Centre Number

Student Number

CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NEW SOUTH WALES

2007
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics
Extension 2

Afternoon Session
Friday 17 August 2007

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided separately
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1-8
- All questions are of equal value

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Marks

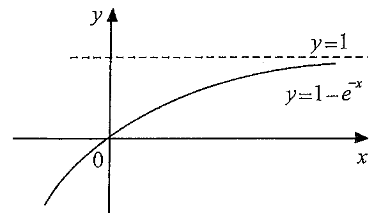
Question 1 (15 marks) Use a SEPARATE writing booklet

(a) It is given that $(2+\cos x)(2-\cos y)=3$, where $0<x<\pi$ and $0<y<\pi$.

(i) Show that $\cos y=\dfrac{1+2\cos x}{2+\cos x}$ and $\sin y=\dfrac{\sqrt{3}\sin x}{2+\cos x}$. **2**

(ii) Hence show that $\dfrac{dy}{dx}=\dfrac{\sqrt{3}}{2+\cos x}$. **3**

(b)



The diagram shows the graph of $f(x)=1-e^{-x}$. On separate diagrams sketch the graphs of the following functions, showing clearly the equations of any asymptotes:

(i) $y=[f(x)]^2$ **1**

(ii) $y=f(x^2)$ **1**

(iii) $y=\dfrac{1}{f(x)}$ **2**

(iv) $y=\ln f(x)$ **1**

(c) The function $f(x)$ is given by $f(x)=a+\dfrac{b\sin x}{x}$, $x\neq 0$ and $f(0)=0$, where a and b are non-zero real numbers.

(i) Show that $f(x)$ is an even function. **1**

(ii) Find the general solution of the equation $f(x)=a$. **2**

(iii) If $\lim\limits_{x\to\infty} f(x)=1$ and $f(x)$ is continuous at $x=0$, find the values of a and b. **2**

Question 2 (15 marks) Use a SEPARATE writing booklet

Marks

(a)(i) Find $\int \frac{1-x^{-2}}{1-x^{-1}}\,dx$. **2**

(ii) Find $\int \left(\sqrt{e^x}+1\right)^2 dx$. **2**

(b) Evaluate $\int_0^{\frac{\sqrt{3}}{2}} \frac{1-x}{\sqrt{1-x^2}}\,dx$. **3**

(c) Use the substitution $t=\tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{4+5\sin x}\,dx$. **4**

(d)(i) If $I_n=\int_1^e (1-\ln x)^n\,dx$, $n=0,1,2,\ldots$ show that $I_n=-1+nI_{n-1}$, $n=1,2,3,\ldots$ **2**

(ii) Hence find the value of I_3. **2**

Question 3 (15 marks) Use a SEPARATE writing booklet

Marks

(a) Find all the complex numbers $z=a+ib$, where a and b are real, such that $|z|^2+5\bar{z}+10i=0$. **3**

(b) $z_1=1+i\sqrt{3}$ and $z_2=1-i$ are two complex numbers.

(i) Express z_1, z_2 and $\frac{z_1}{z_2}$ in modulus / argument form. **2**

(ii) Find the smallest positive integer n such that $\frac{z_1^n}{z_2^n}$ is imaginary. For this value of n, write the value of $\frac{z_1^n}{z_2^n}$ in the form bi where b is a real number. **2**

(c)(i) On an Argand diagram shade the region where both $|z-1|\le 1$ and $0\le \arg z\le \frac{\pi}{6}$. **2**

(ii) Find the perimeter of the shaded region. **2**

(d) On an Argand diagram the points A, B and C represent the complex numbers α, β and γ respectively. ΔABC is equilateral, named with its vertices taken anticlockwise.

(i) Show that $\gamma-\alpha=\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)(\beta-\alpha)$. **2**

(ii) Show that $\alpha^2+\beta^2+\gamma^2=\alpha\beta+\beta\gamma+\gamma\alpha$. **2**

Question 4 (15 marks) Use a SEPARATE writing booklet

Marks

(a) $P(a\cos\theta, b\sin\theta)$ is a point in the first quadrant on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $Q(a\sec\theta, b\tan\theta)$ is a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where $a>b>0$.

(i) Sketch the ellipse, the hyperbola and their common auxiliary circle $x^2+y^2=a^2$ on the same diagram, showing the angle θ and the related points P and Q. Show clearly how the positions of P and Q are determined by the value of θ, $0<\theta<\frac{\pi}{2}$. **2**

(ii) Prove that the tangent to the ellipse at P has equation $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$. **3**

Deduce that this tangent cuts the x-axis vertically below Q.

(iii) Given that the tangent to the hyperbola at Q has equation $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$, show that this tangent and the tangent to the ellipse at P intersect at $T\left(a, b\tan\frac{\theta}{2}\right)$. Show both tangents on your sketch. **4**

(iv) Without any further working, sketch a second diagram showing both curves, the common auxiliary circle, the points P, Q and the corresponding tangents intersecting at T if $\frac{\pi}{2}<\theta<\pi$. **1**

(b) $P\left(2p, \frac{2}{p}\right)$ is a variable point on the hyperbola $xy=4$. The normal to the hyperbola at P meets the hyperbola again at $Q\left(2q, \frac{2}{q}\right)$. M is the midpoint of PQ.

(i) Show that $q=-\frac{1}{p^3}$. **2**

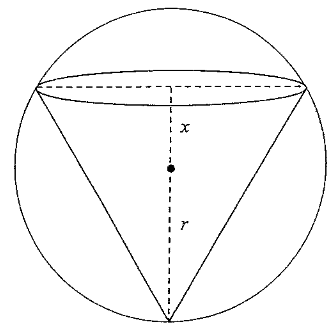
(ii) Show that M has coordinates $\left(\frac{1}{p}\left(p^2-\frac{1}{p^2}\right), p\left(\frac{1}{p^2}-p^2\right)\right)$. **1**

(iii) Show that as P moves on the hyperbola, the locus of M has equation $\left(x^2-y^2\right)^2=-x^3y^3$. **2**

Question 5 (15 marks) Use a SEPARATE writing booklet

Marks

(a) A right circular cone of height $(r+x)$ is inscribed in a sphere of radius r.



(i) Show that the volume V of the cone is given by $V=\frac{\pi}{3}\left(r^3+r^2x-rx^2-x^3\right)$. **2**

(ii) Hence show that V is a maximum when $x=\frac{1}{3}r$ **2**

(iii) Find the ratio of the maximum volume of the cone to the volume of the sphere. **2**

(b)(i) By considering $f'(x)$ where $f(x)=e^x-x$, show that $e^x>x$ for $x\geq 0$. **2**

(ii) Hence use Mathematical Induction to show that for $x\geq 0$, $e^x>\frac{x^n}{n!}$ for all positive integers $n\geq 1$. **3**

(c) The polynomial $P(x)$ is given by $P(x)=x^3+ax^2+bx+c$ where a, b and c are real. The equation $P(x)=0$ has roots α, β and γ. S_n is defined by $S_n=\alpha^n+\beta^n+\gamma^n$ for $n=1, 2, 3, \ldots$, and it is given that $S_1=S_3=3$ and $S_2=7$.

