Centre Number

Student Number

CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NEW SOUTH WALES

**2003**
**TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION**

# Mathematics
# Extension 2

Morning Session
Monday 11 August 2003

## General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided separately
- All necessary working should be shown in every question

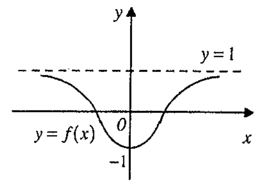
Total marks (120)
- Attempt Questions 1 – 8
- All questions are of equal value



2604 - 1

**Question 1** **Begin a new page** **Marks**

(a) The diagram shows the graph of $y = f(x)$ where $f(x) = 1 - 2e^{-x^2}$.




(i) Find the values of the $x$ intercepts. **1**

(ii) On separate diagrams sketch the graphs of **5**
$y = \{f(x)\}^2$, $y^2 = f(x)$, $y = \cos^{-1} f(x)$,
in each case showing the intercepts on the axes and the equations of any asymptotes.

(b) Consider the function $f(x) = \dfrac{x}{1-x^2}$.

(i) Show that the function is increasing for all values of $x$ in its domain. **1**

(ii) Sketch the graph of $y = f(x)$ showing the intercepts on the axes and the equations of any asymptotes. **2**

(iii) Find the values of $k$ such that the equation $\dfrac{x}{1-x^2} = kx$ has three distinct real roots. **2**

(c) Consider the curve defined by $2x^2 + xy - y^2 = 0$. At the point $(2,4)$ on the curve, find the values of $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. **4**

**Question 2** **Begin a new page**

(a)(i) Find $\displaystyle\int \frac{\cos 2x}{\cos^2 x}\,dx$. **2**

(ii) Find $\displaystyle\int \frac{x^3}{1+x^2}\,dx$. **2**

(b) Use the substitution $u = 1 + e^x$ to find $\displaystyle\int \frac{e^{2x}}{\sqrt{1+e^x}}\,dx$. **3**

(c) Use integration by parts to evaluate $\displaystyle\int_1^e \frac{\ln x}{x^2}\,dx$. **3**

(d)(i) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\displaystyle\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} \frac{1}{2+\cos x}\,dx$. **3**

(ii) Hence use the substitution $u = 4\pi - x$ to evaluate $\displaystyle\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} \frac{x}{2+\cos x}\,dx$. **2**

## Question 3 — Begin a new page

Marks

(a) If $x$ is real and $(x+i)^4$ is imaginary, find the possible values of $x$ in surd form. **3**

(b) $z$ and $w$ are two complex numbers such that $|z|=4$, $\arg z=\frac{5\pi}{6}$, $|w|=2$, $\arg w=\frac{\pi}{3}$.

(i) Express each of $z$ and $w$ in the form $a+ib$, where $a$ and $b$ are real. **2**

(ii) In an Argand diagram the points $P$ and $Q$ represent the complex numbers $z$ and $w$ respectively. Find the distance $PQ$ in simplest exact form. **2**

(c)(i) Express $\sqrt{3}+i$ in modulus / argument form. **1**

(ii) On an Argand diagram sketch the locus of the point $P$ representing the complex number $z$ such that $\left|z-\left(\sqrt{3}+i\right)\right|=1$, and find the set of possible values of $|z|$ and $\arg z$. **3**

(d) In an Argand diagram the points $P$, $Q$ and $R$ represent the complex numbers $z_1$, $z_2$ and $z_2+i(z_2-z_1)$ respectively.

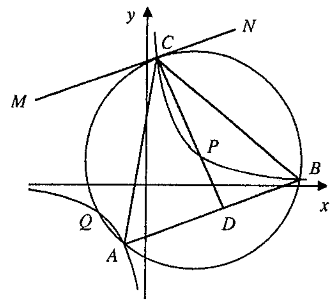
(i) Show that $PQR$ is a right-angled triangle. **2**

(ii) Find in terms of $z_1$ and $z_2$ the complex number represented by the point $S$ such that $PQRS$ is a rectangle. **2**

## Question 4 — Begin a new page

Marks

(a)




$P\left(c\theta,\frac{c}{\theta}\right)$ and $Q\left(-c\theta,-\frac{c}{\theta}\right)$, where $\theta>0$ and $c>0$, are two points on the rectangular hyperbola $xy=c^2$. The circle with centre $P$ and radius $PQ$ cuts the hyperbola again at points $A\left(c\alpha,\frac{c}{\alpha}\right)$, $B\left(c\beta,\frac{c}{\beta}\right)$ and $C\left(c\gamma,\frac{c}{\gamma}\right)$. $CP$ produced meets $AB$ at $D$. $MCN$ is tangent to the circle at $C$.

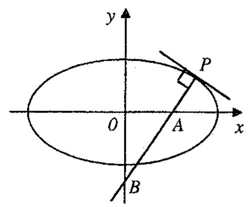
(i) Show that the circle cuts the hyperbola at points $\left(ct,\frac{c}{t}\right)$ where $t$ satisfies the equation $t^4-2t^3\theta-3t^2\left(\theta^2+\frac{1}{\theta^2}\right)-\frac{2}{\theta}t+1=0$. Hence deduce that $\alpha\beta\gamma\theta=-1$. **3**

(ii) Show that $CPD\perp AB$. Hence show that $MCN \parallel AB$. **2**

(iii) Show that $CA=CB$. **3**

(iv) What word best classifies triangle $ABC$? Justify your answer. **1**

(b)




$P(a\cos\theta, b\sin\theta)$, where $0<\theta<\frac{\pi}{2}$, is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a>b>0$. The normal to the ellipse at $P$ has equation $ax\sin\theta-by\cos\theta=\left(a^2-b^2\right)\sin\theta\cos\theta$. This normal cuts the $x$ axis at $A$ and the $y$ axis at $B$.

(i) Show that $\Delta OAB$ has area $\dfrac{\left(a^2-b^2\right)^2}{2ab}\sin\theta\cos\theta$. **3**

(ii) Find the maximum area of $\Delta OAB$ and the coordinates of $P$ when this maximum occurs. **3**

Marks

**Question 5** **Begin a new page**

(a) The equation $x^4 - x^3 + 2x^2 - 2x + 1 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(i) Show that none of $\alpha, \beta, \gamma, \delta$ is an integer. **2**

(ii) Find the monic equation of degree four with roots $\alpha - 1, \beta - 1, \gamma - 1, \delta - 1$, and hence find the value of $(\alpha+\beta+\gamma)(\beta+\gamma+\delta)(\gamma+\delta+\alpha)(\delta+\alpha+\beta)$. **4**

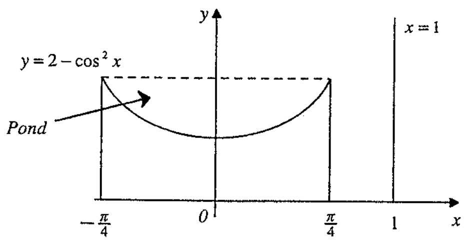
(b) (i) Express the roots of the equation $z^5 + 32 = 0$ in modulus / argument form. **3**

(ii) Hence show that $z^4 - 2z^3 + 4z^2 - 8z + 16 = \left\{z^2 - \left(4\cos\frac{\pi}{5}\right)z + 4\right\}\left\{z^2 - \left(4\cos\frac{3\pi}{5}\right)z + 4\right\}$. **2**

(iii) Hence find the exact values of $\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$ in simplest surd form. **4**

**Question 6** **Begin a new page**

(a)




A mould for a circular fish pond is made by rotating the region bounded by the curve $y = 2 - \cos^2 x$ and the $x$ axis between $x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$ through one complete revolution about the line $x = 1$. All measurements are in metres.

