Centre Number

Student Number

CATHOLIC SECONDARY SCHOOLS ASSOCIATION OF NEW SOUTH WALES

2004
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics
Extension 2

Morning Session
Monday 9 August 2004

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided separately
- All necessary working should be shown in every question

Total marks **(120)**
- Attempt Questions 1 – 8
- All questions are of equal value

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Question 1 **Begin a new page** **Marks**

(a) Consider the function $f(x) = x(x-3)^2$.

(i) Sketch the graph of the curve $y = f(x)$ showing clearly the coordinates and nature of any turning points and the intercepts on the x and y axes. Find the set of possible values of the real number k such that the equation $f(x) = k$ has three real, distinct solutions. 4

(ii) On separate axes, sketch the graphs of the following curves, showing clearly the coordinates and nature of any turning points and the equations of any asymptotes: 5

$y = \{f(x)\}^2 \qquad y = \dfrac{1}{f(x)} \qquad y^2 = f(x)$

(b) A curve is given parametrically in terms of the real number t by the equations
$x = \dfrac{3t}{1+t^3}$ and $y = \dfrac{3t^2}{1+t^3}$.

(i) Express t in terms of x and y. Hence show that the curve has Cartesian equation $x^3 + y^3 = 3xy$. Deduce that the curve is symmetrical about the line $y = x$. 3

(ii) Show that $\dfrac{dy}{dx} = \dfrac{y - x^2}{y^2 - x}$. Hence show that the curve has a horizontal tangent when $x = \sqrt[3]{2}$. 3
Write down the coordinates of a point on the curve where the tangent is vertical.

Question 2 **Begin a new page**

(a) (i) Find $\int (\cos x + \sin x)^2 \, dx$. 2

(ii) Find $\int \dfrac{1}{1-x^2} \, dx$. 2

(b) Use the substitution $u = e^x - 1$ to find $\int \dfrac{e^{2x}}{(e^x - 1)^2} \, dx$. 3

(c) (i) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \dfrac{1}{1+\sin x} \, dx$. 3

(ii) Hence find the value of $\int_0^{\frac{\pi}{2}} \dfrac{\sin x}{1+\sin x} \, dx$. 1

(d) (i) If $I_n = \int_1^e x^3 (\ln x)^n \, dx$ for $n = 0,1,2,\ldots$, show that $I_n = \dfrac{e^4}{4} - \dfrac{n}{4} I_{n-1}$ for $n = 1,2,3,\ldots$. 2

(ii) Hence find the value of $\int_1^e x^3 (\ln x)^2 \, dx$. 2

Question 3 **Begin a new page** **Marks**

(a) Solve the equation $|z|^2 + 2i\bar{z} = 4i + 7$, expressing any answers in the form $z = a + ib$ where a and b are real. 4

(b) A, B and C are the angles of a triangle.
Show that $(\cos A + i\sin A)(\cos B + i\sin B) + (\cos C - i\sin C) = 0$. 3

(c) $z_1 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ and $z_2 = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ are two complex numbers.

(i) On an Argand diagram draw the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OS}$ to represent z_1, z_2 and $z_1 + z_2$ respectively. 1

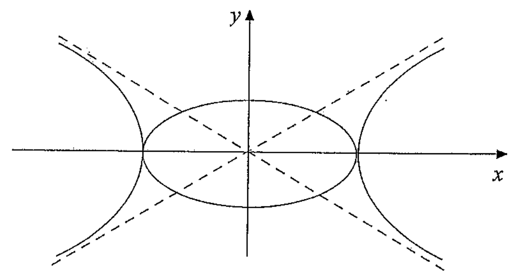
(ii) Hence express $z_1 + z_2$ in modulus / argument form. 2

(d) (i) On an Argand diagram shade the region where both $|z| \le 4$ and $|z-4| \le 4$. 2

(ii) Find the exact area of the shaded region. 3

Question 4 **Begin a new page**

(a)



$P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $a > b > 0$. The tangent to the ellipse at P passes through a focus of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ with eccentricity e.

(i) Show that the tangent to the ellipse at P has equation $bx\cos\theta + ay\sin\theta = ab$. 3

(ii) Show that P lies on a directrix of the hyperbola. 2

(iii) Show that the tangent to the ellipse at P has gradient ± 1. 3

(b) (i) Sketch the graph of the rectangular hyperbola $xy = 1$, showing clearly the coordinates of the foci and the equations of the directrices. 2

(ii) $P\left(p, \frac{1}{p}\right)$ and $Q\left(q, \frac{1}{q}\right)$ are two points on the rectangular hyperbola $xy = 1$.
Show that the chord PQ has equation $x + pqy - (p+q) = 0$. 2

(iii) If O is the origin, show that ΔOPQ has area $\dfrac{|p^2 - q^2|}{2|pq|}$ square units. 3

Marks

Question 5 **Begin a new page**

(a) (i) Use the substitution $x = 10\sqrt{2}\sin\theta$ to show that $\int_{-10}^{10} \sqrt{200 - x^2}\, dx = 100 + 50\pi$, then use a geometrical argument to verify this result. **4**

(ii) A mould for a model railway tunnel is made by rotating the region bounded by the curve $y = \sqrt{200 - x^2}$ and the x axis between the lines $x = -10$ and $x = 10$ through $180°$ about the line $x = 100$ (where all measurements are in cm). Use the method of cylindrical shells to show that the volume V cm^3 of the tunnel is given by $V = \pi \int_{-10}^{10} (100 - x)\sqrt{200 - x^2}\, dx$. Hence find the volume of the tunnel in m^3 correct to 2 significant figures. **4**

(b) (i) Show that the roots of the equation $z^{10} = 1$ are given by $z = \cos\frac{r\pi}{5} + i\sin\frac{r\pi}{5}$, $r = 0, 1, 2, \ldots, 9$. **2**

(ii) Explain why the equation $\left(\frac{z-1}{z}\right)^{10} = 1$ has only nine roots. Show that the roots of $\left(\frac{z-1}{z}\right)^{10} = 1$ are given by $z = \frac{1}{2}\left(1 + i\cot\frac{r\pi}{10}\right)$, $r = 1, 2, 3, \ldots, 9$. **5**

Question 6 **Begin a new page**

(a) A particle of mass m is moving vertically in a resisting medium in which the resistance to motion has magnitude $\frac{1}{10}mv^2$ where the particle has speed v ms^{-1}. The acceleration due to gravity is g ms^{-2}.

