Centre Number

Student Number

CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NEW SOUTH WALES

2005
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics
Extension 2

Morning Session
Monday 8 August 2005

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided separately
- All necessary working should be shown in every question

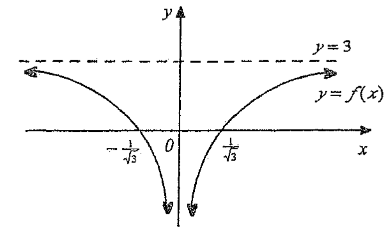
Total marks – 120

- Attempt Questions 1–8
- All questions are of equal value

Marks

Question 1 **(Begin a new page)**

(a) The diagram below shows the graph $y = f(x)$ where $f(x) = 3 - \dfrac{1}{x^2}$.



On separate diagrams, sketch the following graphs, in each case showing any intercepts on the coordinate axes and the equations of any asymptotes:

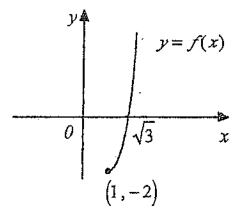
(i) $y = \{f(x)\}^2$ **2**

(ii) $y^2 = f(x)$ **2**

(b)(i) The polynomial equation $P(x) = 0$ has a double root α. Show that α is also a root of the equation $P'(x) = 0$. **2**

(ii) The line $y = mx$ is a tangent to the curve $y = 3 - \dfrac{1}{x^2}$. Show that the equation $mx^3 - 3x^2 + 1 = 0$ has a double root and hence find any values of m. **4**

(c) The diagram below shows the graph $y = f(x)$ where $f(x) = x^3 - 3x$, $x \ge 1$.



(i) Copy the diagram. On your diagram sketch the graph of the inverse function $y = f^{-1}(x)$ showing any intercepts on the coordinate axes and the coordinates of any endpoints. Draw in the line $y = x$. **2**

(ii) Find the coordinates of any points of intersection of the curves $y = f(x)$ and $y = f^{-1}(x)$. Hence find the area of the region in the first quadrant bounded by the curves $y = f(x)$ and $y = f^{-1}(x)$ and the coordinate axes. **3**

Question 2 **(Begin a new page)** Marks

(a) Find $\int \frac{1-\sin x}{\cos^2 x}\,dx$. 2

(b) Find $\int \left(e^x + e^{-\frac{1}{2}x}\right)^2\,dx$. 2

(c) Use the substitution $u=\sqrt{x}$ to evaluate $\int_1^{25} \frac{1}{x+\sqrt{x}}\,dx$, expressing the answer in simplest exact form. 3

(d) Use the substitution $t=\tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{3}} \frac{1}{5-4\cos x}\,dx$, expressing the answer in simplest exact form. 3

(e)(i) If $I_n=\int_0^1 x(1-x)^n\,dx$, $n=0,1,2,\ldots$, show that $I_n=\frac{n}{n+2}I_{n-1}$, $n=1,2,3,\ldots$. 3

(ii) Hence show that $I_n=\frac{1}{2\,{}^{n+2}C_2}$, $n=1,2,3,\ldots$ 2

Question 3 **(Begin a new page)**

(a) Show that the complex number $z=\frac{6-2i}{3+4i}-\frac{6}{5i}$ is real. 2

(b) $z_1=4\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$ and $z_2=2\left(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}\right)$.

(i) On an Argand diagram draw the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ representing z_1, z_2, z_1+z_2 respectively. 2

(ii) Hence find $|z_1+z_2|$ in simplest exact form. 2

(c) The quadratic equation $z^2+kz+4=0$, k real and $-4<k<4$, has two non-real roots α, β.

