Centre Number

Student Number

CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NEW SOUTH WALES

2005
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

Afternoon Session
Tuesday 9 August 2005

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided separately
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1-7
- All questions are of equal value

2603 - 1

Marks

Question 1 **(Begin a new page)**

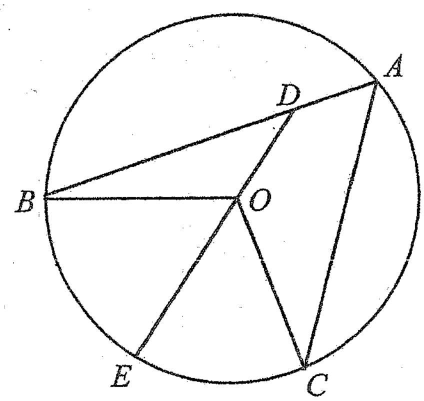
(a) Find the value of $\lim_{x \to 0} \frac{\sin 2x}{5x}$. 2

(b) The polynomial $P(x)$ is given by $P(x) = x^3 + ax + b$ for some real numbers a and b. 2 is a zero of $P(x)$. When $P(x)$ is divided by $(x+1)$ the remainder is -15.

(i) Write down two equations in a and b. 2
(ii) Hence find the values of a and b. 1

(c)(i) Find the exact values of the gradients of the tangents to the curve $y = e^x$ at the points where $x = 0$ and $x = 1$. 1
(ii) Find the acute angle between these tangents correct to the nearest degree. 2

(d)



In the diagram A, B and C are points on a circle with centre O. D is a point on AB such that $ADOC$ is a cyclic quadrilateral. DO produced meets the circle again at E.

(i) Copy the diagram.
(ii) Give a reason why $\angle CAD = \angle COE$. 1
(iii) Show that DOE bisects $\angle COB$. 3

Question 2 **(Begin a new page)** **Marks**

(a) Evaluate $\log_2 7$ correct to two decimal places. **2**

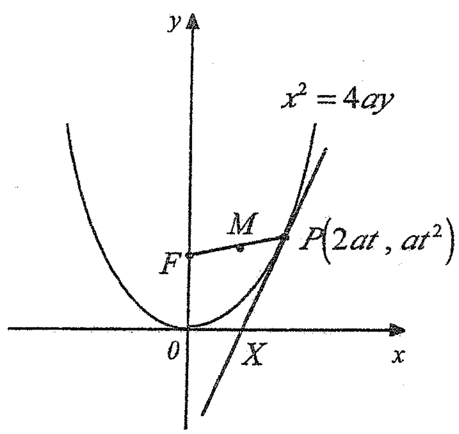
(b)(i) Show that $\dfrac{1}{1-\tan x} - \dfrac{1}{1+\tan x} = \tan 2x$. **2**

(ii) Evaluate $\dfrac{1}{1-\tan\frac{\pi}{6}} - \dfrac{1}{1+\tan\frac{\pi}{6}}$ in simplest exact form. **1**

(c) $A(x, 10)$ and $B(x^2, 6)$ are two fixed points for some real number x.
The point $P(5, 4)$ divides the interval AB externally in the ratio 3 : 1.

(i) Show that $3x^2 - x = 10$. **1**
(ii) Find any values of x. **2**

(d)



$P(2at, at^2)$ is a point on the parabola $x^2 = 4ay$ with focus F. The tangent to the parabola at P cuts the x axis at X. M is the midpoint of PF.

(i) Show that the tangent to the parabola at P has equation $tx - y - at^2 = 0$. **2**
(ii) Show that MX is parallel to the y axis. **2**

Marks

Question 3 **(Begin a new page)**

(a) Consider the function $f(x) = 1 + \ln x$.

(i) Show that the function $f(x)$ is increasing and the curve $y = f(x)$ is concave down for all values of x in the domain of the function. **2**

(ii) Find the equation of the tangent to the curve $y = f(x)$ at the point on the curve where $x = 1$. **2**

(iii) Find the equation of the inverse function $f^{-1}(x)$. **1**

(iv) On the same diagram sketch the graph of the curves $y = f(x)$ and $y = f^{-1}(x)$ Show clearly the coordinates of any points of intersection of the two curves and any intercepts made on the coordinate axes. **3**

(b)(i) Show that $\frac{d}{dx}\left(x\sqrt{1-x^2} + \sin^{-1} x\right) = 2\sqrt{1-x^2}$. **2**

(ii) Evaluate $\int_0^{\frac{1}{2}} \sqrt{1-x^2}\ dx$, giving the answer in simplest exact form. **2**

Question 4 **(Begin a new page)**

(a) The equation $x^3 - 3x - 3 = 0$ has exactly one real root α.