(i) If the particle falls vertically downwards from rest, show that its acceleration a ms^{-2} is given by $a = g - \frac{1}{10}v^2$. Hence show that its terminal speed V ms^{-1} is given by $V = \sqrt{10g}$. **2**

(ii) If the particle is projected vertically upwards with speed $V\tan\alpha$ ms^{-1} (for some $0 < \alpha < \frac{\pi}{2}$), show that its acceleration a ms^{-2} is given by $a = -\left(g + \frac{1}{10}v^2\right)$. Hence show that it reaches a maximum height H metres given by $H = 5\ln\sec^2\alpha$, and that it returns to its point of projection with speed $V\sin\alpha$ ms^{-1}. **8**

(b) The equation $x^3 + 2x + 1 = 0$ has roots α, β and γ.

(i) Find the monic cubic equation with roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$. **2**

(ii) Find the monic cubic equation with roots $\frac{\beta+\gamma}{\alpha^2}$, $\frac{\gamma+\alpha}{\beta^2}$ and $\frac{\alpha+\beta}{\gamma^2}$. **3**

Marks

Question 7 **Begin a new page**

(a)(i) Newton's method is being used to find an approximation to the positive root $x = \sqrt{c}$ of the equation $x^2 - c = 0$. The initial approximation is $x = a$ (for some $a > 0$). The error in this initial approximation is $\varepsilon_0 = \left|a - \sqrt{c}\right|$. Show that the error ε_1 in the next approximation (obtained by one application of Newton's method) is given by $\varepsilon_1 = \frac{\varepsilon_0^{\,2}}{2a}$. **3**

(ii) Find the values of a (in terms of c) such that $\varepsilon_1 = \varepsilon_0$. **2**

(b) (i) If $4 - \tan\theta = 5\sin\theta\cos\theta$, show that $x = \tan\theta$ is a root of the equation $x^3 - 4x^2 + 6x - 4 = 0$. **2**

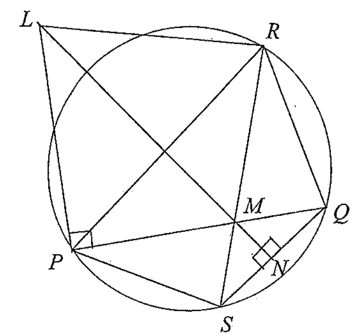
(ii) Solve the equation $4 - \tan\theta = 5\sin\theta\cos\theta$ for $0° \le \theta \le 360°$ giving answers correct to the nearest degree. **3**

(c) (i) By writing $n!$ as a product, show that $n! < (n+1)^n$ for all positive integers n. **2**

(ii) Hence show that $\sqrt[n]{n!} < \sqrt[n+1]{(n+1)!}$ for all positive integers n. **3**

Question 8 **Begin a new page**

PQ and RS are two chords of a circle which intersect at M inside the circle. MN is the perpendicular from M to SQ. L is the point on NM produced such that LP is perpendicular to PQ.



(i) Copy the diagram.

(ii) Show that $\Delta PML \,|||\, \Delta NMQ$ **2**

(iii) Hence show that $LR \perp RS$. **6**

(b) The number x and the real number θ are such that $x + \frac{1}{x} = 2\cos\theta$. Use Mathematical Induction to show that $x^n + \frac{1}{x^n} = 2\cos n\theta$ for all positive integers $n \ge 2$. **7**

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Mathematics Extension 2

Marking guidelines/ solutions

Please note: Mapping grid for this examination is on the last page of these Marking guidelines/solutions

2604 - 2

Question 1

(a) Outcomes Assessed: (i) E6 (ii) E6

Marking Guidelines

Criteria	Marks
i • correct shape and passing through origin	1
• minimum turning point at $(3,0)$	1
• maximum turning point at $(1,4)$	1
• inequality for k	1
ii • shape of $y=\{f(x)\}^2$ with turning points given	1
• shape of graph of $y=\dfrac{1}{f(x)}$; details of turning point and asymptotes	2
• shape of graph of $y^2=f(x)$ for $x<3$, with symmetry in x axis and turning points;	1
vertical tangent at the origin and nature of curve at $(3,0)$.	1

Answer

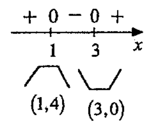
i. $f(x)=x(x-3)^2$

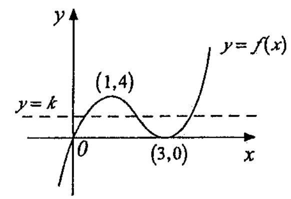
$f'(x)=1.(x-3)^2+x.2(x-3)$

$=3(x-3)(x-1)$

$f'(x)=0 \Rightarrow x=1,\ 3$

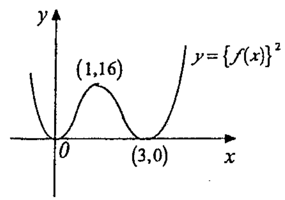
Sign of f':

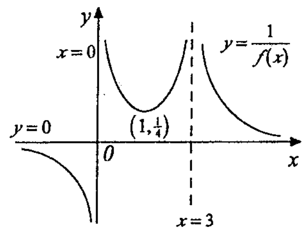


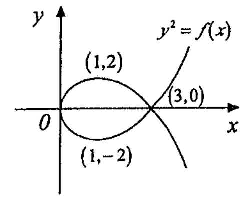


Solutions of $f(x)=k$ are x coordinates of intersection points of curve with horizontal line $y=k$. Hence three distinct real solutions for $\{k: 0<k<4\}$.