(i) Use the method of cylindrical shells to show that the volume of the fish pond is given by $V = \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1-x)\cos 2x \, dx$ . **3**

(ii) Hence find the capacity of the fish pond correct to the nearest litre. **3**

(b) A particle of mass $m$ kilograms is dropped from rest in a medium where the resistance to motion has magnitude $\frac{1}{10}mv^2$ Newtons when the speed of the particle is $v\,\text{ms}^{-1}$. After $t$ seconds, the particle has fallen $x$ metres, and has velocity $v\,\text{ms}^{-1}$ and acceleration $a\,\text{ms}^{-2}$. The particle hits the ground $\ln\left(1+\sqrt{2}\right)$ seconds after it is dropped. Take $g = 10\,\text{ms}^{-2}$.

(i) Draw a diagram showing the forces acting on the particle. Deduce that $a = \frac{1}{10}\left(100 - v^2\right)$. **2**

(ii) Express $v$ as a function of $t$. Hence find the speed with which the particle hits the ground, giving the answer in simplest exact form. **4**

(iii) Find in simplest exact form the distance fallen by the particle before it hits the ground. **3**

Marks

**Question 7** **Begin a new page**

(a) $a, b, c$ denote the lengths of the sides of a triangle.

(i) Express $4b^2c^2 - \left(b^2 + c^2 - a^2\right)^2$ as the product of four factors. **3**

(ii) Hence show that $\left(b^2 + c^2 - a^2\right)^2 < 4b^2c^2$. **1**

(b) Consider the function $f(x) = \cos^{-1} x$.

(i) Show that the function $E(x) = f(x) + f(-x)$ is even, and $O(x) = f(x) - f(-x)$ is odd. **2**

(ii) Hence express $\cos^{-1} x$ as the sum of an even function and an odd function. On the same diagram, sketch the graphs of these two functions. **3**

(c) A sequence $u_1, u_2, u_3, u_4 \ldots$ satisfies the relationship $u_n = u_{n-1} + u_{n-2}$ for $n \geq 3$.

(i) Show that $u_1u_2 + u_2u_3 = u_3^{\,2} - u_1^{\,2}$. **2**

(ii) Use Mathematical Induction to show that **4**

$u_1u_2 + u_2u_3 + u_3u_4 + u_4u_5 + \ldots + u_{2n-1}u_{2n} + u_{2n}u_{2n+1} = u_{2n+1}^{\;2} - u_1^{\,2}$ for $n \geq 1$.

**Question 8** **Begin a new page**

(a) Code numbers of three digits are made from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 where no digit is repeated.

(i) Find the number of different code numbers that can be formed. **1**

(ii) How many of these code numbers are such that the three digits do not occur in increasing order of magnitude, reading from left to right? **3**

(b) Consider the function $f(x) = x - \dfrac{3\sin x}{2 + \cos x}$.

(i) Show that $f'(x) = \left(\dfrac{1 - \cos x}{2 + \cos x}\right)^2$. **2**

(ii) Hence show that $x > \dfrac{3\sin x}{2 + \cos x}$ for $x > 0$. **3**

(c)(i) Show that $\sin(2r+1)\theta - \sin(2r-1)\theta = 2\sin\theta\cos 2r\theta$. Hence show that **3**

$\sin\theta\sum_{r=1}^{n}\cos 2r\theta = \frac{1}{2}\{\sin(2n+1)\theta - \sin\theta\}$.

(ii) Hence evaluate $\sum_{r=1}^{100}\cos^2\left(\frac{r\pi}{100}\right)$. **3**

**1(a) Outcomes Assessed: (i) E6 (ii) E6**

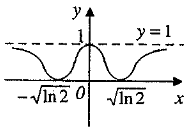
**Marking Guidelines**

| Criteria | Marks |
|---|---|
| (i) • both $x$ intercepts correct | 1 |
| (ii) • graph of $y=\{f(x)\}^2$ | 1 |
| • shape of $y^2=f(x)$ | 1 |
| • asymptotes and $x$ intercepts for $y^2=f(x)$ | 1 |
| • shape of $y=\cos^{-1}f(x)$ | 1 |
| • asymptote and $y$ intercept for $y=\cos^{-1}f(x)$ | 1 |

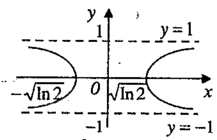
**Answer**

(i) $y=1-2e^{-x^2}$ $\quad y=0 \Rightarrow e^{-x^2}=\frac{1}{2}\Rightarrow -x^2=\ln\frac{1}{2}$ $\quad \therefore x^2=\ln 2$. Hence $x$ intercepts $\pm\sqrt{\ln 2}$

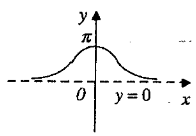
(ii)




$y=\{f(x)\}^2$




$y^2=f(x)$




$y=\cos^{-1}f(x)$

**1(b) Outcomes Assessed: (i) E6 (ii) E6 (iii) E6**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| (i) • find $f'(x)$ and note $f'(x)>0$ for $x\neq\pm1$ | 1 |
| (ii) • shape and position of curve | 1 |
| • asymptotes with equations | 1 |
| (iii) • considering intersections of curve with line $y=kx$ for varying $k$ | 1 |
| • deducing values for $k$ are $k<0$ or $k>1$ | 1 |

**Answer**

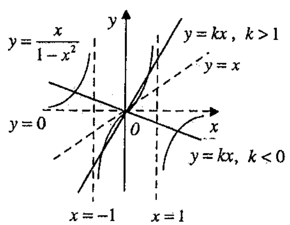
(i) $f(x)=\dfrac{x}{1-x^2}$ has domain $\{x: x\neq\pm1\}$

$$f'(x)=\frac{1.(1-x^2)-x.(-2x)}{(1-x^2)^2}=\frac{1+x^2}{(1-x^2)^2}$$

$\therefore f'(x)>0$ $\quad$ *for* $x\neq\pm1$

Hence $f$ is an increasing function for all $x$ in its domain.

(ii)




(iii) The real roots of $\dfrac{x}{1-x^2}=kx$ are the $x$ coordinates of the points of intersection of the curve with the line $y=kx$, which passes through the origin O and has gradient $k$. As k varies, line $y=kx$ turns around O. The tangent to the curve at the origin has gradient 1, and equation $y=x$. There will be 3 intersection points, and hence 3 distinct real roots for the equation, when $k<0$ or $k>1$.