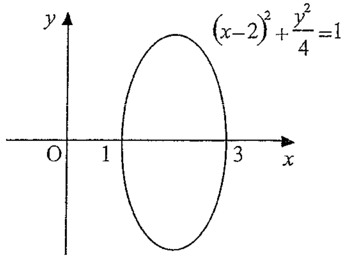
(i) Show that $a=-3$ and $b=1$. **2**

(ii) Find the value of c. **2**

Marks

Question 6 (15 marks) Use a SEPARATE writing booklet

(a)

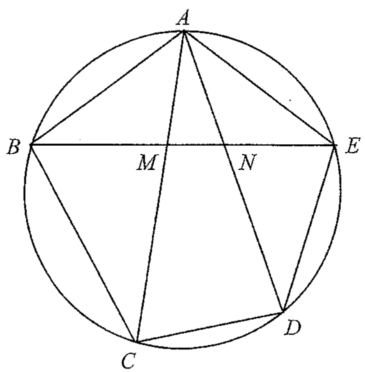


The region enclosed by the ellipse $(x-2)^2+\frac{y^2}{4}=1$ is rotated through one complete revolution about the y-axis.

(i) Use the method of cylindrical shells to show that the volume V of the solid of revolution is given by . $V=8\pi\int_1^3 x\sqrt{1-(x-2)^2}\,dx$ **2**

(ii) Hence find the volume of the solid of revolution in simplest exact form. **4**

(b)



$ABCDE$, where $AB=AE$, is a pentagon inscribed in a circle. BE meets AC and AD at M and N respectively.

(i) Show that $\angle BEA=\angle ACE$. **2**

(ii) Hence show that $CDNM$ is a cyclic quadrilateral. **3**

Marks

(c) The polynomial $P(x)$ is given by $P(x)=x^4+bx^2+1$ where b is a real number. The equation $P(x)=0$ has a real root α, where $\alpha\neq 0$.

(i) Express the other three roots of the equation $P(x)=0$ in terms of α and deduce that all four roots are real. **2**

(ii) Find the set of possible values of b. **2**

Question 7 (15 marks) Use a SEPARATE writing booklet

(a)(i) Show that the equation $z^7-1=0$ has roots $1, \omega, \omega^2, \omega^3, \omega^4, \omega^5$ and ω^6, where $\omega=\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}$. **2**

(ii) Hence show that the equation $z^6+z^5+z^4+z^3+z^2+z+1=0$ has roots $\omega, \omega^2, \omega^3, \omega^4, \omega^5$ and ω^6. **1**

(iii) Find the value of $\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}$. **2**

(iv) Find the monic quadratic equation with numerical coefficients whose roots are $\omega+\omega^2+\omega^4$ and $\omega^3+\omega^5+\omega^6$. **2**

(b) A particle of mass m is moving vertically in a resisting medium in which the resistance to motion has magnitude $\frac{1}{10}mv^2$ when the particle has velocity $v\,\text{ms}^{-1}$. The acceleration due to gravity is $10\,\text{ms}^{-2}$.

(i) The particle is projected vertically upwards with speed $U\,\text{ms}^{-1}$. Show that during its upward motion, its acceleration $a\,\text{ms}^{-2}$ is given by $a=-\frac{1}{10}(100+v^2)$. **1**

(ii) Hence show that its maximum height, H metres, is given by $H=5\ln\left(\frac{U^2+100}{100}\right)$. **3**

(iii) The particle falls vertically from rest. Show that during its downward motion its acceleration $a\,\text{ms}^{-2}$ is given by $a=\frac{1}{10}(100-v^2)$. **1**

(iv) Hence show that it returns to its point of projection with speed $V\,\text{ms}^{-1}$ given by $V=\frac{10U}{\sqrt{U^2+100}}$. **3**

Marks

Question 8 (15 marks) Use a SEPARATE writing booklet

(a) Consider the function $f(x)=\sum_{k=1}^{n}(a_k x-1)^2$ where $a_1>0,\ a_2>0,\dots,\ a_n>0$ are real.

(i) Express $f(x)$ in the form $f(x)=Ax^2+Bx+C$ for real numbers A, B and C. **1**

(ii) Show that $\sum_{k=1}^{n}{a_k}^2 \geq \frac{1}{n}\left(\sum_{k=1}^{n}a_k\right)^2$. **2**

(iii) Hence show that $1^2+3^2+\dots+(2n-1)^2 \geq n^3$ and $1^4+3^4+\dots+(2n-1)^4 \geq n^5$. **2**

(b) Two players A, B play a game of chance comprising several turns in which each player throws a fair 6-sided die. The possible outcomes are a draw (A, B throw the same score), A wins (A's score is higher than B's) or B wins. The game is over when either player first records two wins. Let p_n be the probability the game ends on the n^{th} turn. Let q_n be the probability the game does not end in n or fewer turns.

(i) Explain why, on each turn, the probability that A wins is $\frac{5}{12}$. **1**

(ii) Explain why $p_2+q_2=1$. **1**

(iii) Explain why $p_n+q_n=q_{n-1}$, $n=3,4,5,\dots$ and deduce that $\sum_{k=2}^{n}p_k=1-q_n$, $n\geq 2$. **3**

(iv) Show that $q_n=\dfrac{25n^2-5n+4}{4\times 6^n}$ and $p_n=\dfrac{25(n-1)(5n-8)}{4\times 6^n}$, $n\geq 2$. **3**

(v) What is the probability the game will never end? Justify your answer. **2**

CATHOLIC SECONDARY SCHOOLS ASSOCIATION

2007 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

MATHEMATICS EXTENSION 2

Question 1

a. Outcomes assessed : E6

	Criteria	Marks
i	• rearranges given equation to obtain expression for $\cos y$	1
	• uses trigonometric identity and rearrangement to find expression for $\sin y$	1
ii	• uses implicit differentiation with given equation or with expression for $\cos y$ or $\sin y$	1
	• substitutes for $\cos y$ or $\sin y$ as appropriate in resulting expression for $\frac{dy}{dx}$	1
	• simplifies to obtain required expression for derivative	1

Answer

i. $(2+\cos x)(2-\cos y)=3$

$$2-\cos y=\frac{3}{2+\cos x}$$

$$\cos y=2-\frac{3}{2+\cos x}=\frac{2(2+\cos x)-3}{2+\cos x}=\frac{1+2\cos x}{2+\cos x}$$

$$\sin^2 y=1-\frac{(1+2\cos x)^2}{(2+\cos x)^2}=\frac{(2+\cos x)^2-(1+2\cos x)^2}{(2+\cos x)^2}$$

$$\therefore \sin^2 y=\frac{3-3\cos^2 x}{(2+\cos x)^2}=\frac{3(1-\cos^2 x)}{(2+\cos x)^2}=\frac{3\sin^2 x}{(2+\cos x)^2}$$