ii.







b. **Outcomes Assessed:** (i) E6 (ii) E6

Marking Guidelines

Criteria	Marks
i • finding t in terms of x and y	1
• obtaining the Cartesian equation	1
• deducing symmetry in $y=x$	1
ii • finding derivative in stated form	1
• solving simultaneous equations to find horizontal tangent at $\left(\sqrt[3]{2}, \sqrt[3]{4}\right)$	1
• using symmetry to write down coordinates of a point where tangent is vertical	1

Answer

i. $x=\dfrac{3t}{1+t^3}, \quad y=\dfrac{3t^2}{1+t^3} \Rightarrow \dfrac{y}{x}=t$

$$\therefore x\left\{1+\left(\frac{y}{x}\right)^3\right\}=3\frac{y}{x}$$

$$x+\frac{y^3}{x^2}=3\frac{y}{x}$$

$$x^3+y^3=3xy$$

Interchanging $x \leftrightarrow y$ leaves the equation unchanged, hence curve is symmetrical about the line $y=x$.

ii. $x^3+y^3=3xy$

$$3x^2+3y^2\frac{dy}{dx}=3y+3x\frac{dy}{dx}$$

$$\left(y^2-x\right)\frac{dy}{dx}=y-x^2$$

$$\frac{dy}{dx}=\frac{y-x^2}{y^2-x}$$

Tangent is horizontal when $\dfrac{dy}{dx}=0$, and hence when $y=x^2$ and $y^2 \neq x$.

$$\left.\begin{array}{l} y=x^2 \\ x^3+y^3=3xy \end{array}\right\} \Rightarrow \begin{array}{l} x^3+x^6=3x^3 \\ x^3\left(x^3-2\right)=0 \\ x=0,\ x=\sqrt[3]{2} \end{array}$$

$\therefore$ horizontal tangent at $\left(\sqrt[3]{2}, \sqrt[3]{4}\right)$.

Since the curve is symmetrical about $y=x$, the tangent to the curve will be vertical at the point $\left(\sqrt[3]{4}, \sqrt[3]{2}\right)$.

(At $(0,0)$ both $y^2-x=0$ and $y-x^2=0$, hence nature of tangent at $(0,0)$ is not clear. In fact, the curve has a loop which intersects itself at the origin, giving two tangent lines, one horizontal and one vertical)

Question 2

a. **Outcomes Assessed:** (i) H5 (ii) E8

Marking Guidelines

Criteria	Marks
i • simplification of integrand using trig. identities	1
• primitive function	1
ii • partial fraction decomposition	1
• primitive function	1

Answers

i $(\cos x+\sin x)^2=\cos^2 x+\sin^2 x+2\sin x\cos x$

$$\int(\cos x+\sin x)^2\,dx=\int(1+\sin 2x)\,dx$$
$$=x-\tfrac{1}{2}\cos 2x+c$$

ii $$\int\frac{1}{1-x^2}\,dx=\tfrac{1}{2}\int\frac{1}{1+x}+\frac{1}{1-x}\,dx$$
$$=\tfrac{1}{2}\{\ln|1+x|-\ln|1-x|\}+c$$
$$=\tfrac{1}{2}\ln\left|\frac{1+x}{1-x}\right|+c$$

b. **Outcomes Assessed:** HE6

Marking Guidelines

Criteria	Marks
• substitution to obtain integral in terms of u	1
• finding primitive function in terms of u	1
• writing primitive in terms of x	1

Answer

$u=e^x-1$

$du=e^x\,dx$

$$I=\int\frac{e^{2x}}{\left(e^x-1\right)^2}\,dx=\int\frac{e^x}{\left(e^x-1\right)^2}\cdot e^x\,dx$$
$$=\int\frac{u+1}{u^2}\,du$$
$$=\int\left\{\frac{1}{u}+\frac{1}{u^2}\right\}du$$

$$\therefore I=\ln|u|-\frac{1}{u}+c$$
$$=\ln\left|e^x-1\right|-\frac{1}{e^x-1}+c$$

c. **Outcomes Assessed:** (i) HE6 (ii) E8

Marking Guidelines

Criteria	Marks
i • converting dx to dt and x limits to t limits	1
• expressing integrand in terms of t in simplest form for dt integral	1
• evaluating integral	1
ii • value of integral	1

Answer

i.

$t=\tan\frac{x}{2}$

$dt=\frac{1}{2}\sec^2\frac{x}{2}\,dx$

$=\frac{1}{2}\left(1+t^2\right)dx$

$dx=\dfrac{2}{1+t^2}\,dt$

$x=0 \Rightarrow t=0$

$x=\frac{\pi}{2} \Rightarrow t=1$

$1+\sin x=1+\dfrac{2t}{1+t^2}$

$=\dfrac{(1+t)^2}{1+t^2}$

$$\int_0^{\frac{\pi}{2}}\frac{1}{1+\sin x}\,dx=\int_0^1\frac{1+t^2}{(1+t)^2}\cdot\frac{2}{1+t^2}\,dt$$
$$=-2\left[\frac{1}{1+t}\right]_0^1$$
$$=-2\left(\tfrac{1}{2}-1\right)$$
$$=1$$

ii. $$\int_0^{\frac{\pi}{2}}\frac{\sin x}{1+\sin x}\,dx=\int_0^{\frac{\pi}{2}}\left(1-\frac{1}{1+\sin x}\right)dx=[x]_0^{\frac{\pi}{2}}-1=\frac{\pi}{2}-1$$

d. Outcomes Assessed: (i) E8 (ii) E8

Marking Guidelines

Criteria	Marks
i • applying integration by parts	1
• evaluating and rearranging to obtain required result	1
ii • applying reduction formula to express I_2 in terms of I_0	1
• evaluating I_0 and hence I_2	1