**1(c) Outcomes Assessed: E6**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| • $\frac{dy}{dx}$ in terms of $x$ and $y$ | 1 |
| • value of $\frac{dy}{dx}$ at the point $(2,4)$ | 1 |
| • $\frac{d^2y}{dx^2}$ in terms of $x$, $y$ and $\frac{dy}{dx}$ | 1 |
| • value of $\frac{d^2y}{dx^2}$ at the point $(2,4)$ | 1 |

**Answer**

$$2x^2+xy-y^2=0$$

$$4x+\left(y+x\frac{dy}{dx}\right)-2y\frac{dy}{dx}=0$$

$$(x-2y)\frac{dy}{dx}+(4x+y)=0$$

$$\therefore \frac{dy}{dx}=\frac{4x+y}{2y-x}=\frac{12}{6}=2 \quad at\ (2,4)$$

$$\frac{d^2y}{dx^2}=\frac{\left(4+\frac{dy}{dx}\right)(2y-x)-(4x+y)\left(2\frac{dy}{dx}-1\right)}{(2y-x)^2}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{(4+2).6-12.(4-1)}{36}=0 \quad at\ (2,4).$$

**2(a) Outcomes Assessed: (i) E8 (ii) E8**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| (i) • simplification of integrand to obtain $(2-\sec^2 x)$ | 1 |
| • primitive function | 1 |
| (ii) • rearrangement of integrand to obtain $x+\frac{x}{1+x^2}$ | 1 |
| • primitive function | 1 |

**Answer**

(i) $\dfrac{\cos 2x}{\cos^2 x}=\dfrac{2\cos^2 x-1}{\cos^2 x}=2-\sec^2 x$

$$\int(2-\sec^2 x)dx=2x-\tan x+c$$

(ii) $\dfrac{x^3}{1+x^2}=\dfrac{x(1+x^2)-x}{1+x^2}=x-\dfrac{x}{1+x^2}$

$$\int\left(x-\frac{x}{1+x^2}\right)dx=\tfrac{1}{2}x^2-\tfrac{1}{2}\ln(1+x^2)+c$$

**2(b) Outcomes Assessed: HE6**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| • substitution to get integral in terms of $u$ | 1 |
| • primitive in terms of $u$ | 1 |
| • primitive in terms of $x$. | 1 |

**Answer**

$u=1+e^x$

$du=e^x\,dx$

$du=(u-1)dx$

$dx=\dfrac{1}{u-1}du$

$$I=\int\frac{e^{2x}}{\sqrt{1+e^x}}dx=\int\frac{(u-1)^2}{\sqrt{u}}.\frac{1}{u-1}du$$

$$\therefore I=\int\frac{u-1}{u^{\frac{1}{2}}}du=\int\left(u^{\frac{1}{2}}-u^{-\frac{1}{2}}\right)du$$

$$I=\tfrac{2}{3}u^{\frac{3}{2}}-2u^{\frac{1}{2}}+c$$

$$=\tfrac{2}{3}u^{\frac{1}{2}}(u-3)+c$$

$$I=\tfrac{2}{3}(e^x-2)\sqrt{1+e^x}+c$$

**2(c) Outcomes Assessed: E8**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| • rearrangement to $\left[-\frac{1}{x}\ln x\right]_1^e - \int_1^e -\frac{1}{x^2}\,dx$ using integration by parts | 1 |
| • evaluation of $\left[-\frac{1}{x}\ln x\right]_1^e$ | 1 |
| • value of definite integral | 1 |

**Answer**

$$I=\int_1^e \frac{1}{x^2}\ln x\,dx=\left[-\frac{1}{x}\ln x\right]_1^e-\int_1^e -\frac{1}{x^2}\,dx$$

$$=\left(-\tfrac{1}{e}-0\right)-\left[\frac{1}{x}\right]_1^e$$

$$\therefore I=-\tfrac{1}{e}-\left(\tfrac{1}{e}-1\right)=1-\tfrac{2}{e}$$

**2(d) Outcomes Assessed: (i) HE6 (ii) E8**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| (i) • substitution to obtain integral in terms of $t$. | 1 |
| • primitive in terms of $t$. | 1 |
| • exact value of definite integral | 1 |
| (ii) • substitution to obtain integral in terms of $u$. | 1 |
| • value of definite integral | 1 |

**Answer**

(i)

$t=\tan\frac{x}{2}$

$dt=\frac{1}{2}\sec^2\frac{x}{2}\,dx$

$=\frac{1}{2}\left(1+t^2\right)dx$

$dx=\frac{2}{1+t^2}\,dt$

$x=\frac{3\pi}{2}\Rightarrow t=\tan\frac{3\pi}{4}=-1$

$x=\frac{5\pi}{2}\Rightarrow t=\tan\frac{5\pi}{4}=1$

$2+\cos x=2+\frac{1-t^2}{1+t^2}$

$=\frac{3+t^2}{1+t^2}$

$$I=\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}}\frac{1}{2+\cos x}\,dx=\int_{-1}^{1}\frac{1+t^2}{3+t^2}\cdot\frac{2}{1+t^2}\,dt$$

$$=\frac{2}{\sqrt{3}}\int_{-1}^{1}\frac{\sqrt{3}}{3+t^2}\,dt$$

$$=\frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{t}{\sqrt{3}}\right]_{-1}^{1}$$

$$\therefore I=\frac{2}{\sqrt{3}}\left\{\frac{\pi}{6}-\left(-\frac{\pi}{6}\right)\right\}=\frac{2\pi\sqrt{3}}{9}$$

(ii)

$u=4\pi-x$

$du=-dx$

$x=\frac{3\pi}{2}\Rightarrow u=\frac{5\pi}{2}$

$x=\frac{5\pi}{2}\Rightarrow u=\frac{3\pi}{2}$

$$J=\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}}\frac{x}{2+\cos x}\,dx$$

$$=\int_{\frac{5\pi}{2}}^{\frac{3\pi}{2}}\frac{4\pi-u}{2+\cos u}\cdot -du$$

$$=\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}}\frac{4\pi-u}{2+\cos u}\,du$$

$$=\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}}\frac{4\pi}{2+\cos u}\,du-\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}}\frac{u}{2+\cos u}\,du$$

$$\therefore J=\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}}\frac{4\pi}{2+\cos u}\,du-J$$

$$2J=4\pi\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}}\frac{1}{2+\cos u}\,du$$

$$=4\pi\cdot\frac{2\pi\sqrt{3}}{9}$$

$$\therefore J=\frac{4\pi^2\sqrt{3}}{9}$$

**3(a) Outcomes Assessed: E3**

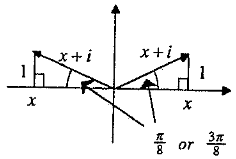
**Marking Guidelines**

| Criteria | Marks |
|---|---|
| • deduction of the possible values of $\arg(x+i)$ | 1 |
| • expressing $x$ in the form $\cot n\frac{\pi}{8}$ | 1 |
| • writing the possible values of $x$ in surd form | 1 |

**Answer**

$(x+i)^4$ imaginary $\Rightarrow \arg(x+i)^4=4\arg(x+i)=n\frac{\pi}{2}$, $n$ odd. $\therefore \arg(x+i)=n\frac{\pi}{8}$, $n$ odd

Since $x$ is real, the point representing $x$ in the Argand diagram lies on the horizontal axis, and the point representing $x+i$ lies in the first or second quadrant.