(i) Explain why α, β are complex conjugates. Hence show that $|\alpha|=|\beta|=2$. 2

(ii) If α, β have arguments $\frac{\pi}{4}$, $-\frac{\pi}{4}$, find the value of k. 2

(d)(i) On an Argand diagram shade the region where both $|z-(1+i)|\le\sqrt{2}$ and $0\le\arg z\le\frac{\pi}{2}$ 2

(ii) Find the exact perimeter and the exact area of the shaded region. 3

Question 4 **(Begin a new page)** Marks

(a) Sketch the graph of the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ showing the intercepts on the axes, the coordinates of the foci and the equations of the directrices. 4

(b) The hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, $a>b>0$, has eccentricity e.

(i) Show that the line through the focus $F(ae,0)$ that is perpendicular to the asymptote $y=\frac{bx}{a}$ has equation $ax+by-a^2e=0$. 1

(ii) Show that this line meets the asymptote at a point on the corresponding directrix. 3

(c) $P\left(p,\frac{1}{p}\right)$ and $Q\left(q,\frac{1}{q}\right)$ are two variable points on the rectangular hyperbola $xy=1$ such that the chord PQ passes through the point $A(0,2)$. M is the midpoint of PQ.

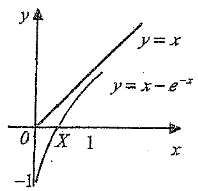
(i) Show that PQ has equation $x+pqy-(p+q)=0$. Hence deduce that $p+q=2pq$. 3

(ii) Deduce that the tangent drawn from the point A to the rectangular hyperbola touches the curve at the point $(1,1)$. 1

(iii) Sketch the rectangular hyperbola showing the points P, Q, A and M. Find the equation of the locus of M and state any restrictions on the domain of this locus. 3

Question 5 **(Begin a new page)**

(a)



The diagram shows the graph of the curve $y=x-e^{-x}$, $x\ge 0$. This curve makes an intercept X on the x-axis, where $0<X<1$. The region bounded by the curve and the line $y=x$ between $x=0$ and $x=X$ is rotated through one complete revolution about the y-axis.

(i) Use the method of cylindrical shells to show that the volume V of the solid of revolution is given by $V=2\pi\int_0^X xe^{-x}\,dx$. 3

(ii) Hence show that $V=2\pi\left(1-X-X^2\right)$ 3

Marks

(b) $z = \cos\theta + i\sin\theta$

(i) Express $1+z$ in modulus argument form. Hence show that $(1+z)^4 = 16\cos^4\frac{\theta}{2}\left(\cos 2\theta + i\sin 2\theta\right)$. **3**

(ii) Use the Binomial Theorem expansion of $(1+z)^4$ to show that $1 + 4\cos\theta + 6\cos 2\theta + 4\cos 3\theta + \cos 4\theta = 16\cos^4\frac{\theta}{2}\cos 2\theta$, and find a corresponding expression for $4\sin\theta + 6\sin 2\theta + 4\sin 3\theta + \sin 4\theta$. **3**

(iii) Hence show that $\dfrac{4\sin\theta + 6\sin 2\theta + 4\sin 3\theta + \sin 4\theta}{1 + 4\cos\theta + 6\cos 2\theta + 4\cos 3\theta + \cos 4\theta} = \tan 2\theta$, **3**

and $\dfrac{4\sin\theta + 4\sin 3\theta + \sin 4\theta}{1 + 4\cos\theta + 4\cos 3\theta + \cos 4\theta} = \tan 2\theta$.

Question 6 (Begin a new page)

(a) A particle of mass m kg is dropped from rest in a medium in which the resistance to motion has magnitude $\frac{1}{10}mv^2$ when the velocity of the particle is $v\ \text{ms}^{-1}$. After t seconds the particle has fallen x metres and has velocity $v\ \text{ms}^{-1}$ and acceleration $a\ \text{ms}^{-2}$. Take the acceleration due to gravity as $10\ \text{ms}^{-2}$.