Answer

i. For $n = 1, 2, 3, \ldots$

$$I_n = \int_1^e x^3 (\ln x)^n \, dx$$

$$= \tfrac{1}{4}\left[x^4 (\ln x)^n\right]_1^e - \int_1^e \tfrac{1}{4}x^4 . n(\ln x)^{n-1}\frac{1}{x}\, dx$$

$$= \tfrac{1}{4}(e^4 - 0) - \tfrac{n}{4}\int_1^e x^3 (\ln x)^{n-1}\, dx$$

$$= \frac{e^4}{4} - \frac{n}{4} I_{n-1}$$

ii. $I_0 = \int_1^e x^3 \, dx = \tfrac{1}{4}\left[x^4\right]_1^e = \dfrac{e^4}{4} - \dfrac{1}{4}$

$$I_2 = \frac{e^4}{4} - \tfrac{2}{4} I_1 = \frac{e^4}{4} - \tfrac{1}{2}\left(\frac{e^4}{4} - \tfrac{1}{4} I_0\right)$$

$$\therefore I_2 = \frac{e^4}{8} + \tfrac{1}{8} I_0 = \frac{e^4}{8} + \tfrac{1}{8}\left(\frac{e^4}{4} - \tfrac{1}{4}\right)$$

$$\therefore \int_1^e x^3 (\ln x)^2 \, dx = \frac{5e^4 - 1}{32}$$

Question 3

a. Outcomes Assessed: E3

Marking Guidelines

Criteria	Marks
• writing equation in terms of real a and b where $z = a + ib$	1
• equating real and imaginary parts to make simultaneous equations	1
• solving these equations to find pairs of values for a, b	1
• stating solutions for z	1

Answer

Let $z = a + ib$ where a and b are real.

$$|z|^2 + 2i\bar{z} = 4i + 7$$

$$a^2 + b^2 + 2i(a - ib) = 4i + 7$$

$$2ai + (a^2 + b^2 + 2b) = 4i + 7$$

Equating real and imaginary parts

$a = 2$ and $b^2 + 2b + 4 = 7$

$(b+3)(b-1) = 0$

$\therefore \ a = 2, \ b = -3 \Rightarrow z = 2 - 3i$

or $a = 2, \ b = 1 \ \Rightarrow z = 2 + i$

b. Outcomes Assessed: E3

Marking Guidelines

Criteria	Marks
• expansion of $(\cos A + i\sin A)(\cos B + i\sin B)$	1
• recognising expansions of cosine and sine of an angle sum	1
• expressing trig. ratios of $A + B$ in terms of trig. ratios of C	1

Answer

$A + B = \pi - C$

$$\therefore \begin{cases} \cos(A+B) = -\cos C \\ \sin(A+B) = \sin C \end{cases}$$

$$(\cos A + i\sin A)(\cos B + i\sin B)$$

$$= (\cos A\cos B - \sin A\sin B) + i(\sin A\cos B + \cos A\sin B)$$

$$= \cos(A+B) + i\sin(A+B)$$

$$= -(\cos C - i\sin C)$$

$$\therefore (\cos A + i\sin A)(\cos B + i\sin B) + (\cos C - i\sin C) = 0$$

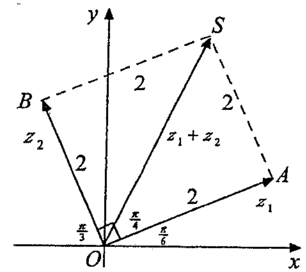
c. Outcomes Assessed: (i) E3 (ii) E3

Marking Guidelines

Criteria	Marks
i • vectors $\vec{OA}$, $\vec{OB}$ with correct lengths and directions, $\vec{OS}$ along diagonal of ‖'ogram	1
ii • modulus of $z_1 + z_2$	1
• argument of $z_1 + z_2$	1

Answer

i.



ii. $OASB$ is a square (parallelogram with $\angle AOB = \frac{\pi}{2}$ and $OA = OB = 2$)

Hence diagonal OS bisects $\angle AOB$ so that $\vec{OS}$ makes angle $\frac{\pi}{4} + \frac{\pi}{6} = \frac{5\pi}{12}$ with the positive x axis.

$\therefore \arg(z_1 + z_2) = \frac{5\pi}{12}$

Using Pythagoras theorem in right ΔOAS,

$|z_1 + z_2| = OS = 2\sqrt{2}$

$\therefore z_1 + z_2 = 2\sqrt{2}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$

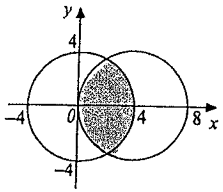
d. Outcomes Assessed: (i) E3 (ii) E3

Marking Guidelines

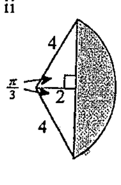
Criteria	Marks
i • two circles of radius 4, centres $(0,0)$ and $(4,0)$.	1
• intersection shaded	1
ii • identifying area as twice area of segment	1
• finding associated sector angle $\frac{2\pi}{3}$	1
• calculation of area	1

Answer

i.



ii



Area is

$$2\times\tfrac{1}{2}\times4^2\left(\tfrac{2\pi}{3}-\sin\tfrac{2\pi}{3}\right)$$

$$=\tfrac{32\pi}{3}-8\sqrt{3}\quad\text{square units}$$

Question 4

a. Outcomes Assessed: (i) E4 (ii) E4 (iii) E4

Marking Guidelines

Criteria	Marks
i • finding gradient of tangent	1
• writing an expression for the equation of the tangent	1
• rearranging and simplifying the equation to produce required form	1
ii • showing $e\cos\theta=\pm1$	1
• deducing P lies on a directrix of the hyperbola	1
iii • stating or using the relationship between a, b and e for the hyperbola	1
• expressing $\frac{dy}{dx}$ (or its square) at P in terms of e and $\cos\theta$	1
• using $e\cos\theta=\pm1$ at P to show that $\frac{dy}{dx}=\pm1$	1

Answer

i.