(i) Show that $2 < \alpha < 3$. **2**

(ii) Starting with an initial approximation $\alpha \approx 2$, use one application of Newton's method to find a further approximation for α correct to one decimal place. **2**

(b) Use the substitution $u = \sin^2 x$ to evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 2x}{1 - \sin^2 x}\ dx$, giving the answer in simplest exact form. **4**

(c) A particle is moving in a horizontal straight line. At time t seconds, the displacement of the particle from a fixed point O on the line is x metres, its velocity is v ms^{-1}, and its acceleration a ms^{-2} is given by $a = 8x - 2x^3$. When the particle is 2 m to the right of O, it is observed to be travelling to the right with a speed of 6 ms^{-1}.

(i) Show that $v^2 = 20 + 8x^2 - x^4$. **2**

(ii) Find the set of possible values of x. **2**

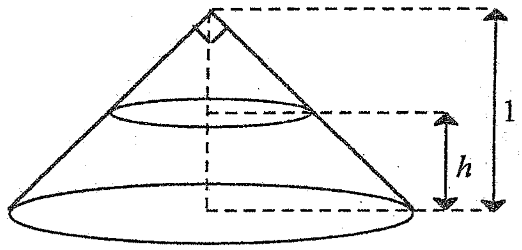
Marks

Question 5 **(Begin a new page)**

(a) A bag contains nine balls labelled 1, 2, 3, … , 9, but otherwise identical. Three balls are chosen at random from the bag. Find the probability that exactly two even numbered balls are chosen

(i) if the balls are selected without replacement. **2**
(ii) if each ball is replaced before the next is selected. **2**

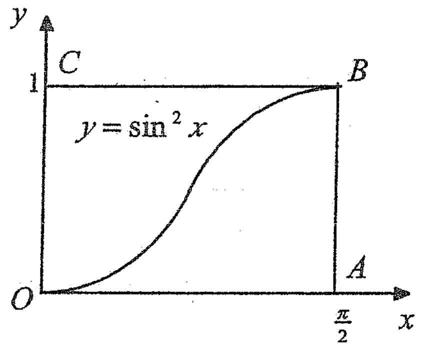
(b)



A closed, right, hollow cone has a height of 1 metre and semi-vertical angle 45°. The cone stands with its base on a horizontal surface. Water is poured into the cone through a hole in its apex at a constant rate of $0 \cdot 1 \text{ m}^3$ per minute.

(i) Show that when the depth of water in the cone is h metres $(0 < h < 1)$ the volume of water $V \text{ m}^3$ in the cone is given by $V = \frac{\pi}{3}(h^3 - 3h^2 + 3h)$. **2**
(ii) Find the rate at which the depth of water in the cone is increasing when $h = 0 \cdot 5$. **2**

(c)



The rectangle $OABC$ has vertices $O(0,0)$, $A\left(\frac{\pi}{2},0\right)$, $B\left(\frac{\pi}{2},1\right)$, and $C(0,1)$. The curve $y = \sin^2 x$ is shown passing through the points O and B. Show that this curve divides the rectangle $OABC$ into two regions of equal area. **4**

Marks

Question 6 **(Begin a new page)**

(a) A particle is performing Simple Harmonic Motion about a fixed point O on a straight line. At time t seconds it has displacement x metres from O given by $x = \cos 2t - \sin 2t$.

(i) Express x in the form $R\cos(2t+\alpha)$ for some $R>0$ and $0<\alpha<\frac{\pi}{2}$. 2

(ii) Find the amplitude and the period of the motion. 2

(iii) Determine whether the particle is initially moving towards O or away from O and whether it is initially speeding up or slowing down. 2

(iv) Find the time at which the particle first returns to its starting point. 2

(b) Use Mathematical Induction to show that for all positive integers $n \geq 1$, 4

$$\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+n} = \frac{2n}{n+1}.$$

Question 7 **(Begin a new page)**

(a) A particle is projected from a point O with velocity $V\,\text{ms}^{-1}$ at an angle θ above the horizontal. At time t seconds it has horizontal and vertical displacements x metres and y metres respectively from O. The acceleration due to gravity is $g\,\text{ms}^{-2}$.

