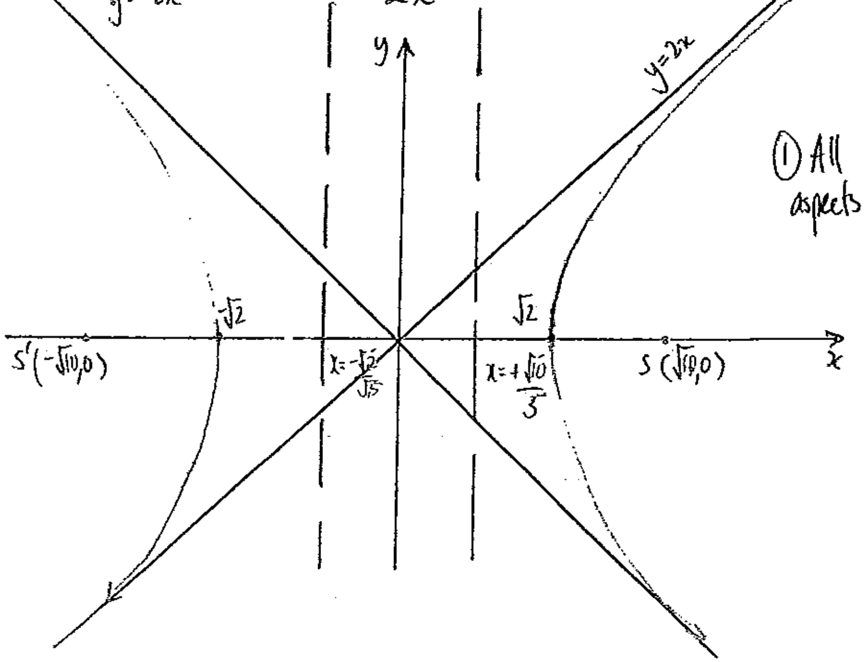
$(\pm\sqrt{10}, 0)$

Directrices

$x = \pm\frac{\sqrt{2}}{\sqrt{5}} = \pm\frac{\sqrt{10}}{5}$

Asymptotes @ $y = \pm\frac{2\sqrt{2}}{\sqrt{2}}x$

$= \pm 2x$



① All aspects

$\therefore + \frac{a^2 y_p}{b^2 x_p} = \frac{y - y_p}{x - x_p}$

$\frac{a^2}{x_p}(x - x_p) = \frac{b^2}{y_p}(y - y_p)$

$\frac{a^2 x}{x_p} - a^2 = \frac{b^2 y}{y_p} - b^2$

$\frac{a^2 x}{x_p} - \frac{b^2 y}{y_p} = a^2 - b^2$

As required.

When given a result to show always put in more rather than less steps.

$$\frac{MY}{OY} = \frac{a^2 \cancel{y_P}}{\cancel{b^2}} \div \frac{\cancel{y_P}(a^2-b^2)}{\cancel{b^2}}$$

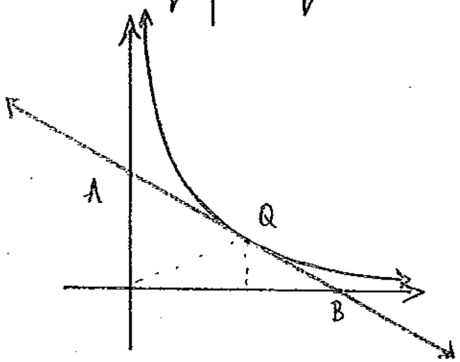
$$= \frac{a^2}{a^2-b^2}$$

use $b^2 = a^2(1-e^2)$, $a^2e^2 = a^2-b^2$

$$\therefore \frac{\text{Area } \Delta MPY}{\text{Area } \Delta OXY} = \frac{a^4}{(a^2-b^2)^2} = \frac{1}{e^4}$$ ☺

This has no x_P or y_P in it so it independent of the position of P!

e) $x + q^2y - 2cq = 0$



$xy = c^2$

let A be the y intercept & B be the x intercept.

A: $q^2 y_A - 2cq = 0$

$q^2 y_A = 2cq$

$y_A = \frac{2c}{q}$

$\therefore A\left(0, \frac{2c}{q}\right)$

Use the diagram and right angled Δ's !!