$y=b\sin\theta\Rightarrow\frac{dy}{d\theta}=b\cos\theta$

$x=a\cos\theta\Rightarrow\frac{dx}{d\theta}=-a\sin\theta$

$\therefore\frac{dy}{dx}=\frac{dy}{d\theta}\div\frac{dx}{d\theta}=-\frac{b\cos\theta}{a\sin\theta}$

Hence tangent to the ellipse at P has equation

$$y-b\sin\theta=-\frac{b\cos\theta}{a\sin\theta}(x-a\cos\theta)$$

$$ay\sin\theta=-bx\cos\theta+ab(\cos^2\theta+\sin^2\theta)$$

$$bx\cos\theta+ay\sin\theta=ab$$

ii. Tangent to ellipse at P passes through either $(ae,0)$ or $(-ae,0)\Rightarrow e\cos\theta=\pm1$ at P.
Then at P, $x=a\cos\theta=\pm\frac{a}{e}$. Hence P lies on the corresponding directrix of the hyperbola.

iii. At P,

$$\left.\begin{aligned}\frac{b^2}{a^2}&=(e^2-1)\\ e\cos\theta&=\pm1\end{aligned}\right\}$$

$$\left(\frac{dy}{dx}\right)^2=(e^2-1)\frac{\cos^2\theta}{\sin^2\theta}=\frac{e^2\cos^2\theta-\cos^2\theta}{1-\cos^2\theta}$$

$$\therefore\left(\frac{dy}{dx}\right)^2=\frac{1-\cos^2\theta}{1-\cos^2\theta}=1$$

Hence tangent to ellipse at P has gradient ±1.

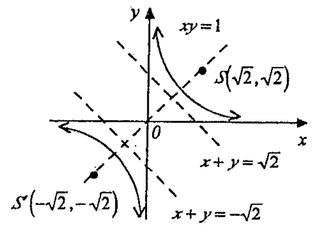
b. Outcomes Assessed: (i) E4 (ii) E4 (iii) E4

Marking Guidelines

Criteria	Marks
i • correct shape and position with coordinates of foci	1
• directrices with equations	1
ii • gradient of chord PQ	1
• equation of chord PQ	1
iii • finding the distance from O to the chord PQ	1
• finding the length of the chord PQ	1
• finding the area of ΔOPQ	1

Answer

i.



ii. $P\left(p,\frac{1}{p}\right)$, $Q\left(q,\frac{1}{q}\right)$

$$\text{gradient } PQ=\frac{\frac{1}{p}-\frac{1}{q}}{p-q}=\frac{q-p}{pq(p-q)}=-\frac{1}{pq}$$

Equation is

$$y-\tfrac{1}{p}=-\tfrac{1}{pq}(x-p)$$

$$pqy-q=-x+p$$

$$x+pqy-(p+q)=0$$

iii. $\perp$distance from O to the chord PQ is

$$d=\frac{|p+q|}{\sqrt{1+(pq)^2}}$$

$$PQ^2=(p-q)^2+\left(\tfrac{1}{p}-\tfrac{1}{q}\right)^2$$

$$=(p-q)^2+\frac{(p-q)^2}{(pq)^2}$$

$$=\frac{(p-q)^2}{(pq)^2}\{(pq)^2+1\}$$

Area of ΔOPQ is given by $A=\frac{1}{2}PQ\times d$

$$\therefore A=\tfrac{1}{2}\left|\frac{p-q}{pq}\right|\sqrt{(pq)^2+1}\times\frac{|p+q|}{\sqrt{1+(pq)^2}}$$

$$=\frac{|p^2-q^2|}{2|pq|}$$

Question 5

a. Outcomes Assessed: (i) HE6 (ii) E7

Marking Guidelines

Criteria	Marks
i • converting dx to $d\theta$ and obtaining integrand for $d\theta$ integral in terms of $\cos^2\theta$	1
• finding primitive function	1
• substituting appropriate limits to evaluate integral	1
• verifying the result geometrically	1
ii • writing the volume δV of the half-cylindrical shell in terms of x	1
• writing V as the limiting sum of these elements of volume to justify the integral for V	1
• evaluating the integral	1
• converting the units to give volume in m^3 to the required accuracy	1

Answer

i. For $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$x = 10\sqrt{2}\sin\theta$

$dx = 10\sqrt{2}\cos\theta \, d\theta$

$x = 10 \Rightarrow \theta = \frac{\pi}{4}$

$x = -10 \Rightarrow \theta = -\frac{\pi}{4}$

$200 - x^2 = 200(1 - \sin^2\theta)$

$= 200\cos^2\theta$

where $\cos\theta \ge 0$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

$$\int_{-10}^{10} \sqrt{200 - x^2}\, dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{200}\cos\theta \,.10\sqrt{2}\cos\theta \, d\theta$$

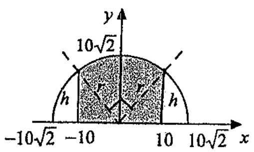
$$= 100\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\cos^2\theta \, d\theta$$

$$= 100\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1 + \cos 2\theta)\, d\theta$$

$$= 100\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

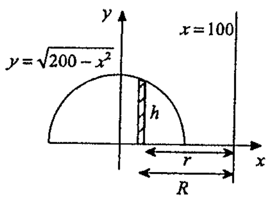
$$= 100\left\{\tfrac{\pi}{4} - (-\tfrac{\pi}{4}) + \tfrac{1}{2}(1 - (-1))\right\}$$

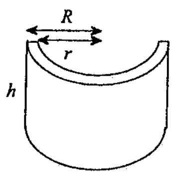
$$= 100 + 50\pi$$



Geometrically,
integral gives shaded area comprising $\frac{1}{4}$ circle ($r = 10\sqrt{2}$) and two right triangles ($h = 10$).
Shaded area is $\frac{1}{4}\pi(10\sqrt{2})^2 + 2\times\frac{1}{2}\times 10\times 10 = 50\pi + 100$

ii.