(i) Write down expressions for x and y in terms of V, θ and t. 2

(ii) Show that $y = x\tan\theta - \frac{gx^2}{2V^2}\left(1+\tan^2\theta\right)$. 2

(b) A particle is projected from O with velocity $60\,\text{ms}^{-1}$ at an angle α above the horizontal. T seconds later, another particle is projected from O with velocity $60\,\text{ms}^{-1}$ at an angle β above the horizontal, where $\beta < \alpha$. The two particles collide 240 metres horizontally from O and at a height of 80 metres above O. Taking $g = 10\,\text{ms}^{-2}$ and using results from (a):

(i) Show that $\tan\alpha = 2$ and $\tan\beta = 1$. 2

(ii) Find the value of T in simplest exact form. 2

(c) The real number x is a solution of the equation $x^2 - x - 1 = 0$. Use the Binomial Theorem to show that the sum S of the series $1 + x + x^2 + \ldots + x^{2n-1}$ $(n = 1, 2, 3\ldots)$ is given by $S = \sum_{r=1}^{n} {}^{n}C_r\, x^{r+1}$. 4

Question 1

a. Outcomes Assessed: H5

Marking Guidelines

Criteria	Marks
• recognises that $\lim_{x\to 0}\frac{\sin 2x}{2x}=1$	1
• uses this result to show the required limit is $\frac{2}{5}$	1

Answer

$$\lim_{x\to 0}\frac{\sin 2x}{5x}=\lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\frac{2}{5}=1\times\tfrac{2}{5}=\tfrac{2}{5}$$

b. Outcomes Assessed: i. PE3 ii. PE4

Marking Guidelines

Criteria	Marks
i. • uses the fact that 2 is a zero of $P(x)$ to write an equation in a and b	1
• uses the remainder theorem to write an equation in a and b	1
ii. • solves simultaneous equations to find the values of a and b	1

Answer

i. $P(2)=0 \Rightarrow 8+2a+b=0 \quad (1)$

$P(-1)=-15 \Rightarrow -1-a+b=-15 \quad (2)$

ii. $(1)-(2) \Rightarrow 9+3a=15$

$\therefore a=2$

Substitution in (2) gives $b=-12$

c. Outcomes Assessed: i. H5 ii. P4

Marking Guidelines

Criteria	Marks
i. • differentiates and substitutes to find both gradients	1
ii. • obtains an expression in terms of e for the tangent of the angle	1
• evaluates this expression then finds the angle correct to the nearest degree	1

Answer

i. $y=e^x$ The tangent at $x=0$ has gradient $e^0=1$

$\frac{dy}{dx}=e^x$ The tangent at $x=1$ has gradient $e^1=e$

ii. $\tan\theta=\left|\frac{e-1}{1+e}\right|\approx 0\cdot 4621$ $\therefore$ required acute angle is $25°$ (to the nearest degree)

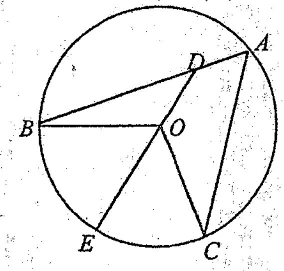
d. Outcomes Assessed: ii. PE3 iii. PE2, PE3

Marking Guidelines

Criteria	Marks
ii. • quotes an appropriate angle property of cyclic quadrilateral $ADOC$	1
iii. • states $\angle COB = 2\angle CAB$	1
• deduces required result	1
• justifies these deductions by quoting an appropriate circle property	1

Answer

i.



ii. Exterior angle COE of cyclic quadrilateral $ADOC$ is equal to the interior opposite angle CAD.

iii. $\angle COB = 2\angle CAB$ (angle at centre is twice angle at circumference subtended by arc BC)

$\therefore \angle COB = 2\angle COE$ ($\angle CAB = \angle CAD = \angle COE$)

$\therefore \angle BOE = \angle COE$ and hence DOE bisects $\angle BOC$

Question 2

a. Outcomes Assessed: H3

Marking Guidelines

Criteria	Marks
• uses the change of base formula to express the logarithm to base e or 10	1
• evaluates the logarithm	1

Answer

$$\log_2 7 = \frac{\log_e 7}{\log_e 2} \approx 2\cdot 81 \text{ (to 2 decimal places)}$$

b. Outcomes Assessed: i. H5 ii. H5

Marking Guidelines

Criteria	Marks
i. • simplifies the expression by taking a common denominator	1
• recognises the expression for $\tan 2x$	1
ii. • uses the result from i. to evaluate the expression in surd form.	1