Q13.

a) i) $xy = c^2$

$y = \frac{c^2}{x}$

$\frac{dy}{dx} = -\frac{c^2}{x^2}$

@P $\frac{dy}{dx} = \frac{-c^2}{(cp)^2}$

$= -\frac{1}{p^2}$

$\therefore$ Gradient of normal is p^2

$$p^2 = \frac{y - \frac{c}{p}}{x - cp}$$

$p^2x - cp^3 = y - \frac{c}{p}$

$p^3x - cp^4 = py - c$

$p^3x - py + c - cp^4 = 0$ As required.

ii) y intercept of normal is...

$-py + c - cp^4 = 0$

$py = c(1-p^4)$

$$y = \frac{c(1-p^4)}{p}$$

iii) $x = -cp$, $y = \frac{c}{p}\left[(1-2p^4)\right]$

$p = -\frac{x}{c}$

$\therefore y = \frac{c}{\frac{-x}{c}}\left[1 - 2\left[\frac{-x}{c}\right]^4\right]$ (1)

$y = -\frac{c^2}{x}\left[1 - \frac{2x^4}{c^4}\right]$

$xy = -c^2\left[\frac{c^4 - 2x^4}{c^4}\right]$

$= -\frac{1}{c^2}\left[2x^4 - c^4\right]$ (1)

$c^2xy = 2x^4 - c^4$ As required.

$b = -(\alpha - \alpha^2 + \alpha^3 - \alpha^4)$

NB $\alpha^5 = -1$

$\therefore b = -(\alpha + (\alpha^5 . \alpha^2) + \alpha^3 + (\alpha^4 \times \alpha^5))$

$= -(\alpha + \alpha^7 + \alpha^3 + \alpha^9)$

From $z^5 - 1 = 0$

$\sum \alpha = 0$

$\therefore \alpha + \alpha^3 + \alpha^7 + \alpha^9 - 1 = 0$

$\alpha + \alpha^3 + \alpha^7 + \alpha^9 = 1$

$\therefore b = -1$ (1)

$\therefore c = \alpha^4 - \alpha^5 - \alpha^5 + \alpha^6$

$= \alpha^4 + \alpha^6 + 2$

$= -(\alpha^5 . \alpha^4)\ 4 - (\alpha^5 . \alpha^1) + 2$

$= -[\alpha^9 + \alpha] + 2$

These are conjugates.

$\therefore C = -\cos\frac{\pi}{5} + 2$ (1)

$\therefore$ Quadratic $x - x + 2 - \cos\frac{\pi}{5} = 0$

$$\left(x - a\cos\theta\right)\frac{\cos\theta}{a} = -\frac{\sin\theta}{b}\left(y - b\sin\theta\right)$$

$$\frac{x\cos\theta}{a} - \cos^2\theta = -\frac{y\sin\theta}{b} + \sin^2\theta$$

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = \cos^2\theta + \sin^2\theta$$

$$= 1$$ As required.

ii. If $MM'S'S$ is a circle, diameter MM

then $\angle MSM' = \angle MS'M' = 90^\circ$

$\therefore$ Prove $M_{ms} \times M_{sm'} = -1$

$M_{ms'} \times M_{m's'} = -1$

Coordinates of M: (x_m, y_m)

$x_m = a$

$$\cancel{a}\frac{\cos\theta}{\cancel{a}} + \frac{y_m\sin\theta}{b} = 1$$

$$y_m = \frac{b}{\sin\theta}\left(1 - \cos\theta\right)$$

$$M_{ms} \times M_{sm'} = \frac{\frac{b^2}{\sin^2\theta}\left(1+\cos\theta\right)\left(1-\cos\theta\right)}{-\left(a-ae\right)\left(a+ae\right)}$$

$$= \frac{\frac{b^2}{\sin^2\theta}\left(1-\cos^2\theta\right)}{-a^2\left(1-e^2\right)}$$

$$= \frac{\frac{b^2\cancel{\sin^2\theta}}{\cancel{\sin^2\theta}}}{-a^2\left(1-e^2\right)}$$

Also $b^2 = a^2\left(1-e^2\right)$

$$= \frac{b^2}{-b^2}$$

$$= -1$$

$\therefore$ Right angled.

$\therefore$ $MM'S'S$ is a circle with diameter MM' as MM' subtend a right angle at both S & S'.