$h = \sqrt{200 - x^2}$

$r = 100 - x$

$R = 100 - x + \delta x$

$\delta V = \frac{1}{2}\pi\,(R^2 - r^2)\,h$

$= \frac{1}{2}\pi\,(R + r)(R - r)\,h$

$= \frac{1}{2}\pi\,\{2(100 - x) + \delta x\}\,\delta x\, h$

Ignoring terms in $(\delta x)^2$,

$\delta V = \pi(100 - x)\sqrt{200 - x^2}\,\delta x$

$$V = \pi \lim_{\delta x \to 0} \sum_{x=-10}^{x=10} (100 - x)\sqrt{200 - x^2}\,\delta x$$

$$V = \pi \int_{-10}^{10} (100 - x)\sqrt{200 - x^2}\, dx$$

$$\therefore V = 100\pi \int_{-10}^{10} \sqrt{200 - x^2}\, dx - \pi \int_{-10}^{10} x\sqrt{200 - x^2}\, dx = 100\pi\,(100 + 50\pi) + 0$$

(Since $f(x) = x\sqrt{200 - x^2}$ is an odd function, the second integral is 0.)

Hence the volume of the tunnel is $5000\pi(2 + \pi)\times 10^{-6}\text{m}^3 \approx 0\cdot 081\text{m}^3$ (to 2 sig. fig.)

b. **Outcomes Assessed:** (i) **E3** (ii) **E3**

Marking Guidelines

Criteria	Marks
i • noting the root 1 can be written in this form with $r = 1$	1
• identifying the moduli and arguments of the remaining tenth roots of 1	1
ii • explaining why the equation has only nine roots	1
• identifying the nine roots as solutions of $\frac{z-1}{z} = \cos\frac{r\pi}{5} + i\sin\frac{r\pi}{5}$, $r = 1, 2, 3, \ldots, 9$	1
• finding an expression for $\frac{1}{z}$ (or z) in terms of sin and cos of $\frac{r\pi}{10}$	1
• taking the reciprocal of a complex number using De Moivre	1
• rearrangement of expressions involving complex numbers and trig. functions	1

Answer

i. One of the ten roots of $z^{10} = 1$ is 1. Using De Moivre's theorem, the other roots are equally spaced by $\frac{2\pi}{10} = \frac{\pi}{5}$ around the unit circle in the Argand diagram, each having modulus 1, with arguments $\frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \ldots, \frac{9\pi}{5}$.
Hence the roots are
$z = \cos\frac{r\pi}{5} + i\sin\frac{r\pi}{5}$, $r = 0, 1, 2, \ldots, 9$
(where $r = 0$ gives the root $z = 1$)

ii. $\left(\frac{z-1}{z}\right)^{10} = 1 \Rightarrow (z-1)^{10} = z^{10}$. After expanding and simplifying, this polynomial equation in z has degree 9 and hence only 9 roots. Clearly $\frac{z-1}{z} \ne 1$, hence the roots are given by

$\frac{z-1}{z} = \cos\frac{r\pi}{5} + i\sin\frac{r\pi}{5}$, $r = 1, 2, 3, \ldots, 9$

$1 - \frac{1}{z} = \cos\frac{r\pi}{5} + i\sin\frac{r\pi}{5}$

$\frac{1}{z} = \left(1 - \cos\frac{r\pi}{5}\right) - i\sin\frac{r\pi}{5}$

$= 2\sin^2\frac{r\pi}{10} - i\,2\sin\frac{r\pi}{10}\cos\frac{r\pi}{10}$

$= 2\sin\frac{r\pi}{10}\left(\sin\frac{r\pi}{10} - i\cos\frac{r\pi}{10}\right)$

$= -2i\sin\frac{r\pi}{10}\left(\cos\frac{r\pi}{10} + i\sin\frac{r\pi}{10}\right)$

$z = \frac{1}{2}i\frac{1}{\sin\frac{r\pi}{10}}\left(\cos\frac{-r\pi}{10} + i\sin\frac{-r\pi}{10}\right)$

$z = \frac{1}{2}i\left(\cot\frac{r\pi}{10} - i\right)$

$z = \frac{1}{2}\left(1 + i\cot\frac{r\pi}{10}\right)$, $r = 1, 2, 3, \ldots, 9$

Question 6

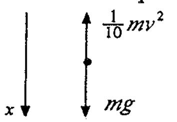
a. **Outcomes Assessed:** (i) **E5** (ii) **E5**

Marking Guidelines

Criteria	Marks
i • showing forces on particle and using Newton s second law to justify expression for a	1
• stating $a \to 0$ as $v \to \sqrt{10g}$ to explain expression for terminal velocity V	1
ii • showing forces on particle and using Newton s second law to justify expression for a	1
• writing a as a derivative with respect to x then inverting	1
• finding the primitive function	1
• evaluating the constant of integration using the initial conditions	1
• finding an expression for x in terms of v and rearranging to find H in required form	1
• using expression for a as derivative appropriate for downward journey	1
• finding primitive function and evaluating constant from new initial conditions	1
• finding v after falling H on downward journey	1

Answer

i. *Forces on particle*



By Newton's second law

$m\ddot{x} = mg - \frac{1}{10}mv^2$

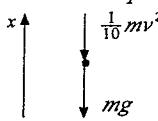
$\ddot{x} = g - \frac{1}{10}v^2$

$\therefore a = g - \frac{1}{10}v^2$

$a \to 0$ as $v \to \sqrt{10g}$

$\therefore$Terminal velocity $V = \sqrt{10g}$.

ii. *Forces on particle*



Initial conditions

$t = 0,\ x = 0,\ v = V\tan\alpha$

By Newton's second law

$m\ddot{x} = -mg - \frac{1}{10}mv^2$

$\therefore a = -\left(g + \frac{1}{10}v^2\right)$

$\frac{1}{2}\frac{dv^2}{dx} = -\left(g + \frac{1}{10}v^2\right)$

$= -\frac{10g + v^2}{10}$

$2\frac{dx}{d(v^2)} = -\frac{10}{(v^2) + 10g}$

$2x = -10\ln\left(v^2 + 10g\right)A$

for some constant A.