Answer

i

$$\frac{1}{1-\tan x} - \frac{1}{1+\tan x} = \frac{1+\tan x-(1-\tan x)}{1-\tan^2 x}$$
$$= \frac{2\tan x}{1-\tan^2 x}$$
$$= \tan 2x$$

ii.

$$\frac{1}{1-\tan\frac{\pi}{6}} - \frac{1}{1+\tan\frac{\pi}{6}} = \tan\frac{\pi}{3}$$
$$= \sqrt{3}.$$

c. Outcomes Assessed: i. P4 ii. P4

Marking Guidelines

Criteria	Marks
i. • finds equation for x	1
ii. • solves a quadratic equation to find two values for x (award 1 mark for correct process with a minor error in application)	2

Answer

i.

$A(x, 10) \quad B(x^2, 6)$

$3 : -1$

$$\left(\frac{3x^2 - x}{2}, \frac{18-10}{2}\right) \equiv P(5, 4)$$

$$\therefore \frac{3x^2 - x}{2} = 5$$

ii.

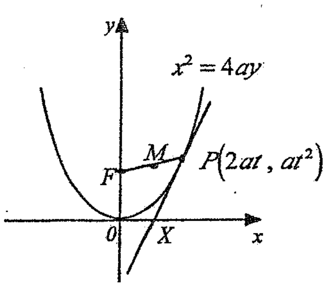
$$3x^2 - x = 10$$
$$3x^2 - x - 10 = 0$$
$$(3x+5)(x-2) = 0$$
$$\therefore x = -\tfrac{5}{3} \text{ or } x = 2$$

d. Outcomes Assessed: i. PE4 ii. PE3

Marking Guidelines

Criteria	Marks
i. • shows that the gradient of the tangent is t	1
• uses gradient t and coordinates of P to find the equation of the tangent	1
ii. • finds the x coordinate of X	1
• finds the x coordinate of M and deduces MX parallel to the y axis.	1

Answer



i. $x = 2at \qquad y = at^2$

$$\frac{dx}{dt} = 2a \qquad \frac{dy}{dt} = 2at$$

$$\therefore \frac{dy}{dx} = \frac{2at}{2a} = t$$

Tangent at P has gradient t and equation

$$y - at^2 = t(x - 2at)$$
$$y - at^2 = tx - 2at^2$$
$$tx - y - at^2 = 0$$

ii. $F(0, a) \quad \therefore \quad M\left(at, \dfrac{a(1+t^2)}{2}\right)$

At X, $y = 0 \Rightarrow tx - at^2 = 0 \quad \therefore x = at$

Since both M and X lie on the vertical line $x = at$, MX is parallel to the y axis.

Question 3

a. Outcomes Assessed: i. **H5** ii. **H5** iii. **HE4** iv. **HE4**

Marking Guidelines

Criteria	Marks
i. • shows first derivative positive for $x>0$ and deduces f increasing	1
• shows second derivative negative and deduces curve concave down	1
ii. • finds gradient of tangent by evaluating first derivative at $x=1$	1
• finds y coordinate when $x=1$; deduces $y=x$ is equation of required tangent	1
iii. • uses an appropriate algebraic method to find the equation of the inverse function	1
iv. • shows curve $y=f(x)$ rising and concave down with negative y axis as asymptote	1
• shows curve $y=f^{-1}(x)$ as reflection in $y=x$ touching $y=f(x)$ at $(1,1)$	1
• shows x intercept for $y=f(x)$ and y intercept for $y=f^{-1}(x)$	1

Answer

i. $f(x)=1+\ln x$ has domain $\{x: x>0\}$.

$f'(x)=\frac{1}{x}>0$ *for* $x>0$ $\therefore f$ is increasing for all x in its domain

$f''(x)=-\frac{1}{x^2}<0$ *for all* x $\therefore$ curve $y=f(x)$ is concave down for all x in the domain of f

ii. When $x=1$, $y=f(1)=1$.

$f'(1)=1$ $\therefore$ tangent at $(1,1)$ has gradient 1 and equation $y=x$

iii. $y=1+\ln x$

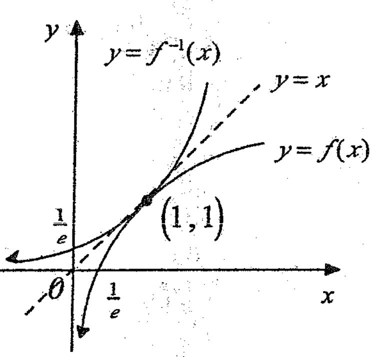
$y-1=\ln x$

$e^{y-1}=x$

Interchanging $x \leftrightarrow y$, inverse function has equation $y=e^{x-1}$.