(i) Draw a diagram showing the forces acting on the particle. Hence show that $a = \dfrac{100 - v^2}{10}$. **2**

(ii) Show that $t = \frac{1}{2}\ln\left(\dfrac{10+v}{10-v}\right)$. **2**

(iii) Find expressions in terms of t for v and x. **3**

(iv) Show that the terminal velocity is $10\ \text{ms}^{-1}$. Hence find the exact time taken and the exact distance fallen by the particle in reaching a speed equal to 80% of its terminal velocity. **3**

(b) The equation $x^3 + px + q = 0$ (where p, q real) has roots α, β, γ.

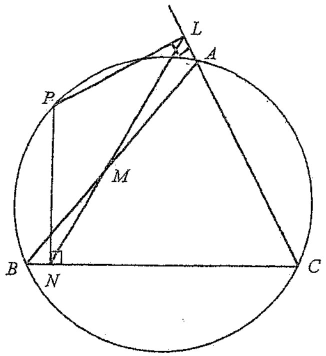
(i) Show that the monic cubic equation with roots α^2, β^2, γ^2 is $x^3 + 2px^2 + p^2x - q^2 = 0$. **2**

(ii) Show that $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = \dfrac{\gamma^2}{q}$. Hence find a cubic equation with roots $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$, $\dfrac{1}{\beta} + \dfrac{1}{\gamma}$ and $\dfrac{1}{\gamma} + \dfrac{1}{\alpha}$. **3**

Marks

Question 7 (Begin a new page)

(a)



ABC is an acute-angled triangle inscribed in a circle. P is a point on the minor arc AB of the circle. PL and PN are the perpendicuars from P to CA (produced) and CB respectively. LN cuts AB at M.

(i) Copy the diagram

(ii) Explain why $PNCL$ is a cyclic quadrilateral. **1**

(iii) Show that $\angle PBM = \angle PNM$. **3**

(iv) Hence show that PM is perpendicular to AB. **3**

(b) The equation $x^2 + x + 1 = 0$ has roots α, β. $T_n = \alpha^n + \beta^n$, $n = 1, 2, 3, \ldots$

(i) Show that $T_1 = T_2 = -1$. **2**

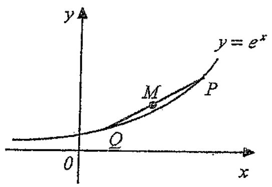
(ii) Show that $T_n = -T_{n-1} - T_{n-2}$, $n = 3, 4, 5, \ldots$ **2**

(iii) Hence use Mathematical Induction to show that $T_n = 2\cos\frac{2n\pi}{3}$, $n = 1, 2, 3, \ldots$ **4**

Question 8 (Begin a new page) — Marks

(a) A die is biased so that on any single roll the probability of getting an even score is p where $p \neq 0 \cdot 5$. In 12 rolls of this die the probability of getting exactly 4 even scores is three times the probability of getting exactly 3 even scores. Find the value of p. **3**

(b)



$P(a, e^a)$ and $Q(b, e^b)$, where $a > b$, are two points on the curve $y = e^x$.
M is the midpoint of PQ.

(i) Use the diagram to show that $e^a + e^b > 2e^{\frac{1}{2}(a+b)}$. **2**

(ii) Hence show that if $a > b > c > d$ then $e^a + e^b + e^c + e^d > 4e^{\frac{1}{4}(a+b+c+d)}$. **2**

(c) A closed hollow right cone with radius r and height h has volume V and surface area A.