$\left.\begin{matrix} x = 0 \\ v = V\tan\alpha \end{matrix}\right\} \Rightarrow A = \frac{1}{V^2\tan^2\alpha + 10g}$

$\therefore A = \frac{1}{10g\left(\tan^2\alpha + 1\right)} = \frac{1}{10g\sec^2\alpha}$

$\therefore x = -5\ln\left(\frac{v^2 + 10g}{10g\sec^2\alpha}\right)$

$= 5\ln\left(\frac{10g\sec^2\alpha}{v^2 + 10g}\right)$

At highest point, $v = 0,\ x = H$

$\therefore H = 5\ln\sec^2\alpha$

After this time, the particle falls from rest.

For this part of the motion, let $t = 0, x = 0$ when $v = 0$ at height H.

$a = g - \frac{1}{10}v^2$

$\frac{1}{2}\frac{dv^2}{dx} = \frac{10g - v^2}{10}$

$2\frac{dx}{d(v^2)} = \frac{10}{10g - (v^2)}$

$2x = -10\ln\left(10g - v^2\right)B$

for some constant B.

$x = 0,\ v = 0 \Rightarrow B = \frac{1}{10g}$

$\therefore x = -5\ln\left(\frac{10g - v^2}{10g}\right)$

On return to point of projection $x = H$, hence

$5\ln\sec^2\alpha = -5\ln\left(1 - \frac{v^2}{10g}\right)$

$\ln\cos^2\alpha = \ln\left(1 - \frac{v^2}{10g}\right)$

$\cos^2\alpha = 1 - \frac{v^2}{10g}$

$\therefore \frac{v^2}{10g} = 1 - \cos^2\alpha = \sin^2\alpha$

$\therefore v^2 = 10g\sin^2\alpha$

$v = V\sin\alpha$

b. Outcomes Assessed: (i) E4 (ii) E4

Marking Guidelines

Criteria	Marks
i • transforming the equation $x \to \frac{1}{x}$	1
• rearranging into required form	1
ii • writing roots in simplest form	1
• transforming equation obtained in i. $x \to -x$	1
• rearranging into required form	1

Answer

i. $x^3 + 2x + 1 = 0$ has roots $\alpha,\ \beta,\ \gamma$,

Hence $\frac{1}{\alpha},\ \frac{1}{\beta},\ \frac{1}{\gamma}$ satisfy

$\left(\frac{1}{x}\right)^3 + 2\left(\frac{1}{x}\right) + 1 = 0$

$1 + 2x^2 + x^3 = 0$

$\therefore x^3 + 2x^2 + 1 = 0$ has roots $\frac{1}{\alpha},\ \frac{1}{\beta},\ \frac{1}{\gamma}$

ii. $\alpha + \beta + \gamma = 0 \Rightarrow \frac{\beta + \gamma}{\alpha^2} = \frac{-\alpha}{\alpha^2} = -\frac{1}{\alpha}$

Hence required equation has roots $-\frac{1}{\alpha},\ -\frac{1}{\beta},\ -\frac{1}{\gamma}$.

But these satisfy $(-x)^3 + 2(-x)^2 + 1 = 0$.

Hence required equation is $x^3 - 2x^2 - 1 = 0$.

Question 7

a. Outcomes Assessed: (i) PE3 , E9 (ii) P4

Marking Guidelines

Criteria	Marks
i • applying Newton s method once to get next approximation in terms of a and c	1
• simplifying $a_1 - \sqrt{c}$	1
• substituting $\varepsilon_0, \varepsilon_1$ to obtain required result	1
ii •finding values of ε_0 such that $\varepsilon_1 = \varepsilon_0$	1
• finding the positive values of a such that $\varepsilon_1 = \varepsilon_0$	1

Answer

i. $f(x) = x^2 - c$

$f'(x) = 2x$

Let next approximation from Newton's method be a_1.

$a_1 = a - \frac{a^2 - c}{2a}$

$= \frac{a^2 + c}{2a}$

$a_1 - \sqrt{c} = \frac{a^2 + c - 2a\sqrt{c}}{2a}$

$= \frac{\left(a - \sqrt{c}\right)^2}{2a}$

$\therefore a_1 - \sqrt{c} = \frac{\varepsilon_0^{\,2}}{2a}$

$\therefore \varepsilon_1 = \left|a_1 - \sqrt{c}\right| = \frac{\varepsilon_0^{\,2}}{2a}$

ii.

$\varepsilon_1 = \varepsilon_0 \Rightarrow \frac{\varepsilon_0^{\,2}}{2a} = \varepsilon_0$

$\therefore \varepsilon_0\left(\varepsilon_0 - 2a\right) = 0$

$\varepsilon_0 = 0$ or $\varepsilon_0 = 2a$

$\therefore a = \sqrt{c}$ or $\left|a - \sqrt{c}\right| = 2a$

$a - \sqrt{c} = 2a$ or $a - \sqrt{c} = -2a$

$a = -\sqrt{c}$ $\quad$ $3a = \sqrt{c}$

But $a > 0$, hence $a = \sqrt{c}$ or $a = \frac{1}{3}\sqrt{c}$

b. Outcomes Assessed: (i) H5 (ii) PE3

Marking Guidelines

Criteria	Marks
i • writing $\sin\theta\cos\theta$ in terms of x	1
• rearranging equation into required form	1
ii • Factoring equation into linear and quadratic	1
• finding one solution for θ to nearest degree	1
• finding second solution to nearest degree.	1

Answer

i. $x=\tan\theta \Rightarrow \sin 2\theta = \dfrac{2x}{1+x^2}$

Then

$$4-\tan\theta = 5\sin\theta\cos\theta$$
$$4-\tan\theta = \tfrac{5}{2}\sin 2\theta$$
$$4-x=\frac{5x}{1+x^2}$$
$$(4-x)(1+x^2)=5x$$
$$-x^3+4x^2-x+4=5x$$
$$x^3-4x^2+6x-4=0$$

ii. Let $P(x)=x^3-4x^2+6x-4$

$P(2)=8-16+12-4=0$

Then $P(x)=(x-2)(x^2-2x+2)$

But $x^2-2x+2=0$ has no real roots, hence $x^3-4x^2+6x-4=0$ has exactly one real root $x=2$.