$\therefore f^{-1}(x)=e^{x-1}$

iv.



At the x intercept of curve $y=f(x)$

$1+\ln x=0$

$\ln x=-1$

$x=e^{-1}$

Curves $y=f(x)$, $y=f^{-1}(x)$ are reflections of each other in the line $y=x$. Since this line is tangent to the curve $y=f(x)$ at the point $(1,1)$, the curves touch at this point

b. Outcomes Assessed: i. HE4 ii. HE4

Marking Guidelines

Criteria	Marks
i. • uses the product rule to differentiate the first term	1
• differentiates $\sin^{-1} x$ and simplifies the expression for the derivative	1
ii. • writes an expression for the integral using the derivative in i.	1
• evaluates this expression in simplest exact form.	1

Answer

i.

$$\frac{d}{dx}\left(x\sqrt{1-x^2}+\sin^{-1}x\right)=\left\{1.\sqrt{1-x^2}+x.\tfrac{1}{2}(1-x^2)^{-\frac{1}{2}}.(-2x)\right\}+\frac{1}{\sqrt{1-x^2}}$$

$$=\sqrt{1-x^2}+(1-x^2)^{-\frac{1}{2}}\left(-x^2+1\right)$$

$$=\sqrt{1-x^2}+(1-x^2)^{\frac{1}{2}}$$

$$\therefore \frac{d}{dx}\left(x\sqrt{1-x^2}+\sin^{-1}x\right)=2\sqrt{1-x^2}$$

ii.

$$\left[x\sqrt{1-x^2}+\sin^{-1}x\right]_0^{\frac{1}{2}}=\int_0^{\frac{1}{2}}2\sqrt{1-x^2}\ dx$$

$$\left(\tfrac{1}{2}\sqrt{1-\tfrac{1}{4}}-0\right)+\left(\sin^{-1}\tfrac{1}{2}-\sin^{-1}0\right)=2\int_0^{\frac{1}{2}}\sqrt{1-x^2}\ dx$$

$$\tfrac{\sqrt{3}}{4}+\tfrac{\pi}{6}-0=2\int_0^{\frac{1}{2}}\sqrt{1-x^2}\ dx$$

$$\therefore \int_0^{\frac{1}{2}}\sqrt{1-x^2}\ dx=\tfrac{\sqrt{3}}{8}+\tfrac{\pi}{12}$$

Question 4

a. Outcomes Assessed: i. PE3 ii. PE3

Marking Guidelines

Criteria	Marks
i. • shows that $f(x)=x^3-3x-3$ changes sign between $x=2$ and $x=3$	1
• notes that f is continuous to deduce $2<\alpha<3$	1
ii. • evaluates $f'(2)$	1
• applies Newton's method once with a first approximation $\alpha \approx 2$	1

Answer

i. Let $f(x)=x^3-3x-3$.
$f(x)$ is continuous,
$f(2)=-1<0$ and $f(3)=15>0$.
$\therefore 2<\alpha<3$

ii. $f'(x)=3x^2-3$

$$\alpha \approx 2-\frac{f(2)}{f'(2)}=2-\frac{-1}{9}$$

$$\alpha \approx 2\cdot 1$$

b. Outcomes Assessed: HE6

Marking Guidelines

Criteria	Marks
• expresses du in terms of dx	1
• finds limits for integral with respect to u	1
• finds primitive function as a function of u	1
• evaluates the integral in simplest exact form	1

Answer

$$u = \sin^2 x$$
$$du = 2\sin x \cos x \, dx$$
$$du = \sin 2x \, dx$$

$$x = \tfrac{\pi}{4} \Rightarrow u = \tfrac{1}{2}$$
$$x = \tfrac{\pi}{3} \Rightarrow u = \tfrac{3}{4}$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{1-\sin^2 x} \sin 2x \, dx = \int_{\frac{1}{2}}^{\frac{3}{4}} \frac{1}{1-u} \, du$$
$$= \Big[-\ln(1-u)\Big]_{\frac{1}{2}}^{\frac{3}{4}}$$
$$= -\ln\tfrac{1}{4} + \ln\tfrac{1}{2}$$
$$= \ln(\tfrac{1}{2} \div \tfrac{1}{4})$$
$$= \ln 2$$

c. Outcomes Assessed: i. HE5 ii. HE5

Marking Guidelines

Criteria	Marks
i. • finds an expression for $\frac{1}{2}v^2$ by integration of a with respect to x	1
• evaluates the constant of integration and obtains v^2 as a function of x	1
ii. • uses an algebraic method to deduce $x^2 \le 10$ since $v^2 \ge 0$	1
• writes an inequality for x.	1