(i) Show that $9V^2 = r^2A^2 - 2\pi r^4 A$. **3**

(ii) Hence show that if A is fixed then the maximum value of V is $\sqrt{\dfrac{A^3}{72\pi}}$. **5**

Question 1

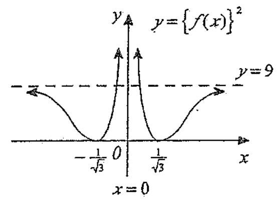
a. Outcomes Assessed: i. E6 ii. E6

Marking Guidelines

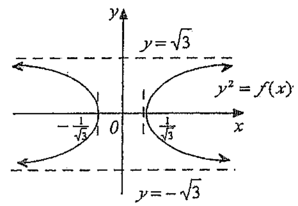
Criteria	Marks
i. • shows curve symmetric about y axis with minimum turning points at $\left(\pm\frac{1}{\sqrt{3}}, 0\right)$	1
• shows asymptotes as positive y axis as $x \to 0$ and $y = 9$ as $x \to \pm\infty$	1
ii. • shows curve symmetric about both x and y axes with vertical tangents at $\left(\pm\frac{1}{\sqrt{3}}, 0\right)$	1
• shows asymptotes as $y = \sqrt{3}$ and $y = -\sqrt{3}$ as $x \to \pm\infty$	1

Answer

i.



ii.



b. Outcomes Assessed: i. E4 ii. E4

Marking Guidelines

Criteria	Marks
i. • writes $P(x)$ in the form $(x-\alpha)^2 Q(x)$	1
• shows $P'(\alpha) = 0$	1
ii. • solves simultaneously to deduce double root of given equation	1
• uses result from i. to show $3m\alpha^2 - 6\alpha = 0$	1
• deduces $\frac{2}{m}$ is the double root of the given equation	1
• finds both values of m	1

Answer

i. $P(x) = (x-\alpha)^2 Q(x)$

$P'(x) = 2(x-\alpha)Q(x) + (x-\alpha)^2 Q'(x)$

$= (x-\alpha)\{2Q(x) + (x-\alpha)Q'(x)\}$

$\therefore P'(\alpha) = 0$

ii. The line $y = mx$ meets the curve where

$$mx = 3 - \frac{1}{x^2}$$
$$mx^3 = 3x^2 - 1$$
$$mx^3 - 3x^2 + 1 = 0$$

This equation has a double root if the line is tangent to the curve.
Let $P(x) = mx^3 - 3x^2 + 1$
Then $P'(x) = 3mx^2 - 6x = 3x(mx - 2)$
Since $P(0) \neq 0$, the double root of $P(x) = 0$ must be $\frac{2}{m}$. Then

$$m\left(\tfrac{2}{m}\right)^3 - 3\left(\tfrac{2}{m}\right)^2 + 1 = 0$$
$$-\tfrac{4}{m^2} + 1 = 0$$
$$m^2 = 4$$
$$m = \pm 2$$

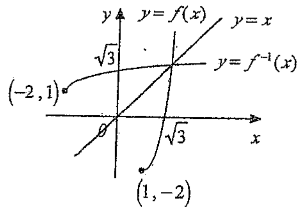
c. **Outcomes Assessed:** i. HE4 ii. H8

Marking Guidelines

Criteria	Marks
i. • shows shape of graph of inverse function as reflection in $y=x$	1
• gives y intercept and endpoint of graph of inverse function.	1
ii. • finds intersection point of curves	1
• uses symmetry to express area as a definite integral	1
• evaluates the integral to find the area	1

Answer

i.



ii. The curves $y=f(x)$, $y=f^{-1}(x)$ intersect on the line $y=x$ where

$$x^3-3x=x,\ x\ge 1$$
$$x^3-4x=0$$
$$x(x^2-4)=0$$

$\therefore x=2$ and intersection point is $(2,2)$.

By symmetry, the required area is twice the area bounded by the line $y=x$, the x-axis and the curve $y=x^3-3x$.

Hence area is A square units, where

$$A=2\left\{\int_0^2 x\ dx-\int_{\sqrt3}^2 (x^3-3x)\ dx\right\}$$
$$=2\left\{\left[\tfrac12x^2\right]_0^2-\left[\tfrac14x^4-\tfrac32x^2\right]_{\sqrt3}^2\right\}$$
$$=2\left\{2-\left(\tfrac14(16-9)-\tfrac32(4-3)\right)\right\}$$
$$=2\left\{2-\left(\tfrac74-\tfrac32\right)\right\}$$
$$=\tfrac72$$

$\therefore$ area is $3\frac12$ sq. units.