For $x>0$, $\arg(x+i)=\frac{\pi}{8},\frac{3\pi}{8}$ $\Rightarrow x=\cot\frac{\pi}{8},\ \cot\frac{3\pi}{8}$

For $x<0$, $\arg(x+i)=\frac{5\pi}{8},\frac{7\pi}{8}\Rightarrow -x=\cot\frac{\pi}{8},\ \cot\frac{3\pi}{8}$

$\frac{\pi}{8}+\frac{3\pi}{8}=\frac{\pi}{2}\Rightarrow\begin{cases}\cot\frac{\pi}{8}=\tan\frac{3\pi}{8}\\ \cot\frac{3\pi}{8}=\tan\frac{\pi}{8}\end{cases}$. Hence $x=\pm\tan\frac{\pi}{8},\ \pm\tan\frac{3\pi}{8}$.

If $t=\tan\frac{\pi}{8}$, $\frac{2t}{1-t^2}=\tan\frac{\pi}{4}=1$. If $t=\tan\frac{5\pi}{8}$, $\frac{2t}{1-t^2}=\tan\frac{5\pi}{4}=1$

Hence $\tan\frac{\pi}{8}$, $\tan\frac{5\pi}{8}$ are the roots of $t^2+2t-1=0$.

$\left.\begin{array}{l}\tan\frac{\pi}{8}>0\Rightarrow\tan\frac{\pi}{8}=-1+\sqrt{2}\\ \tan\frac{5\pi}{8}<0\Rightarrow\tan\frac{5\pi}{8}=-1-\sqrt{2}\end{array}\right\}\Rightarrow x=\pm\left(\sqrt{2}-1\right),\ \pm\left(\sqrt{2}+1\right)$. (Since $\tan\frac{3\pi}{8}=-\tan\frac{5\pi}{8}=1+\sqrt{2}$)

**3(b) Outcomes Assessed: (i) E3 (ii) E3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| (i) • expression for $z$ in form $a+ib$ | 1 |
| • expression for $w$ in form $a+ib$ | 1 |
| (ii) • coordinates of $P$ and $Q$ | 1 |
| • distance $PQ$ in simplest surd form | 1 |

**Answer**

(i) $z=4\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)=4\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=-2\sqrt{3}+2i$

$w=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=1+\sqrt{3}\,i$

(ii) $P\left(-2\sqrt{3},2\right)$, $Q\left(1,\sqrt{3}\right)$

$PQ^2=\left(1+2\sqrt{3}\right)^3+\left(\sqrt{3}-2\right)^2=20$

$\therefore PQ=2\sqrt{5}$

**3(c) Outcomes Assessed: (i) E3 (ii) E3**

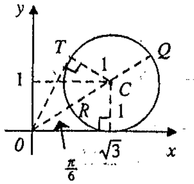
**Marking Guidelines**

| Criteria | Marks |
|---|---|
| (i) • modulus / argument form | 1 |
| (ii) • circle with correct centre and radius | 1 |
| • possible values for $\lvert z\rvert$ | 1 |
| • possible values for $\arg z$ | 1 |

**Answer**

(i) $\sqrt{3}+i=2\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

(ii) Locus of $P$ is a circle, centre $C\left(\sqrt{3},1\right)$ and radius 1.




$|z|$ takes its minimum and maximum values for $P$ at points $R$ and $Q$ respectively. $OC=2$, $RC=1 \Rightarrow OR=1$, $OQ=3$. Hence $1 \le |z| \le 3$.

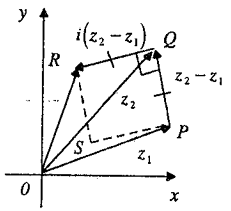
$OT$ is tangent to the circle at $T$. Then $O\hat{T}C = 90°$ and $\sin T\hat{O}C = \frac{1}{2} \Rightarrow T\hat{O}C = \frac{\pi}{6}$. Hence $0 \le \arg z \le \frac{\pi}{3}$.

**3(d) Outcomes Assessed: (i) E3 (ii) E3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| (i) • $\overrightarrow{PQ}$ represents $z_2 - z_1$ | |
| • deduction $\overrightarrow{QR}$ represents $i(z_2 - z_1)$ and hence is perpendicular to $\overrightarrow{PQ}$. | 1<br>1 |
| (ii) • deduction $\overrightarrow{PS}$ represents $i(z_2 - z_1)$ | 1 |
| • $S$ represents $z_1 + i(z_2 - z_1)$ | 1 |

**Answer**




(i) The vector $\overrightarrow{PQ}$ represents $z_2 - z_1$. $\overrightarrow{PQ}$ rotated anticlockwise by $\frac{\pi}{2}$ represents $i(z_2 - z_1)$. Now $\overrightarrow{OR}$ is the vector sum of the vectors representing $z_2$ and $i(z_2 - z_1)$, as shown in the diagram. Clearly $P\hat{Q}R = \frac{\pi}{2}$, and $\Delta PQR$ is right angled at $Q$.

(ii) If $PQRS$ is a rectangle, $\overrightarrow{PS}$ is parallel and equal to $\overrightarrow{QR}$, and hence $\overrightarrow{PS}$ also represents $i(z_2 - z_1)$. Now $\overrightarrow{OS}$ is the vector sum of $\overrightarrow{OP}$ and $\overrightarrow{PS}$. Hence $S$ represents $z_1 + i(z_2 - z_1)$.

**4(a) Outcomes Assessed: (i) E3, E4 (ii)H5, E4 (iii) PE3 (iv) H5**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| (i) • $PQ^2$ in terms of $\theta$ | 1 |
| • substitution of $(ct, \frac{c}{t})$ into equation of circle and simplification | 1 |
| • deduction $\alpha\beta\gamma\theta = -1$ (with explanation) | 1 |
| (ii) • product of gradients $CP$ and $AB$ is $-1$ | 1 |
| • deduction $MCN$ perpendicular to $CD$ and hence parallel to $AB$ | 1 |
| (iii)• correct sequence of geometric deductions, giving reasons, to show $CA = CB$ (award 2 marks if sequence of deductions is correct, but one reason missing; award 1 mark if sequence of deductions is correct, but two or more reasons missing) | 3 |
| (iv) • deduction that triangle $ABC$ is equilateral with reasons | 1 |

**Answers**

(i) $PQ^2 = 4c^2\left(\theta^2 + \frac{1}{\theta^2}\right)$ Hence circle has equation $(x - c\theta)^2 + \left(y - \frac{c}{\theta}\right)^2 = 4c^2\left(\theta^2 + \frac{1}{\theta^2}\right)$.