Answer

i.
$$a = 8x - 2x^3$$
$$\frac{d}{dx}\left(\tfrac{1}{2}v^2\right) = 8x - 2x^3$$
$$\tfrac{1}{2}v^2 = 4x^2 - \tfrac{1}{2}x^4 + c$$
$$\left.\begin{array}{l} x = 2 \\ v = 6 \end{array}\right\} \Rightarrow \begin{array}{l} 18 = 16 - 8 + c \\ \therefore c = 10 \end{array}$$

$$\therefore \tfrac{1}{2}v^2 = 4x^2 - \tfrac{1}{2}x^4 + 10$$
$$v^2 = 20 + 8x^2 - x^4$$

ii. $v^2 = (10 - x^2)(2 + x^2)$

But $v^2 \ge 0, \quad \therefore x^2 \le 10$

Hence $-\sqrt{10} \le x \le \sqrt{10}$

Question 5

a. Outcomes Assessed: i. PE3 ii. HE3

Marking Guidelines

Criteria	Marks
i. • counts how many ways two even are chosen from four, and one odd from five	1
• completes the factors in numerator and denominator to evaluate the probability	1
ii. • counts how many ways two even are chosen from four, and one odd from five	1
• completes the factors in numerator and denominator to evaluate the probability	1

Answer

In each case there are three orders of E, E, O, and the odd number can be selected in 5 ways.

i. $P(exactly\ 2\ even) = \dfrac{3\times5\times4\times3}{9\times8\times7} = \frac{5}{14}$

(Select two evens in order in 4×3 ways
Select three balls in order in $9\times8\times7$ ways)

ii. $P(exactly\ 2\ even) = \dfrac{3\times5\times4^2}{9^3} = \frac{80}{243}$

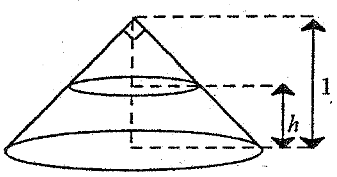
(Select two evens in order in 4×4 ways
Select three balls in order in $9\times9\times9$ ways)

b. Outcomes Assessed: i. **P4** ii. **HE5**

Marking Guidelines

Criteria	Marks
i. • finds the radius and volume of the conical vessel	1
• considers the volume of the conical empty space to express V in terms of h.	1
ii. • expresses $\dfrac{dV}{dt}$ in terms of h and $\dfrac{dh}{dt}$	1
• calculates the rate of increase of the depth by substitution in this expression	1

Answer



i. The conical vessel has height 1 m, radius 1 m and volume $\frac{\pi}{3}\ \text{m}^3$.
When the depth of the water is h, the empty space is a similar cone, the ratio of heights being $(1-h):1$.

$\therefore V = \frac{\pi}{3}\left\{1-(1-h)^3\right\} = \frac{\pi}{3}\left(h^3-3h^2+3h\right)$

ii.

$V = \frac{\pi}{3}(h^3-3h^2+3h)$

$\dfrac{dV}{dt} = \frac{\pi}{3}(3h^2-6h+3)\dfrac{dh}{dt}$

$0\cdot1 = \pi\,(h^2-2h+1)\dfrac{dh}{dt}$

When $h=0\cdot5$, $\quad 0\cdot1 = \pi\times0\cdot25\times\dfrac{dh}{dt}$

$\therefore \dfrac{dh}{dt} = \dfrac{2}{5\pi} \approx 0\cdot13$

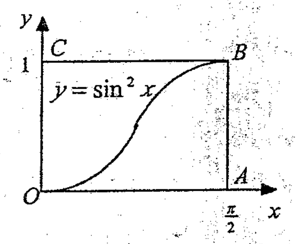
The depth is increasing at a rate $\frac{2}{5\pi}\approx 0\cdot13$ m per minute.