Also $0<x<\pi$ and $0<y<\pi$,
hence $\sin x>0,\ \sin y>0$.
Clearly $2+\cos x>0$:

$$\therefore \sin y=\frac{\sqrt{3}\sin x}{2+\cos x}$$

ii. $(2+\cos x)(2-\cos y)=3$

$$-\sin x(2-\cos y)+(2+\cos x)\sin y\,\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx}=\frac{\sin x(2-\cos y)}{\sin y(2+\cos x)}$$

Substituting for $\sin y$

$$\frac{dy}{dx}=\left(\frac{2+\cos x}{\sqrt{3}}\right)\left(\frac{2-\cos y}{2+\cos x}\right)$$

But $(2+\cos x)(2-\cos y)=3$

$$\therefore \frac{dy}{dx}=\frac{\sqrt{3}}{2+\cos x}$$

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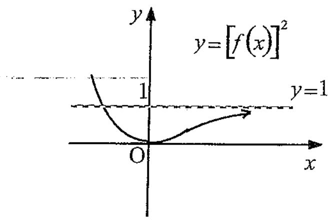
b. Outcomes assessed : E6

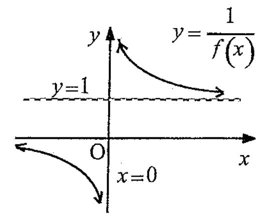
	Criteria	Marks
i	• sketches curve with correct shape and position, showing asymptote with equation	1
ii	• sketches curve with correct shape and position, showing asymptote with equation	1
iii	• sketches branch of curve with correct shape and position for $x<0$, showing asymptotes	1
	• sketches branch of curve with correct shape and position for $x>0$, showing asymptotes	1
iv	• sketches curve with correct shape and position, showing asymptotes	1

Answer

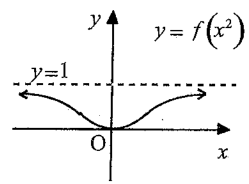
i.



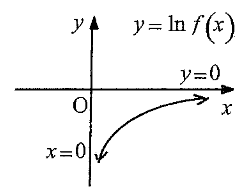
iii.



ii.



iv.



c. Outcomes assessed : E6

Criteria	Marks
i • applies criterion for determining a function is even	1
ii • identifies solutions as non-zero solutions of $\sin x = 0$	1
• writes general solution	1
iii • evaluates appropriate limit to find value of a	1
• evaluates appropriate limit to find value of b	1

Answer

i. For $x \neq 0$, $f(-x) = a + \dfrac{b\sin(-x)}{(-x)}$

$$= a + \frac{-b\sin x}{-x}$$

$$= a + \frac{b\sin x}{x}$$

$$= f(x)$$

Hence f is an even function.

ii. $f(x) = a \Rightarrow \sin x = 0,\ x \neq 0$

$\therefore x = n\pi,\ n = \pm1, \pm2, \ldots$

iii. $\left|\dfrac{b\sin x}{x}\right| \leq \left|\dfrac{b}{x}\right| \to 0$ as $x \to \infty$

$\therefore \lim\limits_{x\to\infty} f(x) = a \qquad \therefore a = 1$

$\lim\limits_{x\to 0} f(x) = a + b\lim\limits_{x\to 0}\dfrac{\sin x}{x} = a + b \qquad \therefore b = -1$

Question 2

a. Outcomes assessed : H5

Criteria	Marks
i • rearranges integrand into terms involving standard integrals	1
• finds primitive	1
ii • rearranges integrand into terms involving standard integrals	1
• finds primitive	1

Answer

i. $\displaystyle\int \frac{1-x^{-2}}{1-x^{-1}}\,dx = \int \frac{(1-x^{-1})(1+x^{-1})}{1-x^{-1}}\,dx$

$$= \int \left(1 + \frac{1}{x}\right) dx$$

$$= x + \ln|x| + c$$

ii. $\displaystyle\int \left(\sqrt{e^x}+1\right)^2 dx = \int \left(e^x + 2e^{\frac{1}{2}x} + 1\right) dx$

$$= e^x + 4e^{\frac{1}{2}x} + x + c$$

$$= e^x + 4\sqrt{e^x} + x + c$$

b. Outcomes assessed : HE4, HE6

Criteria	Marks
• rearranges integrand into terms involving standard integrals	1
• writes $\sin^{-1} x$ as one term in the primitive, then substitutes limits to find term $\frac{\pi}{3}$ in answer	1
• finds and evaluates remaining term in the primitive to complete value of the integral.	1

Answer

$$\int_0^{\frac{\sqrt{3}}{2}} \frac{1-x}{\sqrt{1-x^2}}\,dx = \int_0^{\frac{\sqrt{3}}{2}} \left(\frac{1}{\sqrt{1-x^2}} + \tfrac{1}{2}(-2x)(1-x^2)^{-\frac{1}{2}}\right) dx$$

$$= \left[\sin^{-1} x + (1-x^2)^{\frac{1}{2}}\right]_0^{\frac{\sqrt{3}}{2}}$$

$$= \left(\sin^{-1}\tfrac{\sqrt{3}}{2} - \sin^{-1} 0\right) + \left(\sqrt{1-\tfrac{3}{4}} - \sqrt{1-0}\right)$$

$$= \tfrac{\pi}{3} - \tfrac{1}{2}$$

c. Outcomes assessed : HE6, E8

Criteria	Marks
• writes dx in terms of dt and converts limits to t values	1
• expresses original integrand as a function of t	1
• writes new integrand in partial fraction form	1
• finds primitive then evaluates by substitution of limits	1

Answer

$$t = \tan\tfrac{x}{2}$$

$$dt = \tfrac{1}{2}\sec^2\tfrac{x}{2}\,dx$$

$$2dt = \left(1+\tan^2\tfrac{x}{2}\right)dx$$

$$= (1+t^2)dx$$

$$dx = \frac{2}{1+t^2}\,dt$$

$$x = 0 \Rightarrow t = 0$$

$$x = \tfrac{\pi}{2} \Rightarrow t = 1$$

$$4 + 5\sin x = 4 + 5\frac{2t}{1+t^2}$$

$$= \frac{4t^2+10t+4}{1+t^2}$$

$$= \frac{2(2t^2+5t+2)}{1+t^2}$$

$$= \frac{2(2t+1)(t+2)}{1+t^2}$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{4+5\sin x}\,dx$$

$$= \int_0^1 \frac{1+t^2}{2(2t+1)(t+2)}\cdot\frac{2}{1+t^2}\,dt$$

$$= \int_0^1 \frac{1}{(2t+1)(t+2)}\,dt$$

$$= \tfrac{1}{3}\int_0^1 \left(\frac{2}{(2t+1)} - \frac{1}{(t+2)}\right) dt$$

$$= \tfrac{1}{3}\left[\ln\left(\frac{2t+1}{t+2}\right)\right]_0^1$$

$$= \tfrac{1}{3}\left(\ln 1 - \ln\tfrac{1}{2}\right)$$

$$= \tfrac{1}{3}\ln 2$$

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d. Outcomes assessed : E8

Criteria	Marks
i • applies integration by parts	1
• evaluates and simplifies to obtain recurrence formula	1
ii • applies recurrence formula to express I_3 in terms of I_0	1
• evaluates I_0 and hence I_3	1