Circle intersects hyperbola at $(ct, \frac{c}{t})$ where

$$(ct - c\theta)^2 + \left(\frac{c}{t} - \frac{c}{\theta}\right)^2 = 4c^2\left(\theta^2 + \frac{1}{\theta^2}\right)$$

Dividing both sides by $c^2$ and expanding

$$t^2 - 2t\theta + \theta^2 + \frac{1}{t^2} - \frac{2}{t\theta} + \frac{1}{\theta^2} = 4\left(\theta^2 + \frac{1}{\theta^2}\right)$$

$$t^2 - 2t\theta - 3\left(\theta^2 + \frac{1}{\theta^2}\right) + \frac{1}{t^2} - \frac{2}{t\theta} = 0$$

$$t^4 - 2t^3\theta - 3t^2\left(\theta^2 + \frac{1}{\theta^2}\right) - \frac{2}{\theta}t + 1 = 0$$

But the circle cuts the hyperbola at $A$, $B$, $C$ and $Q$. Hence this equation has roots $\alpha, \beta, \gamma$ and $-\theta$.

$\therefore \alpha\beta\gamma(-\theta) = 1$ and $\therefore \alpha\beta\gamma\theta = -1$.



(ii) $gradient\ CP = \dfrac{c\left(\frac{1}{\theta} - \frac{1}{\gamma}\right)}{c(\theta - \gamma)} = -\dfrac{1}{\theta\gamma}$

Similarly $gradient\ AB = \dfrac{-1}{\alpha\beta}$

$\therefore gradient\ CP \,.\, gradient\ AB = \dfrac{1}{\alpha\beta\gamma\theta} = -1$

Hence $CPD \perp AB$

Also $MCN$ is tangent to the circle at $C$, hence $MCN \perp CP$ ( *tangent* $\perp$ *radius at pt. of contact*).

$\therefore M\hat{C}D + A\hat{D}C = 90° + 90° = 180°$

Hence $MCN \parallel AB$ (*supplementary cointerior* $\angle$*s on transversal CD*)

(iii) $M\hat{C}A = C\hat{A}B$ (*Alt.* $\angle$*s equal,* $MN \parallel AB$)

$M\hat{C}A = C\hat{B}A$ ($\angle$ *between tangent and chord equal to* $\angle$ *in alternate segment*)

$\therefore C\hat{B}A = C\hat{A}B$

$\therefore \Delta ABC$ is isosceles with $CA = CB$

(*equal sides opp. equal* $\angle$*s in* $\Delta ABC$)

(iv) Similarly, by constructing tangent to circle at $A$, and producing $AP$ to meet $BC$, $AB = AC$. Hence triangle $ABC$ is equilateral

**4(b) Outcomes Assessed: (i) E4 (ii) H5**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| (i) • $x$ intercept in terms of $\theta$ | 1 |
| • $y$ intercept in terms of $\theta$ | 1 |
| • explanation for area | 1 |
| (ii) • deduction that maximum occurs for $\theta = \frac{\pi}{4}$. | 1 |
| • maximum value of area | 1 |
| • coordinates of $P$ in surd form | 1 |

**Answer**

(i) At $A$: $y = 0 \Rightarrow x = \dfrac{a^2 - b^2}{a}\cos\theta$

At $B$: $x = 0 \Rightarrow y = -\dfrac{a^2 - b^2}{b}\sin\theta$

$Area\ \Delta AOB = \frac{1}{2} \,.\, OA \,.\, OB$

$$= \frac{1}{2}\left(\frac{a^2 - b^2}{a}\cos\theta\right)\left(\frac{a^2 - b^2}{b}\sin\theta\right)$$

$$= \frac{(a^2 - b^2)^2}{2ab}\sin\theta\cos\theta$$

(ii) $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$

$\therefore \sin\theta\cos\theta$ has a maximum value of $\frac{1}{2}$ when $\theta = \frac{\pi}{4}$.

Hence maximum area of $\Delta AOB$ is $\dfrac{(a^2 - b^2)^2}{4ab}$ when $P$ has coordinates $\left(\dfrac{a}{\sqrt{2}}, \dfrac{b}{\sqrt{2}}\right)$.

**5(a) Outcomes Assessed: (i) E4 (ii) E4**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| (i) • deduce integer solution would be factor of 1 | 1 |
| • state neither 1 nor $-1$ is a root and make required deduction | 1 |
| (ii) • application of a suitable method to find equation with explanation | 1 |
| • equation simplified with all coefficients correct | 1 |
| • expression of required product of factors as product of roots of new equation | 1 |
| • value of required product of factors | 1 |

**Answer**

(i) Any integer root of $x^4 - x^3 + 2x^2 - 2x + 1 = 0$ would have to be a factor of 1. But neither 1 nor $-1$ satisfies the equation. Hence none of $\alpha, \beta, \gamma, \delta$ is an integer.

(ii) $(x+1)^4 - (x+1)^3 + 2(x+1)^2 - 2(x+1) + 1 = 0$ has roots $\alpha - 1, \beta - 1, \gamma - 1, \delta - 1$.

Expansion and simplification gives the monic equation $x^4 + 3x^3 + 5x^2 + 3x + 1 = 0$. **

From the original equation, $\alpha + \beta + \gamma + \delta = 1$. Hence

$$(\alpha+\beta+\gamma)(\beta+\gamma+\delta)(\gamma+\delta+\alpha)(\delta+\alpha+\beta) = (1-\delta)(1-\alpha)(1-\beta)(1-\gamma) = (\alpha-1)(\beta-1)(\gamma-1)(\delta-1) = 1$$

since for equation **, product of the roots is 1.

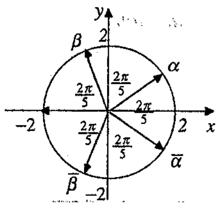
**5(b) Outcomes Assessed: (i) E3 (ii) E3 (iii) E4**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| (i) • deduction that there are 5 roots, each with modulus 2, one being $-2$ | 1 |
| • one pair of complex conjugate roots in mod / arg form (with explanation) | 1 |
| • second pair of complex conjugate roots in mod / arg form | 1 |
| (ii) • removing factor $(z+2)$ and identifying zeros of remaining factor | 1 |
| • expansion of $(z-\alpha)(z-\overline{\alpha})$ to explain factorisation | 1 |
| (iii) • sum and product of $\cos\frac{\pi}{5}$, $\cos\frac{3\pi}{5}$ | 2 |
| • quadratic equation with roots $\cos\frac{\pi}{5}$, $\cos\frac{3\pi}{5}$ | 1 |
| • $\cos\frac{\pi}{5}$, $\cos\frac{3\pi}{5}$ in simplest surd form | 1 |

**Answers**

(i)




The five 5th roots of $-32$ have modulus 2, and are equally spaced by $\frac{2\pi}{5}$ around the circle of radius 2 centred on the origin in the Argand diagram. One of these 5th roots is $-2$. The others are

$\alpha = 2\{\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\}$, $\overline{\alpha} = 2\{\cos(-\frac{\pi}{5}) + i\sin(-\frac{\pi}{5})\}$,

$\beta = 2\{\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\}$, $\overline{\beta} = 2\{\cos(-\frac{3\pi}{5}) + i\sin(-\frac{3\pi}{5})\}$

(ii) $0 = z^5 + 32 = (z+2)(z^4 - 2z^3 + 4z^2 - 8z + 16)$

Hence $\alpha, \overline{\alpha}, \beta, \overline{\beta}$ are roots of $z^4 - 2z^3 + 4z^2 - 8z + 16 = 0$.