c. Outcomes Assessed: **H8**

Marking Guidelines

Criteria	Marks
• writes an integral to give the area under the curve; states the area of the rectangle	1
• writes $\sin^2 x$ in terms of $\cos 2x$	1
• finds the primitive function	1
• substitutes to evaluate the definite integral	1

Answer



$$\int_0^{\frac{\pi}{2}} \sin^2 x \;\; dx = \tfrac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos 2x) \;\; dx$$
$$= \tfrac{1}{2}\left[x - \tfrac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$$
$$= \tfrac{1}{2}\left\{\left(\tfrac{\pi}{2}-0\right) - \tfrac{1}{2}\left(\sin\pi - \sin 0\right)\right\}$$
$$= \tfrac{\pi}{4}$$
$$= \tfrac{\pi}{2} \div 2$$

Area of rectangle $OABC$ is $\frac{\pi}{2}$ sq. units

Area under curve is given by

Hence curve divides rectangle $OABC$ into two regions of area $\frac{\pi}{4}$ sq. units.

Question 6

a. Outcomes Assessed: i. **HE3** ii. **HE3** iii. **HE3** iv. **HE3, HE7**

Marking Guidelines

Criteria	Marks
i. • finds the value of R	1
• finds the value of α	1
ii. • states the amplitude	1
• states the period	1
iii. • finds initial values of x and v and deduces particle initially travelling towards O.	1
• finds initial value of a and deduces particle is speeding up.	1
iv. • writes a trigonometric equation for t.	1
• finds the required value of t.	1

Answer

i. $x = \cos 2t - \sin 2t$

$$x = \sqrt{2}\left(\tfrac{1}{\sqrt{2}}\cos 2t - \tfrac{1}{\sqrt{2}}\sin 2t\right)$$
$$= \sqrt{2}\left(\cos\tfrac{\pi}{4}\cos 2t - \sin\tfrac{\pi}{4}\sin 2t\right)$$
$$= \sqrt{2}\cos\left(2t+\tfrac{\pi}{4}\right)$$

ii. Amplitude is $\sqrt{2}$ m

Period is π seconds

iii. $v = -2\sqrt{2}\sin\left(2t+\frac{\pi}{4}\right)$

$a = -4\sqrt{2}\cos\left(2t+\frac{\pi}{4}\right)$

When $t=0$, $v=-2$, $a=-4$, $x=1$

Initial velocity is 2 ms^{-1} to the left

Initial acceleration is 4 ms^{-2} to the left

Particle is initially 1 m to the right of O.

Initially particle is moving towards O and speeding up.

iv. On first return to its initial position

$$\sqrt{2}\cos\left(2t+\tfrac{\pi}{4}\right) = 1$$
$$\cos\left(2t+\tfrac{\pi}{4}\right) = \tfrac{1}{\sqrt{2}}$$
$$2t+\tfrac{\pi}{4} = \tfrac{\pi}{4}, \tfrac{7\pi}{4}, \ldots$$
$$2t = 0, \tfrac{3\pi}{2}, \ldots$$

Hence particle first returns to its starting point after $\frac{3\pi}{4}$ s.

b. Outcomes Assessed: HE4

Marking Guidelines

Criteria	Marks
• establishes the truth of $S(1)$	1
• substitutes for sum of first k terms in expression for LHS of $S(k+1)$ if $S(k)$ is true	1
• uses the sum of an A.P. to simplify $(k+1)$th term of expression for LHS of $S(k+1)$	1
¥ rearranges expression to show if $S(k)$ is true then $S(k+1)$ is true	1

Answer

Let $S(n)$ be the statement $\frac{1}{1}+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots+n}=\frac{2n}{n+1}$, $n=1,2,3,\ldots$

Consider $S(1)$: $LHS=\frac{1}{1}=1$; $RHS=\frac{2\times 1}{1+1}=1$. Hence $S(1)$ is true.

If $S(k)$ is true : $\frac{1}{1}+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots+k}=\frac{2k}{k+1}$ **

Consider $S(k+1)$:

$$LHS=\frac{1}{1}+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots+k}+\frac{1}{1+2+3+\ldots+(k+1)}$$

$$=\frac{2k}{k+1}+\frac{1}{1+2+3+\ldots+(k+1)} \quad \textit{if } S(k) \textit{ is true, using}^{**}$$

$$=\frac{2k}{k+1}+\frac{2}{(k+1)\{1+(k+1)\}} \quad \textit{sum of } (k+1) \textit{ terms of A.P}:\ a=1,\ d=1$$

$$=\frac{2}{(k+1)}\left\{k+\frac{1}{k+2}\right\}$$

$$=\frac{2(k^2+2k+1)}{(k+1)(k+2)}$$

$$=\frac{2\,(k+1)^2}{(k+1)(k+2)}$$

$$=\frac{2\,(k+1)}{\{(k+1)+1\}}$$

$$=RHS$$

Hence if $S(k)$ is true, then $S(k+1)$ is true. But $S(1)$ is true, hence $S(2)$ is true, and then $S(3)$ is true and so on. Therefore by Mathematical Induction, $S(n)$ is true for all positive integers n.