Answer

i.

$$I_n = \int_1^e (1-\ln x)^n \, dx$$

$$= \left[x(1-\ln x)^n\right]_1^e - \int_1^e x \, . \, n\,(1-\ln x)^{n-1}\left(\tfrac{-1}{x}\right)dx$$

$$= -1 + n\int_1^e (1-\ln x)^{n-1}\,dx$$

$$\therefore I_n = -1 + n\,I_{n-1}, \quad n = 1, 2, 3, \ldots$$

ii.

$$I_3 = -1 + 3I_2$$

$$= -1 + 3(-1 + 2I_1)$$

$$= -4 + 6I_1$$

$$= -4 + 6(-1 + I_0)$$

$$= -10 + 6I_0$$

But $I_0 = \int_1^e 1\,dx = e - 1$

$$\therefore I_3 = 6e - 16$$

Question 3

a. Outcomes assessed : E3

Criteria	Marks
• writes equation in terms of a and b	1
• equates real and imaginary parts to find b and a quadratic equation for a	1
• solves this quadratic equation to find two values for z	1

Answer

$z = a + i\,b$

$$|\,z\,|^2 + 5\bar{z} + 10i = 0$$

$$a^2 + b^2 + 5(a - i\,b) + 10i = 0$$

$$a^2 + b^2 + 5a + 5i\,(2-b) = 0$$

Equating real and imaginary parts

$b = 2$ and $a^2 + 5a + b^2 = 0$

$$(a+4)(a+1) = 0$$

$$a = -4,\ -1$$

$$\therefore z = -4 + 2i \text{ or } z = -1 + 2i$$

b. Outcomes assessed : E3

Criteria	Marks
i • expresses both complex numbers in modulus/argument form	1
• expresses the quotient in modulus/argument form	1
ii • realises $\frac{7n\pi}{12}$ is an odd multiple of $\frac{\pi}{2}$ and deduces smallest positive integer n is 6	1
• evaluates quotient in form bi	1

Answer

i. $z_1 = 2\left(\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}\right) \quad z_2 = \sqrt{2}\left(\tfrac{1}{\sqrt{2}} - i\tfrac{1}{\sqrt{2}}\right)$

$$z_1 = 2\left(\cos\tfrac{\pi}{3} + i\sin\tfrac{\pi}{3}\right)$$

$$z_2 = \sqrt{2}\left(\cos\left(-\tfrac{\pi}{4}\right) + i\sin\left(-\tfrac{\pi}{4}\right)\right)$$

$$\left|\frac{z_1}{z_2}\right| = \frac{2}{\sqrt{2}} = \sqrt{2}, \quad \arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{3} - \left(-\frac{\pi}{4}\right) = \frac{7\pi}{12}$$

$$\therefore \frac{z_1}{z_2} = \sqrt{2}\left(\cos\tfrac{7\pi}{12} + i\sin\tfrac{7\pi}{12}\right)$$

ii. $\arg\left(\dfrac{z_1^{\,n}}{z_2^{\,n}}\right) = \dfrac{7n\pi}{12}$. $\quad \therefore \dfrac{z_1^{\,n}}{z_2^{\,n}}$ is imaginary if

$$\frac{7n\pi}{12} = \frac{m\pi}{2}, \quad m = \pm 1, \pm 3, \pm 5, \ldots$$

The smallest such positive integer n is 6.

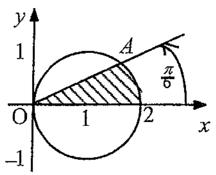
$$\arg\left(\frac{z_1^{\,6}}{z_2^{\,6}}\right) = \frac{7\pi}{2} = 4\pi - \frac{\pi}{2}, \quad \left|\frac{z_1^{\,6}}{z_2^{\,6}}\right| = (\sqrt{2})^6 = 8$$

$$\therefore \frac{z_1^{\,6}}{z_2^{\,6}} = 8\left(\cos\left(-\tfrac{\pi}{2}\right) + i\sin\left(-\tfrac{\pi}{2}\right)\right) = -8i$$

c. Outcomes assessed : E3

Criteria	Marks
i • realises that the region lies inside the circle with radius 1 centred at $(1, 0)$	1
• shades the part of this region which lies between two appropriate rays from the origin	1
ii • uses circle property to find length of boundary arc	1
• finds the length of the chord and hence the perimeter.	1

Answer



ii. Boundary arc subtends angle $2\times\frac{\pi}{6}=\frac{\pi}{3}$ at the centre of the circle, and hence has length $\frac{\pi}{3}$.

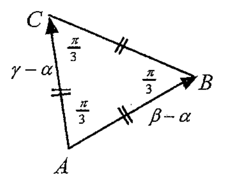
$OA^2 = 1^2 + 1^2 - 2\cos\frac{2\pi}{3} = 3$

Hence perimeter is $2+\frac{\pi}{3}+\sqrt{3}$

. Outcomes assessed : E3

Criteria	Marks
i • relates the complex numbers $\gamma-\alpha$, $\beta-\alpha$ to the vectors $\overrightarrow{AC}$, $\overrightarrow{AB}$	1
• uses the properties of an equilateral triangle to deduce the result	1
ii • writes a similar result considering a second pair of triangle sides as vectors	1
• uses both results to obtain required equality	1

nswer



ABC is equilateral. Hence $AC = AB$
nd the angle between these sides is $\frac{\pi}{3}$.
$\overrightarrow{AC}$ is the rotation of $\overrightarrow{AB}$ anticlockwise by $\frac{\pi}{3}$.
$\gamma-\alpha=\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)(\beta-\alpha)$

ii. Similarly $\overrightarrow{CB}$ is the rotation of $\overrightarrow{CA}$ anticlockwise by $\frac{\pi}{3}$.

$\therefore \beta-\gamma=\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)(\alpha-\gamma)$

Hence $\dfrac{\gamma-\alpha}{\beta-\alpha}=\dfrac{\beta-\gamma}{\alpha-\gamma}=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$

$\therefore (\gamma-\alpha)(\alpha-\gamma)=(\beta-\gamma)(\beta-\alpha)$

$-\alpha^2-\gamma^2+2\gamma\alpha=\beta^2-\alpha\beta-\beta\gamma+\gamma\alpha$

Hence $\alpha^2+\beta^2+\gamma^2=\alpha\beta+\beta\gamma+\gamma\alpha$

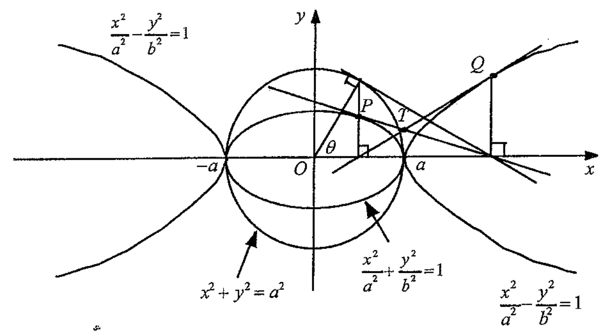
Question 4

a. Outcomes assessed : E3, E4

Criteria	Marks
i • sketches auxiliary circle and ellipse, showing how θ determines position of P	1
• sketches hyperbola, showing how θ determines position of Q	1
ii • finds gradient of tangent by differentiation	1
• finds equation of tangent	1
• finds x intercept of tangent	1
iii • solves equations simultaneously to show $x=a$ at T	1
• finds the y coordinate of T in the required form	2
• shows the tangents on the sketch with x intercepts and T correctly positioned	1
iv • sketches second diagram	1

Answer

i.



ii.