$\therefore z^4 - 2z^3 + 4z^2 - 8z + 16 = (z-\alpha)(z-\overline{\alpha})(z-\beta)(z-\overline{\beta})$

But $(z-\alpha)(z-\overline{\alpha}) = z^2 - (\alpha+\overline{\alpha})z + \alpha\overline{\alpha} = z^2 - 2\text{Re}(\alpha).z + |\alpha|^2$

$\therefore z^4 - 2z^3 + 4z^2 - 8z + 16 = \{z^2 - (4\cos\frac{\pi}{5})z + 4\}\{z^2 - (4\cos\frac{3\pi}{5})z + 4\}$

(iii) Equating coefficients of $z$: $\quad -8 = -16(\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}) \Rightarrow \cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$

Equating coefficients of $z^2$: $\quad 4 = 4 + 4 + 16\cos\frac{\pi}{5}\cos\frac{3\pi}{5} \Rightarrow \cos\frac{\pi}{5}\cos\frac{3\pi}{5} = -\frac{1}{4}$

Hence $\cos\frac{\pi}{5}$, $\cos\frac{3\pi}{5}$ are roots of the quadratic equation $4x^2 - 2x - 1 = 0$.

This equation has solutions $x = \dfrac{2 \pm 2\sqrt{5}}{8} = \dfrac{1 \pm \sqrt{5}}{4}$.

Hence $\cos\frac{\pi}{5} > 0,\ \cos\frac{3\pi}{5} < 0 \Rightarrow \cos\frac{\pi}{5} = \frac{1}{4}(1+\sqrt{5}),\quad \cos\frac{3\pi}{5} = \frac{1}{4}(1-\sqrt{5})$

**6(a) Outcomes Assessed: (i) E7 (ii) E8**

**Marking Guidelines**

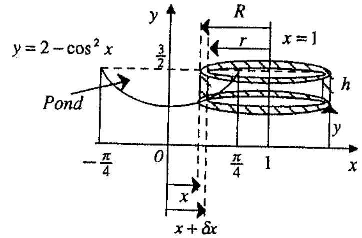
| Criteria | Marks |
|---|---|
| (i) • inner and outer radius and height of cylindrical shell in terms of $x$ | 1 |
| • volume of cylindrical shell in terms of $x$ | 1 |
| • volume of pond as limiting sum of cylindrical shells and then integral | 1 |
| (ii) • primitive function and substitution of limits | 2 |
| • conversion to capacity in litres | 1 |

**Answer**

(i) $y = 2 - \cos^2 x$

$x = \pm\frac{\pi}{4} \Rightarrow y = 2 - \frac{1}{2} = \frac{3}{2}$ $\quad$ Then height of cylindrical shell is given by

$h = \frac{3}{2} - (2 - \cos^2 x) = \cos^2 x - \frac{1}{2}$




$\left.\begin{array}{l} h = \cos^2 x - \frac{1}{2} \\ R = 1 - x \\ r = 1 - x - \delta x \end{array}\right\} \Rightarrow$

$\delta V = \pi(R^2 - r^2)h$

$= \pi(R+r)(R-r)(\cos^2 x - \frac{1}{2})$

$= \pi\{2(1-x) - \delta x\}(\delta x)(\frac{1}{2}\cos 2x)$

$\delta V = \pi(1-x)\cos 2x\ \delta x$

*(ignoring second order terms in $(\delta x)^2$)*

$\therefore V = \lim_{\delta x \to 0} \sum_{x=-\frac{\pi}{4}}^{x=\frac{\pi}{4}} \pi(1-x)\cos 2x\ \delta x$

$= \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1-x)\cos 2x\ dx$

(ii) $V = \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1-x)\cos 2x\ dx = \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2x\ dx - \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x\cos 2x\ dx$

$\therefore V = \frac{\pi}{2}[\sin 2x]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} - 0 \quad$ *(since $x\cos 2x$ is an odd function)*

$\therefore V = \frac{\pi}{2}\{1 - (-1)\} = \pi$

Hence the pond has volume $\pi$ m$^3$ and capacity $1000\pi$ L $\approx 3142$L.

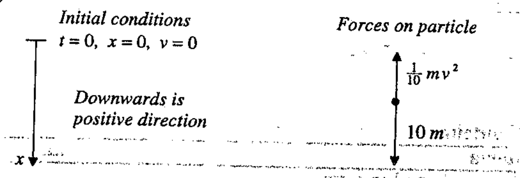
**6(b) Outcomes Assessed: (i) E5 (ii) E5 (iii) E5**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| (i) • diagram showing weight and resistance forces | 1 |
| • use of Newton's second law to deduce $a$ in terms of $v$ | 1 |
| (ii) • $\frac{dt}{dv}$ in partial fraction form | 1 |
| • primitive function, then substitution to obtain $t$ as a function $v$ | 1 |
| • $v$ as a function of $t$ | 1 |
| • substitution and simplification to obtain simplest exact value of speed on impact | 1 |
| (iii) • using either $a = v\frac{dv}{dx}$ or $a = \frac{1}{2}\frac{dv^2}{dx}$ to obtain $\frac{dx}{dv}$ or $\frac{dx}{d(v^2)}$ in terms of $v$ | 1 |
| • primitive function, then substitution to find $x$ in terms of $v$ | 1 |
| • substitution and simplification to obtain simplest exact value of distance fallen | 1 |

**Answer**

(i)




By Newton's second law

$ma = 10m - \frac{1}{10}mv^2$

$a = \frac{1}{10}(100 - v^2)$

(ii) $\dfrac{dv}{dt} = \frac{1}{10}(100 - v^2)$

$\dfrac{dt}{dv} = \dfrac{10}{(10+v)(10-v)}$

$= \frac{1}{2}\left\{\dfrac{1}{(10+v)} + \dfrac{1}{(10-v)}\right\}$

$t = \frac{1}{2}\{\ln(10+v) - \ln(10-v)\} + c$

$t = 0,\ v = 0 \Rightarrow c = 0$

$2t = \ln\left(\frac{10+v}{10-v}\right) \quad \therefore e^{2t} = \frac{10+v}{10-v}$

$10e^{2t} - ve^{2t} = 10 + v \Rightarrow v = 10\dfrac{e^{2t} - 1}{e^{2t} + 1}$

$t = \ln(1+\sqrt{2}) \Rightarrow e^{2t} = (e^t)^2 = (1+\sqrt{2})^2 = 3 + 2\sqrt{2}$

$\therefore v = 10\dfrac{2(1+\sqrt{2})}{2(2+\sqrt{2})} = 5(1+\sqrt{2})(2-\sqrt{2}) = 5\sqrt{2}$

Hence particle hits ground with speed $5\sqrt{2}$ ms$^{-1}$.