Question 7

a. Outcomes Assessed: i. HE3 ii. HE3

Marking Guidelines

Criteria	Marks
i. • writes expression for x	1
• writes expression for y	1
ii. • substitutes for t to get y in terms of x, $\tan\theta$, $\cos\theta$	1
• uses trig identities to obtain required expression for y	1

Answer

i. $x = Vt\cos\theta$

$y = Vt\sin\theta - \frac{1}{2}gt^2$

ii. $y = Vt\cos\theta\,\tan\theta - \frac{1}{2}g\left(\frac{x}{V\cos\theta}\right)^2$

$$y = x\tan\theta - \frac{gx^2}{2V^2}\sec^2\theta$$

$$= x\tan\theta - \frac{gx^2}{2V^2}\left(1+\tan^2\theta\right)$$

b. Outcomes Assessed: i. HE3 ii. HE3

Marking Guidelines

Criteria	Marks
i. • substitutes values into equation of path of projectile from (a)	1
• solves resulting quadratic for $\tan\theta$ and evaluates $\tan\alpha$, $\tan\beta$	1
ii. • uses the horizontal displacements at collision to obtain equation for T	1
• evaluates T in simplest surd form	1

Answer

i. If a particle fired at an angle θ above the horizontal with a speed of 60ms^{-1} passes through the point where $x = 240,\ y = 80$ then, using the result in (a) ii.,

$$240\tan\theta - \frac{10\times 240^2}{2\times 60^2}\left(1+\tan^2\theta\right) = 80$$

$$3\tan\theta - \left(1+\tan^2\theta\right) = 1$$

$$\tan^2\theta - 3\tan\theta + 2 = 0$$

$$(\tan\theta - 1)(\tan\theta - 2) = 0$$

$$\therefore \tan\theta = 1 \ \ or \ \ \tan\theta = 2$$

If particles fired at angles α, β collide where $x = 240,\ y = 80$, then α, β are roots of this equation in θ.
Since $\beta < \alpha$, $\tan\alpha = 2$ and $\tan\beta = 1$.

ii. Let the particles collide t seconds after the second particle is fired. Then, using expression for x from (a) i.,

$$240 = 60(t+T)\cos\alpha \Rightarrow 4\sec\alpha = t + T$$

$$240 = 60t\cos\beta \Rightarrow 4\sec\beta = t$$

$$\therefore T = 4\left(\sec\alpha - \sec\beta\right)$$

$$= 4\left(\sqrt{5} - \sqrt{2}\right)$$

c. Outcomes Assessed: PE3, H5, HE1

Marking Guidelines

Criteria	Marks
• finds an expression for S as the sum of a G.P	1
• uses the quadratic equation to express x^{2n} in the form $(x+1)^n$	1
• uses the binomial theorem to expand $(x+1)^n-1$	1
• uses the quadratic expression to show $x-1=\frac{1}{x}$, then obtains required expression for S.	1

Answer

S is the sum of $2n$ terms of a G.P. $\therefore\ S=\dfrac{x^{2n}-1}{x-1}$

$$\begin{aligned} x^2-x-1&=0\\ x^2&=x+1\\ x^{2n}&=(x+1)^n\\ &=\sum_{r=0}^{n} {}^nC_r\, x^r\\ x^{2n}-1&=\sum_{r=1}^{n} {}^nC_r\, x^r \end{aligned}$$

$$\begin{aligned} x^2-x-1&=0\\ x^2-x&=1\\ x-1&=\frac{1}{x}\\ \frac{1}{x-1}&=x \end{aligned}$$

$$\begin{aligned} \therefore S&=\frac{1}{x-1}\left(x^{2n}-1\right)\\ &=x.\sum_{r=1}^{n} {}^nC_r\, x^r\\ &=\sum_{r=1}^{n} {}^nC_r\, x^{r+1} \end{aligned}$$