$y = b\sin\theta \Rightarrow \dfrac{dy}{d\theta} = b\cos\theta$

$x = a\cos\theta \Rightarrow \dfrac{dx}{d\theta} = -a\sin\theta$

$\therefore \dfrac{dy}{dx} = \dfrac{dy}{d\theta} \div \dfrac{dx}{d\theta} = -\dfrac{b\cos\theta}{a\sin\theta}$ is gradient of tangent at P.

Hence tangent to ellipse at P has equation $\quad y - b\sin\theta = -\dfrac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

Rearrangement gives $\quad x\,b\cos\theta + y\,a\sin\theta = ab(\cos^2\theta + \sin^2\theta) = ab$

Hence equation of tangent is $\quad \dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$

$y = 0 \Rightarrow \dfrac{x\cos\theta}{a} = 1 \therefore x = a\sec\theta$

Hence tangent cuts x axis vertically below Q.

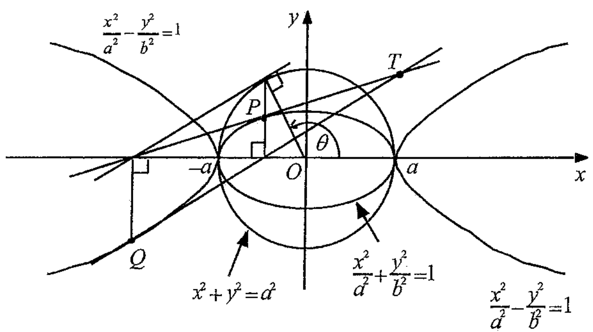
iii. $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1 \qquad (1)$

$\dfrac{x\sec\theta}{a} - \dfrac{y\tan\theta}{b} = 1 \qquad (2)$

$(1) \times \sec\theta + (2) \Rightarrow \dfrac{x}{a}(1 + \sec\theta) + 0 = \sec\theta + 1 \qquad \therefore x = a$

Sub. for x in (1): $\quad y = b\left(\dfrac{1-\cos\theta}{\sin\theta}\right) = b\left(\dfrac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right) \qquad \therefore y = b\tan\frac{\theta}{2}$

iv.



b. Outcomes assessed : E3, E4

	Criteria	Marks
i	• finds the gradient of the normal at P	1
	• equates this to the gradient of PQ and rearranges	1
ii	• writes the coordinates of M in terms of p and q, then substitutes for q and rearranges	1
iii	• finds expression for p^2 in terms of x and y	1
	• substitutes for p^2 in expression for y^2 then rearranges to get required equation	1

Answer

i. $y = \dfrac{4}{x} \Rightarrow \dfrac{dy}{dx} = \dfrac{-4}{x^2} = -\dfrac{1}{p^2}$ at P.

Hence normal at P has gradient p^2.

$\therefore$ PQ has gradient p^2.

$\dfrac{\frac{2}{q} - \frac{2}{p}}{2q - 2p} = p^2$

$\dfrac{p-q}{pq} = p^2(q-p)$

$\therefore \quad q = -\dfrac{1}{p^3}$

ii. At M, $\quad x = p + q = p - \frac{1}{p^3} = \frac{1}{p}\left(p^2 - \frac{1}{p^2}\right)$

$y = \frac{1}{p} + \frac{1}{q} = \frac{1}{p} - p^3 = p\left(\frac{1}{p^2} - p^2\right)$

iii. At M, $\dfrac{y}{x} = -p^2$

Hence $\quad y^2 = -\dfrac{y}{x}\left(-\dfrac{x}{y} + \dfrac{y}{x}\right)^2$

$xy = -\left(\dfrac{y^2 - x^2}{xy}\right)^2$

$-xy = \dfrac{(y^2 - x^2)^2}{x^2y^2}$

$\therefore (x^2 - y^2)^2 = -x^3y^3$

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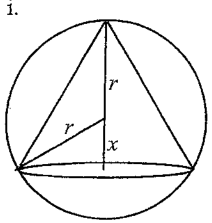
Question 5

a. Outcomes assessed : H5

Criteria	Marks
i • finds radius of base of cone	1
• finds volume of cone and rearranges into required form	1
ii • differentiates with respect to x and solves $\frac{dV}{dx}=0$	1
• verifies this stationary value is a maximum	1
iii • finds the maximum volume in terms of r	1
• finds the ratio in simplest terms	1

Answer

i.



The base of the cone has radius R where $R^2=r^2-x^2$.

$\therefore V=\frac{1}{3}\pi\left(r^2-x^2\right)\left(r+x\right)$

$=\frac{\pi}{3}\left(r^3+r^2x-rx^2-x^3\right)$

ii.

$\frac{dV}{dx}=\frac{\pi}{3}\left(r^2-2rx-3x^2\right)$

$=\frac{\pi}{3}\left(r-3x\right)\left(r+x\right)$

$\frac{dV}{dx}=0 \Rightarrow x=\frac{1}{3}r$

(since $x>0$ and $r>0$)

Then

$\frac{d^2V}{dx^2}=\frac{\pi}{3}\left(-2r-6x\right)=-\frac{4\pi}{3}r<0$

Hence maximum V for $x=\frac{1}{3}r$

iii.

$V_{\max}=\frac{\pi}{3}\left(r^2-\frac{1}{9}r^2\right)\left(r+\frac{1}{3}r\right)$

$=\frac{32\pi}{81}r^3$

$V_{sphere}=\frac{4\pi}{3}r^3$

$V_{\max}:V_{sphere}=8:27$

b. Outcomes assessed : HE2

Criteria	Marks
i • shows $f(x)$ is stationary at $x=0$ and monotonic increasing for $x\ge 0$	1
• uses $f(0)=1$ to deduce $f(x)>0$ for $x\ge 0$	1
ii • defines an appropriate sequence of statements $S(n)$, using (i) to establish the truth of $S(1)$	1
• writes an inequality involving definite integrals conditional on the truth of $S(k)$	1
• finds the primitives and evaluates to establish that if $S(k)$ is true then $S(k+1)$ is true	1

Answer

i. $f(x)=e^x-x$

$f'(x)=e^x-1$

$f'(0)=0,\ f'(x)>0$ for $x>0$

Function f is stationary at $x=0$ and monotonic increasing for $x>0$. But $f(0)=1$. $\therefore f(x)\ge 1$ for $x\ge 0$.