(iii)

$$\frac{1}{2}\frac{dv^2}{dx}=\frac{1}{10}\left(100-v^2\right)$$

$$\frac{dx}{d\left(v^2\right)}=\frac{5}{\left(100-v^2\right)}$$

$$x=-5\ln A\left(100-v^2\right),\ A \text{ constant}$$

$$x=0,\ v=0 \Rightarrow A=\tfrac{1}{100}$$

$$x=-5\ln\left(1-\frac{v^2}{100}\right)$$

$$v=5\sqrt{2}\Rightarrow x=-5\ln\left(1-\tfrac{50}{100}\right)$$

$$\therefore x=-5\ln\tfrac{1}{2}=5\ln 2$$

Alternatively

$$v\frac{dv}{dx}=\frac{1}{10}\left(100-v^2\right)$$

$$\frac{1}{v}\frac{dx}{dv}=10\frac{1}{\left(100-v^2\right)}$$

$$\frac{dx}{dv}=5\frac{2v}{\left(100-v^2\right)}$$

$$x=-5\ln\left(100-v^2\right)+b$$

$$x=0, v=0 \Rightarrow b=5\ln 100$$

$$x=-5\ln\left(\frac{100-v^2}{100}\right)=-5\ln\left(1-\frac{v^2}{100}\right)$$

Particle falls $5\ln 2$ metres before it hits the ground.

**7(a) Outcomes Assessed: (i) P4 (ii) PE3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| (i) • initial factorisation by difference of squares | 1 |
| • rearrangement of each factor as a difference of squares | 1 |
| • final factorisation | 1 |
| (ii) • required deduction made from triangle property | 1 |

**Answer**

(i)

$$4b^2c^2-\left(b^2+c^2-a^2\right)^2=\left\{2bc-\left(b^2+c^2-a^2\right)\right\}\left\{2bc+\left(b^2+c^2-a^2\right)\right\}$$
$$=\left\{a^2-(b-c)^2\right\}\left\{(b+c)^2-a^2\right\}$$
$$=\{a+(b-c)\}\{a-(b-c)\}\{(b+c)+a\}\{(b+c)-a\}$$
$$=(a+b-c)(a-b+c)(a+b+c)(b+c-a)$$

(ii) $a>0, b>0, c>0 \Rightarrow (a+b+c)>0$

Also in a triangle, the sum of any two sides is greater than the third side.
Hence the other three factors are also strictly positive.

$$\therefore 4b^2c^2-\left(b^2+c^2-a^2\right)^2>0 \qquad \therefore \left(b^2+c^2-a^2\right)^2<4b^2c^2$$

**7(b) Outcomes Assessed: (i) P5 (ii) P5**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| (i) • use definition of even function to show $E(x)$ is even | 1 |
| • use definition of odd function to show $O(x)$ is odd | 1 |
| (ii) • expression for $\cos^{-1}x$ in terms of $E(x)$ and $O(x)$ | 1 |
| • graph of $y=\frac{1}{2}E(x)$, with domain restricted to $-1\le x\le 1$ | 1 |
| • graph of $y=\frac{1}{2}O(x)$ | 1 |

**Answer**

(i) $E(-x)=f(-x)+f(-(-x))=f(-x)+f(x)=f(x)+f(-x)=E(x)$ Hence $E(x)$ is even.

$O(-x)=f(-x)-f(-(-x))=f(-x)-f(x)=-\{f(x)-f(-x)\}=-O(x)$ Hence $O(x)$ is odd.

(ii) $\frac{1}{2}\{E(x)+O(x)\}=f(x) \Rightarrow f(x)=\frac{1}{2}E(x)+\frac{1}{2}O(x)$ where $\frac{1}{2}E(x)$ is even, and $\frac{1}{2}O(x)$ is odd.

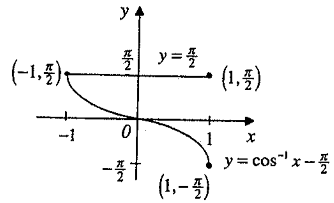
$$E(x)=\cos^{-1}x+\cos^{-1}(-x)$$
$$=\cos^{-1}x+\pi-\cos^{-1}x$$
$$\therefore E(x)=\pi,\ -1\le x\le 1$$

$$O(x)=\cos^{-1}x-\cos^{-1}(-x)$$
$$=\cos^{-1}x-\left(\pi-\cos^{-1}x\right)$$
$$\therefore O(x)=2\cos^{-1}x-\pi$$

$$\cos^{-1}x=\tfrac{1}{2}E(x)+\tfrac{1}{2}O(x)$$

where $g(x)=\frac{1}{2}E(x)=\frac{\pi}{2},\ -1\le x\le 1$ is an even function

and $h(x)=\frac{1}{2}O(x)=\cos^{-1}x-\frac{\pi}{2}$ is an odd function.




**7(c) Outcomes Assessed: (i) H5 (ii) HE2, E9**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| (i) • using the recurrence relationship to write $u_2$ in terms of $u_1, u_3$ | 1 |
| • rearrangement and simplification to obtain required result | 1 |
| (ii) • use (i) to establish truth of statement for $n=1$ | |
| • write LHS of $S(k+1)$ as $\left(u_{2k+1}{}^2-u_1{}^2\right)+u_{2k+1}u_{2k+2}+u_{2k+2}u_{2k+3}$ if $S(k)$ is true | 1 |
| • use same procedure as in (i) to complete proof that $S(k)$ true $\Rightarrow S(k+1)$ true | 1 |
| • deduce truth of $S(n)$ for all integers $n\ge 1$ | 1 |
| | 1 |

**Answers**

(i) $u_3=u_2+u_1$ Hence

$$u_1u_2+u_2u_3=u_2\left(u_3+u_1\right)$$
$$=\left(u_3-u_1\right)\left(u_3+u_1\right)$$
$$=u_3{}^2-u_1{}^2$$

(ii) Let $S(n),\ n=1,2,3,\ldots$ be the sequence of statements

$$u_1u_2+u_2u_3+u_3u_4+u_4u_5+\ \ldots\ +u_{2n-1}u_{2n}+u_{2n}u_{2n+1}=u_{2n+1}{}^2-u_1{}^2,\quad n=1,2,3,\ldots$$