$\tan\theta=2,\ 0°\le\theta\le 360°$

$\theta \approx 63°, 243°$

c. Outcomes Assessed: (i) PE3 (ii) PE3 , E9

Marking Guidelines

Criteria	Marks
i • writing $n!$ as a product, and noting that each factor is less than $n+1$	1
• deducing the required result	1
ii • multiplying both sides by $n!$	1
• rearranging both sides in terms of powers of $n!$, $(n+1)!$	1
• deducing the required result	1

Answer

i. $n! = n(n-1)(n-2)\ldots 2.1$ for positive integers n.

Each of the n factors on the right is less than $n+1$.

$\therefore n! < (n+1)^n$

ii.
$$n! < (n+1)^n$$
$$(n!)^n . n! < (n!)^n (n+1)^n$$
$$(n!)^{n+1} < \{n!(n+1)\}^n = \{(n+1)!\}^n$$
$$(n!) < \{(n+1)!\}^{\frac{n}{n+1}}$$
$$(n!)^{\frac{1}{n}} < \{(n+1)!\}^{\frac{1}{n+1}}$$

Hence $\sqrt[n]{n!} < \sqrt[n+1]{(n+1)!}$

Question 8

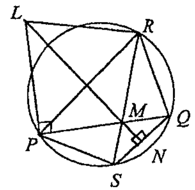
a. Outcomes Assessed: (ii) H5 (iii) PE2 , PE3

Marking Guidelines

Criteria	Marks
i • applying an appropriate test for similar triangles	1
• stating reasons to justify deductions	1
ii • sequence of deductions to show $\angle PRS = \angle PLM$	1
• justifying these deductions using geometric properties	2
• deducing $PMRL$ is a cyclic quadrilateral	1
• quoting an appropriate test for a cyclic quadrilateral	1
• justifying deduction that $LR \perp RS$	1

Answer

i.



ii. In ΔPML, ΔNMQ

$\angle MPL = \angle MNQ$ *(both given as right angles)*

$\angle PML = \angle NMQ$ *(vertically opp. $\angle$ s are equal)*

$\therefore \Delta PML \,|||\, \Delta NMQ$ *(2 prs $\angle$ s are equal)*

iii. $\angle PRS = \angle PQS$ *($\angle$ s subtended at the circumference by arc PS are equal)*

$\angle PQS = \angle PLM$ *(equal $\angle$ s in similar Δ' s NMQ, PML)*

$\therefore \angle PRS = \angle PLM$

$\therefore PMRL$ is a cyclic quadrilateral *(PM subtends equal angles at R, L on same side of PM)*

$\therefore \angle LRM = 90°$ *(opp. $\angle$ s of cyclic quad. are supplementary)*

$\therefore LR \perp RS$

b. Outcomes Assessed: H5 , HE2

Marking Guidelines

Criteria	Marks
• showing that statement is true for $n=2$	1
• writing $x^{k+1}+\frac{1}{x^{k+1}}$ in terms of similar expressions in x^k and x^{k-1}	1
• expressing $x^{k+1}+\frac{1}{x^{k+1}}$ in terms of cosine functions if statements are true for $n\le k$	1
• simplifying trigonometric expression	2
• deducing that statement true for $n=k+1$ if true for $n\le k$, provided $k\ge 2$.	1
• using the truth of two consecutive statements to infer the truth of the next one in the final explanation	1

Answer

Let $S(n)$ be the statement $x^n + \frac{1}{x^n} = 2\cos n\theta$ for $n = 1, 2, 3, \ldots$

Clearly $S(1)$ *is true (by the stated relationship between x and* θ*)*

Consider $S(2)$: $x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2 = 4\cos^2\theta - 2 = 2\left(2\cos^2\theta - 1\right) = 2\cos 2\theta$

$\therefore S(2)$ is true.

If $S(n)$ *is true for* $n \le k$: $x^n + \frac{1}{x^n} = 2\cos n\theta, \quad n = 1, 2, 3, \ldots, k$

Consider $S(k+1)$, *where* $k \ge 2$:

$$x^{k+1} + \frac{1}{x^{k+1}} = \left(x + \frac{1}{x}\right)\left(x^k + \frac{1}{x^k}\right) - \left(x^{k-1} + \frac{1}{x^{k-1}}\right)$$

$$= 2\cos\theta \,.\, 2\cos k\theta - 2\cos(k-1)\theta \qquad \text{if } S(n) \text{ is true, } n = 1, 2, \ldots, k$$

$$= 4\cos\theta \,.\, \cos k\theta - 2\{\cos k\theta \cos\theta + \sin k\theta \sin\theta\}$$

$$= 2\cos k\theta \cos\theta - 2\sin k\theta \sin\theta$$

$$= 2\cos(k+1)\theta$$

Hence if $S(n)$ is true for $1 \le n \le k$, then $S(k+1)$ is true. But $S(1)$ and $S(2)$ are true, hence $S(3)$ is true. Then $S(2)$ and $S(3)$ are true, hence $S(4)$ is true, and so on. Hence by Mathematical Induction, $S(n)$ is true for all positive integers n.

$\therefore x^n + \frac{1}{x^n} = 2\cos n\theta$ for all positive integers $n \ge 2$.