$\therefore f(x)>0$, and hence $e^x>x$, for $x\ge 0$.

ii. Let $S(n)$, $n=1,2,3,\ldots$ be the sequence of statements $e^x>\frac{x^n}{n!}$ for $x\ge 0$

Clearly $S(1)$ is true since $e^x>\frac{x^1}{1!}$ using (i).

If $S(k)$ is true $e^x>\frac{x^k}{k!}$ for $x\ge 0$

Then for $x\ge 0$, $\int_0^x e^t\,dt \ge \int_0^x \frac{t^k}{k!}\,dt$

$$\left[e^t\right]_0^x \ge \left[\frac{t^{k+1}}{(k+1)!}\right]_0^x$$

$$e^x-1\ge \frac{x^{k+1}}{(k+1)!}$$

$$e^x\ge \frac{x^{k+1}}{(k+1)!}+1$$

$$e^x>\frac{x^{k+1}}{(k+1)!}$$

Hence if $S(k)$ is true, then $S(k+1)$ is true. But $S(1)$ is true, hence $S(2)$ is true, and then $S(3)$ is true and so on. Hence $S(n)$ is true for all positive integers $n\ge 1$.

c. Outcomes assessed : PE3

Criteria	Marks
i • expresses S_2 in terms of a and b	1
• uses the given values of S_1 and S_2 to find values of a and b	1
ii • writes an equation for c in terms of S_1, S_2, S_3	1
• uses the given values of these sums to evaluate c	1

Answer

i. $P(x)=x^3+ax^2+bx+c$

$S_n=\alpha^n+\beta^n+\gamma^n$

$\therefore a=-S_1=-3$

$S_2=\left(\alpha+\beta+\gamma\right)^2-2\left(\alpha\beta+\beta\gamma+\gamma\alpha\right)$

$=\left(-a\right)^2-2b$

$\therefore 7=9-2b$

$\therefore b=1$

ii. $P(\alpha)=P(\beta)=P(\gamma)=0$

$\alpha^3-3\alpha^2+\alpha+c=0$

$\beta^3-3\beta^2+\beta+c=0$

$\gamma^3-3\gamma^2+\gamma+c=0$

$\therefore\ S_3-3S_2+S_1+3c=0$

$3-\ 21\ +\ 3+3c=0$

$\therefore c=5$

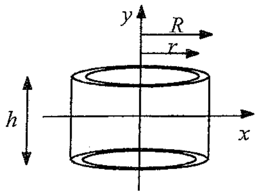
Question 6

a. Outcomes assessed : HE6, E7

Criteria	Marks
i • finds an expression for the volume of a typical cylindrical shell	1
• takes the limiting sum of such shell volumes to express V as the integral of a function of x	1
ii • makes an appropriate substitution	1
• recognises that the integral of an odd function between -1 and 1 is zero	1
• recognises that the remaining integral is given by the area of a semi circle of radius 1	1
• evaluates V, giving an exact value	1

Answer

i.



$R = x + \delta x$

$r = x$

$h = 2y$

Cylindrical shell has volume

$\delta V = \pi\left(R^2 - r^2\right)h$

$= \pi h\left(R + r\right)\left(R - r\right)$

$= 2\pi y\left(2x + \delta x\right)\delta x$

Ignoring second order terms in $(\delta x)^2$:

$V = \lim_{\delta x \to 0} \sum_{x=1}^{x=3} 4\pi xy\, \delta x$

$= \int_1^3 4\pi xy\, dx$

$= 4\pi \int_1^3 x \,.\, 2\sqrt{1-(x-2)^2}\, dx$

$= 8\pi \int_1^3 x\sqrt{1-(x-2)^2}\, dx$

ii.

$u = x - 2$

$du = dx$

$x = 1 \Rightarrow u = -1$

$x = 3 \Rightarrow u = 1$

$V = 8\pi \int_{-1}^{1} (u+2)\sqrt{1-u^2}\; du$

$= 8\pi \int_{-1}^{1} u\sqrt{1-u^2}\; du + 16\pi \int_{-1}^{1} \sqrt{1-u^2}\; du$

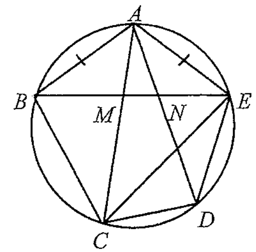
But $f(u) = u\sqrt{1-u^2}$ is an odd function, and $\int_{-1}^{1}\sqrt{1-u^2}\; du$ is the area of a semi circle of radius 1.

$\therefore V = 0 + 16\pi\left(\tfrac{1}{2}\pi \,.\, 1^2\right) = 8\pi^2$

b. Outcomes assessed : PE3

Criteria	Marks
i • explains why $\angle BEA = \angle ABE$	1
• explains why $\angle ABE = \angle ACE$	1
ii • explains why $\angle EAD = \angle ECD$	1
• explains why $\angle BEA + \angle EAD = \angle END$	1
• uses these deductions with the result from i. to apply a test for a cyclic quadrilateral	1

Answer



i. $AB = AE$ *(given)*

$\therefore \angle BEA = \angle ABE$ *($\angle$'s opp. equal sides are equal in ΔABE)*

But $\angle ABE = \angle ACE$ *($\angle$'s subtended at the circumference by same arc AE are equal)*

$\therefore \angle BEA = \angle ACE$

ii. $\angle EAD = \angle ECD$ *($\angle$'s subtended at the circumference by same arc ED are equal)*

$\therefore \angle BEA + \angle EAD = \angle ACE + \angle ECD$

But $\angle BEA + \angle EAD = \angle END$ *(Exterior $\angle$ is sum of interior opp. $\angle$'s in ΔAEN)*

and $\angle ACE + \angle ECD = \angle ACD$ *(By addition of adjacent $\angle$'s)*

$\therefore \angle END = \angle ACD$

$\therefore CDNM$ is a cyclic quadrilateral (*Exterior $\angle$ equal to interior opp. $\angle$*)

c. Outcomes assessed : PE3

Criteria	Marks
i • shows the roots come in pairs of opposites	1
• uses the product of the roots to write the other roots in terms of α and deduce they are real	1
ii • solves the equation as a quadratic in x^2	1
• uses the fact that b and all the roots are real to find the possible values for b	1

Answer

i. $P(-\alpha) = \alpha^4 + b\,\alpha^2 + 1 = P(\alpha)$

Hence if α is a root, then $-\alpha$ is also a root.

Let the roots be $\alpha, -\alpha, \beta, -\beta$.

Then considering the product of the roots, $\alpha^2\beta^2 = 1 \Rightarrow \beta = \pm\frac{1}{\alpha}$

Hence roots are $\alpha, -\alpha, \frac{1}{\alpha}, -\frac{1}{\alpha}$, and all four roots are real since α is real.

ii. Considering $x^4 + bx^2 + 1 = 0$ as a quadratic equation in x^2, solutions are $x^2 = \dfrac{-b \pm \sqrt{b^2-4}}{2}$

Hence the roots of this equation are $\pm\sqrt{\dfrac{-b+\sqrt{b^2-4}}{2}}$, $\pm\sqrt{\dfrac{-b-\sqrt{b^2-4}}{2}}$.