Consider $S(1)$: $u_1u_2+u_2u_3=u_3{}^2-u_1{}^2$ has been proved in (i) $\therefore S(1)$ is true

If $S(k)$ is true : then $u_1u_2+u_2u_3+u_3u_4+u_4u_5+\ \ldots\ +u_{2k-1}u_{2k}+u_{2k}u_{2k+1}=u_{2k+1}{}^2-u_1{}^2$ **

Consider $S(k+1)$: $LHS=\left(u_1u_2+u_2u_3+u_3u_4+u_4u_5+\ \ldots\ +u_{2k-1}u_{2k}+u_{2k}u_{2k+1}\right)+u_{2k+1}u_{2k+2}+u_{2k+2}u_{2k+3}$

$=\left(u_{2k+1}{}^2-u_1{}^2\right)+u_{2k+2}\left(u_{2k+3}+u_{2k+1}\right)$ if $S(k)$ is true using **

$=\left(u_{2k+1}{}^2-u_1{}^2\right)+\left(u_{2k+3}-u_{2k+1}\right)\left(u_{2k+3}+u_{2k+1}\right)$ since $u_{2k+3}=u_{2k+2}+u_{2k+1}$

$=\left(u_{2k+1}{}^2-u_1{}^2\right)+\left(u_{2k+3}{}^2-u_{2k+1}{}^2\right)$

$=u_{2k+3}{}^2-u_1{}^2$

$=u_{2(k+1)+1}{}^2-u_1{}^2$

$=RHS$

Hence If $S(k)$ is true, then $S(k+1)$ is true. But $S(1)$ is true, hence $S(2)$ is true, and then $S(3)$ is true and so on. Hence by Mathematical Induction, $S(n)$ is true for all integers $n\ge 1$.

**8(a) Outcomes Assessed: (i) PE3 (ii) PE3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| (i) • correct answer | 1 |
| (ii) • counting the code numbers with digits in order as ${}^9C_3$ | |
| • evaluating ${}^9C_3$ | 1 |
| • subtraction of value of ${}^9C_3$ from (i) to count required code numbers | 1 |
| | 1 |

**Answers**

(i) Number of code numbers is $9\times 8\times 7=504$

(ii) There are ${}^9C_3=84$ selections of three digits, each giving rise to $3!=6$ code numbers, only one of which has the digits in increasing order. Hence 84 code numbers have their digits in increasing order. Hence $504-84=420$ code numbers do not have their digits in increasing order.

**8(b) Outcomes Assessed: (i) H5 (ii) PE3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| (i) • apply quotient rule | 1 |
| • rearrange and simplify to obtain required result | 1 |
| (ii) • deduce that $f(x)$ is increasing or stationary for $x>0$ | 1 |
| • compare $f(x)$ with $f(0)$ for $x>0$ | 1 |
| • evaluate $f(0)$ and deduce required result | 1 |

**Answers**

(i) $f(x)=x-\dfrac{3\sin x}{2+\cos x}$

$$f'(x)=1-\frac{3\{\cos x(2+\cos x)-\sin x(-\sin x)\}}{(2+\cos x)^2}$$

$$=1-\frac{3(\cos^2 x+\sin^2 x+2\cos x)}{(2+\cos x)^2}$$

$$=\frac{(2+\cos x)^2-3-6\cos x}{(2+\cos x)^2}$$

$$=\frac{1-2\cos x+\cos^2 x}{(2+\cos x)^2}$$

$$=\left(\frac{1-\cos x}{2+\cos x}\right)^2$$

(ii) $f'(0)=0$ and $f'(x)\geq 0$ for all $x>0$.
Hence $f(x)$ is stationary at $x=0$, $x=2n\pi$ and increasing for $x>0$, $x\neq 2n\pi$.

$\therefore f(x)>f(0)$ for $x>0$.

But $f(0)=0-\dfrac{0}{2+1}=0$.

Hence $f(x)>0$ for $x>0$.

$$\therefore x-\frac{3\sin x}{2+\cos x}>0$$

$$x>\frac{3\sin x}{2+\cos x} \quad \text{for } x>0$$

**8(c) Outcomes Assessed: (i) H5 (ii) H5, E9**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| (i) • Expansion and simplification to show first equality | 1 |
| • substitution into sum to obtain sum of differences of terms | 1 |
| • rearrangement and simplification to obtain second equality | 1 |
| (ii) • rearrangement so that result from (i) is applicable | 1 |
| • applying result from (i) | 1 |
| • completing evaluation to obtain numerical answer. | 1 |

**Answer**

(i)

$$\sin(2r+1)\theta=\sin 2r\theta\cos\theta+\cos 2r\theta\sin\theta$$

$$\sin(2r-1)\theta=\sin 2r\theta\cos\theta-\cos 2r\theta\sin\theta$$

$$\therefore \sin(2r+1)\theta-\sin(2r-1)\theta=2\sin\theta\cos 2r\theta$$

$$\sum_{r=1}^{n}2\sin\theta\cos 2r\theta=\sum_{r=1}^{n}\{\sin(2r+1)\theta-\sin(2r-1)\theta\}$$

$$=(\sin 3\theta-\sin\theta)+(\sin 5\theta-\sin 3\theta)+(\sin 7\theta-\sin 5\theta)+\ldots+\{\sin(2n+1)\theta-\sin(2n-1)\theta\}$$

$$=\sin(2n+1)\theta-\sin\theta$$

$$\therefore \sin\theta\sum_{r=1}^{n}\cos 2r\theta=\tfrac{1}{2}\{\sin(2n+1)\theta-\sin\theta\}$$

(ii)

$$\sum_{r=1}^{100}\cos^2\left(\tfrac{r\pi}{100}\right)=\sum_{r=1}^{100}\tfrac{1}{2}\left\{1+\cos\left(\tfrac{2r\pi}{100}\right)\right\}$$

$$=\tfrac{1}{2}\sum_{r=1}^{100}1+\tfrac{1}{2}\sum_{r=1}^{100}\cos\left(\tfrac{2r\pi}{100}\right)$$

$$=\tfrac{1}{2}(100)+\tfrac{1}{2}\cdot\frac{1}{\sin\left(\frac{\pi}{100}\right)}\cdot\tfrac{1}{2}\left\{\sin\left(\tfrac{201\pi}{100}\right)-\sin\left(\tfrac{\pi}{100}\right)\right\} \qquad \textit{(using (i) with } n=100,\ \theta=\tfrac{\pi}{100})$$

$$=50+\frac{1}{4\sin\left(\frac{\pi}{100}\right)}\left\{\sin\left(2\pi+\tfrac{\pi}{100}\right)-\sin\left(\tfrac{\pi}{100}\right)\right\}$$

$$=50+\frac{1}{4\sin\left(\frac{\pi}{100}\right)}\left\{\sin\left(\tfrac{\pi}{100}\right)-\sin\left(\tfrac{\pi}{100}\right)\right\}$$

Hence $\displaystyle\sum_{r=1}^{100}\cos^2\left(\tfrac{r\pi}{100}\right)=50$