Since b is real, and all four roots are non zero real numbers,

$b^2 \ge 4$ *and* $-b - \sqrt{b^2-4} > 0$

$|b| \ge 2$ *and* $b < -\sqrt{b^2-4} < 0$

$\therefore b \le -2$

Question 7

a. Outcomes assessed : E3

Criteria	Marks
i • describes the position of the roots on an Argand diagram, referring to DeMoivre	1
• lists the roots and relates the non-real roots to powers of ω	1
ii • writes $z^6+z^5+z^4+z^3+z^2+z+1$ as a factor of z^7-1 to deduce required result	1
iii • realises the sum of the roots, grouped into complex conjugate pairs, is -1	1
• simplifies the sum of each complex conjugate pair to evaluate sum of cosines	1
iv • shows the product of the stated roots is 2	1
• writes the sum of the stated roots and hence finds the required monic quadratic equation	1

Answer

i. $z^7=1 \Rightarrow |z|=1$. Clearly one root of $z^7=1$ is 1. Using DeMoivre's theorem, the seven roots of $z^7=1$ are equally spaced by $\frac{2\pi}{7}$ around the unit circle in the Argand diagram. Hence the roots of $z^7-1=0$ are

$$1,\quad \cos\tfrac{2\pi}{7}+i\sin\tfrac{2\pi}{7}=\omega,\quad \cos\tfrac{4\pi}{7}+i\sin\tfrac{4\pi}{7}=\omega^2,\quad \cos\tfrac{6\pi}{7}+i\sin\tfrac{6\pi}{7}=\omega^3,$$
$$\cos\tfrac{12\pi}{7}+i\sin\tfrac{12\pi}{7}=\omega^6,\quad \cos\tfrac{10\pi}{7}+i\sin\tfrac{10\pi}{7}=\omega^5,\quad \cos\tfrac{8\pi}{7}+i\sin\tfrac{8\pi}{7}=\omega^4$$

ii. $z^7-1=(z-1)(z^6+z^5+z^4+z^3+z^2+z+1)$. Hence $\omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6$ are the roots of $z^6+z^5+z^4+z^3+z^2+z+1=0$.

iii. $\omega^6=\cos\left(-\frac{2\pi}{7}\right)+i\sin\left(-\frac{2\pi}{7}\right)$, $\omega^5=\cos\left(-\frac{4\pi}{7}\right)+i\sin\left(-\frac{4\pi}{7}\right)$, $\omega^4=\cos\left(-\frac{6\pi}{7}\right)+i\sin\left(-\frac{6\pi}{7}\right)$

But $(\omega+\omega^6)+(\omega^2+\omega^5)+(\omega^3+\omega^4)=-1$

Hence $2\cos\frac{2\pi}{7}+2\cos\frac{4\pi}{7}+2\cos\frac{6\pi}{7}=-1$

$\therefore \cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}=-\frac{1}{2}$

iv. Let $\alpha=\omega+\omega^2+\omega^4$, $\beta=\omega^3+\omega^5+\omega^6$

Then $\alpha+\beta=-1$

$$\alpha\beta=\omega^4+\omega^6+\omega^7+\omega^5+\omega^7+\omega^8+\omega^7+\omega^9+\omega^{10}$$
$$=\omega^4+\omega^6+1+\omega^5+1+\omega+1+\omega^2+\omega^3$$
$$=2$$

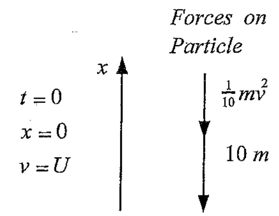
Hence α, β are roots of $z^2+z+2=0$

b. Outcomes assessed : E5

Criteria	Marks
i • identifies forces on particle and uses Newton's second law to find acceleration	1
ii • selects appropriate derivative for a then integrates to find x as a function of v	1
• uses initial conditions to evaluate constant of integration	1
• substitutes $v=0$ to find H in terms of U	1
iii • identifies forces on particle and uses Newton's second law to find acceleration	1
iv • selects appropriate derivative for a then integrates to find x as a function of v	1
• uses initial conditions to evaluate constant of integration	1
• substitutes $x=H$ and expression for H to find V as a function of U	1

Answer

i.



By Newton's second law :

$$ma=-\left(10m+\tfrac{1}{10}mv^2\right)$$
$$\therefore a=-\tfrac{1}{10}\left(100+v^2\right)$$

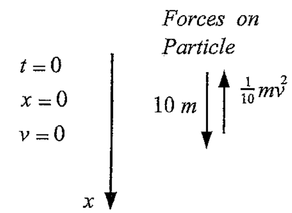
ii. $\frac{1}{2}\frac{dv^2}{dx}=-\frac{1}{10}\left(100+v^2\right)$

$$\frac{dx}{d(v^2)}=-5\frac{1}{100+(v^2)}$$
$$x=-5\ln\left[\left(100+v^2\right)A\right],\quad A \text{ constant}$$
$$\left.\begin{array}{l}x=0\\ v=U\end{array}\right\}\Rightarrow A=\frac{1}{100+U^2}$$
$$\therefore x=5\ln\left(\frac{100+U^2}{100+v^2}\right)$$

At maximum height: $v=0$, $x=H$

$$\therefore H=5\ln\left(\frac{U^2+100}{100}\right)$$

iii.



By Newton's second law :

$$ma=10m-\tfrac{1}{10}mv^2$$
$$\therefore a=\tfrac{1}{10}\left(100-v^2\right)$$

iv. $\frac{1}{2}\frac{dv^2}{dx}=\frac{1}{10}\left(100-v^2\right)$

$$\frac{dx}{d\left(v^2\right)}=5\,\frac{1}{100-\left(v^2\right)}$$

$$x=-5\ln\left[\left(100-v^2\right)B\right], \qquad B \text{ constant}$$

$$\left.\begin{array}{l}x=0\\ v=0\end{array}\right\}\Rightarrow B=\frac{1}{100}$$

$$\therefore x=5\ln\left(\frac{100}{100-v^2}\right)$$

$\therefore$ returns to point of projection with speed V if

$$5\ln\left(\frac{U^2+100}{100}\right)=5\ln\left(\frac{100}{100-V^2}\right)$$

$$\frac{100-V^2}{100}=\frac{100}{U^2+100}$$

$$1-\frac{100-V^2}{100}=1-\frac{100}{U^2+100}$$

$$\frac{V^2}{100}=\frac{U^2}{U^2+100}$$

$$\therefore V=\frac{10U}{\sqrt{U^2+100}}$$

Question 8

a. Outcomes assessed : PE3

Criteria	Marks
i • expands square and regroups terms to obtain required form	1
ii • deduces that the quadratic expression in real x has a negative discriminant	1
• uses this result to obtain required inequality	1
iii • selects $a_k=2k-1, \quad k=1,2,\dots,n$ and deduces first inequality from (ii)	1
• replaces a_k by $a_k^{\,2}$ to deduce second inequality from (ii)	1

Answer

i. $f(x)=\sum_{k=1}^{n}\left(a_k x-1\right)^2$

$$=\sum_{k=1}^{n}\left(a_k^{\,2}x^2-2a_k\,x+1\right)$$

$$=\left(\sum_{k=1}^{n}a_k^{\,2}\right)x^2-2\left(\sum_{k=1}^{n}a_k\right)x+n$$

ii. Since a_k is real, $k=1,2,\dots,n$

$f(x)\ge 0$ for all real x

The quadratic function $f(x)$ has $\Delta\le 0$.

$$\therefore 4\left(\sum_{k=1}^{n}a_k\right)^2-4n\sum_{k=1}^{n}a_k^{\,2}\le 0$$

$$4n\sum_{k=1}^{n}a_k^{\,2}\ge 4\left(\sum_{k=1}^{n}a_k\right)^2$$

$$\sum_{k=1}^{n}a_k^{\,2}\ge \frac{1}{n}\left(\sum_{k=1}^{n}a_k\right)^2$$

iii. Let $a_k=2k-1, \quad k=1,2,\dots,n$

Then $\sum_{k=1}^{n}a_k=\frac{n}{2}\left[1+(2n-1)\right]=n^2$

$$\therefore \frac{1}{n}\left(\sum_{k=1}^{n}a_k\right)^2=n^3$$

$$\therefore 1^2+3^2+\dots+(2n-1)^2\ge n^3$$

Also

$$\sum_{k=1}^{n}\left(a_k^{\,2}\right)^2\ge\frac{1}{n}\left(\sum_{k=1}^{n}a_k^{\,2}\right)^2\ge\frac{1}{n}\left\{\frac{1}{n}\left(\sum_{k=1}^{n}a_k\right)^2\right\}^2$$

$$\therefore \sum_{k=1}^{n}a_k^{\,4}\ge\frac{1}{n}\left(n^3\right)^2$$

$$\therefore 1^4+3^4+\dots+(2n-1)^4\ge n^5$$

b. Outcomes assessed : H5

Criteria	Marks
i • explains why the probability A wins a turn is $\frac{5}{12}$	1
ii • explains why $p_2+q_2=1$	1
iii • explains why $p_n+q_n=q_{n-1}$ for $n\ge 3$	1
• takes the sum of a sequence of such equations $p_k+q_k=q_{k-1}$, $k=2,3,4,\dots,n$	1
• simplifies this to obtain required result	1
iv • finds probability of possible outcomes if the game does not end in n or fewer turns	1
• adds all such probabilities to find q_n	1
• finds p_n	1
v • writes this probability as a limiting sum of p_k, $k=2,3,\dots$	1
• deduces this is the limiting value of q_n as $n\to\infty$ and justifies a limiting value of 0	1

Answer

i. $P(draw)=\frac{6}{36}=\frac{1}{6}$

$\therefore P(A\ wins)=P(B\ wins)=\frac{1}{2}\left(1-\frac{1}{6}\right)=\frac{5}{12}$

ii. The game cannot end on the first turn.

$$q_2=P\left(\text{game does not end on } 1^{st} \text{ nor } 2^{nd} \text{ turn}\right)$$

$$=1-P\left(\text{game ends on } 1^{st} \text{ or } 2^{nd} \text{ turn}\right)$$

$$=1-P\left(\text{game ends on } 2^{nd} \text{ turn}\right)$$

$$=1-p_2$$

$$\therefore p_2+q_2=1$$

iii. Consider the sample space of possible outcomes for the first n turns, $n\ge 3$.

The event E defined as

E : *the game does not end in* $n-1$ *or fewer turns*

is the union of the mutually exclusive events

E_1 : *the game ends on the* n^{th} *turn*

E_2 : *the game does not end in* n *or fewer turns*

$$\therefore P(E_1)+P(E_2)=P(E)\Rightarrow p_n+q_n=q_{n-1}, \quad n\ge 3$$

Then

$$p_2+q_2=1$$
$$p_3+q_3=q_2$$
$$p_4+q_4=q_3$$
$$\dots$$
$$p_n+q_n=q_{n-1}$$

$$\therefore \sum_{k=2}^{n}p_k+q_n=1 \qquad \therefore \sum_{k=2}^{n}p_k=1-q_n, \quad n\ge 2$$

iv. If the game does not end in n or fewer turns, consider the outcomes for the first n turns :

Wins for A	*Wins for B*	*draws*	*probability*
0	0	n	$\left(\frac{1}{6}\right)^n$
1	0	$n-1$	$n\left(\frac{5}{12}\right)\left(\frac{1}{6}\right)^{n-1}$
0	1	$n-1$	$n\left(\frac{5}{12}\right)\left(\frac{1}{6}\right)^{n-1}$
1	1	$n-2$	$n(n-1)\left(\frac{5}{12}\right)^2\left(\frac{1}{6}\right)^{n-2}$

Hence

$$q_n = \frac{1}{6^n}\left(1+\frac{5n}{2}+\frac{5n}{2}+\frac{25n(n-1)}{4}\right)$$
$$=\frac{25n^2-5n+4}{4\times 6^n}$$

Then $p_n = q_{n-1} - q_n$

$$4\times 6^n\, p_n = 6\left\{25(n-1)^2 - 5(n-1)+4\right\} - \left(25n^2-5n+4\right)$$
$$=125n^2-325n+200$$
$$=25\left(5n^2-13n+8\right)$$
$$\therefore p_n = \frac{25(n-1)(5n-8)}{4\times 6^n}$$

v. $$P(\textit{game never ends}) = 1-\lim_{n\to\infty}\sum_{k=2}^{n} p_k$$
$$=1-\lim_{n\to\infty}(1-q_n)$$
$$=1-1+\lim_{n\to\infty} q_n$$
$$=\lim_{n\to\infty}\frac{25n^2-5n+4}{4\times 6^n}$$
$$=\frac{25}{4}\lim_{n\to\infty}\frac{n^2}{6^n} - \frac{5}{4}\lim_{n\to\infty}\frac{n}{6^n} + \lim_{n\to\infty}\frac{1}{6^n}$$
$$=\frac{25}{4}\lim_{n\to\infty}\frac{n^2}{6^n} - \frac{5}{4}\lim_{n\to\infty}\frac{n}{6^n} + 0$$

Considering the behaviour of the familiar graphs $y = xe^{-x}$ and $y = x^2e^{-x}$:

$x\left(\frac{1}{6}\right)^x < x\,e^{-x} \to 0 \;\; as \;\; x\to+\infty$ and $x^2\left(\frac{1}{6}\right)^x < x^2\,e^{-x} \to 0 \;\; as \;\; x\to+\infty$

Hence $\lim_{n\to\infty}\frac{n}{6^n} = \lim_{n\to\infty}\frac{n^2}{6^n} = 0$ $\quad\therefore P(\textit{game never ends}) = 0$