Question 2

a. **Outcomes Assessed:** H5

Marking Guidelines

Criteria	Marks
• rearranges integrand using appropriate trig. identities	1
• obtains primitive function	1

Answer

$$\int\frac{1-\sin x}{\cos^2x}\ dx=\int\left(\frac{1}{\cos^2x}-\frac{\sin x}{\cos^2x}\right)\ dx$$
$$=\int\left(\sec^2x-\sec x\tan x\right)\ dx$$
$$=\tan x-\sec x+c$$

b. **Outcomes Assessed:** H5

Marking Guidelines

Criteria	Marks
• expands square	1
• finds primitive	1

Answer

$$\int\left(e^x+e^{-\frac12x}\right)^2\ dx=\int\left(e^{2x}+2e^{\frac12x}+e^{-x}\right)\ dx$$
$$=\tfrac12e^{2x}+4e^{\frac12x}-e^{-x}+c$$

c. **Outcomes Assessed:** HE6

Marking Guidelines

Criteria	Marks
• expresses definite integral in terms of u	1
• finds primitive function in terms of u	1
• evaluates definite integral using appropriate limits and simplifies answer	1

Answer

$$u=\sqrt x$$
$$u^2=x,\ u>0$$
$$2u\ du=dx$$

$$x=1\Rightarrow u=1$$
$$x=25\Rightarrow u=5$$

$$\frac{1}{x+\sqrt x}=\frac{1}{u^2+u}$$

$$\int_1^{25}\frac{1}{x+\sqrt x}\ dx=\int_1^5\frac{1}{u(u+1)}\,2u\,du$$
$$=2\int_1^5\frac{1}{u+1}\ du$$
$$=2\left[\ln(u+1)\right]_1^5$$
$$=2(\ln6-\ln2)$$
$$=2\ln3$$

d. **Outcomes Assessed:** HE6

Marking Guidelines

Criteria	Marks
• expresses $5-4\cos x$ in terms of t	1
• converts definite integral and finds primitive in terms of t	1
• evaluates and simplifies exact answer	1

Answer

$$t=\tan\tfrac{x}{2}$$
$$dt=\tfrac12\sec^2\tfrac{x}{2}\ dx$$
$$2dt=(1+t^2)\ dx$$
$$\therefore\ \frac{2}{1+t^2}\ dt=dx$$

$$x=0\Rightarrow t=0$$
$$x=\tfrac{\pi}{3}\Rightarrow t=\tfrac{1}{\sqrt3}$$

$$5-4\cos x$$
$$=5-4\,\frac{1-t^2}{1+t^2}$$
$$=\frac{1+9t^2}{1+t^2}$$

$$\int_0^{\frac{\pi}{3}}\frac{1}{5-4\cos x}\ dx=\int_0^{\frac{1}{\sqrt3}}\frac{1+t^2}{1+9t^2}\cdot\frac{2}{1+t^2}\ dt$$
$$=\tfrac23\int_0^{\frac{1}{\sqrt3}}\frac{3}{1+9t^2}\ dt$$
$$=\tfrac23\left[\tan^{-1}3t\right]_0^{\frac{1}{\sqrt3}}$$
$$=\tfrac23\left(\tfrac{\pi}{3}-0\right)$$
$$=\tfrac{2\pi}{9}$$

e. **Outcomes Assessed:** i. E8 ii. E8

Marking Guidelines

Criteria	Marks
i. • applies integration by parts	1
• rearranges integrand to form I_n, I_{n-1}	1
• rearranges further to obtain required recurrence relation	1
ii. • applies recurrence relation to find I_n in terms of I_0	1
• evaluates I_0 and expresses I_n in required form	1

Answer

i. For $n = 1, 2, 3, \ldots$,

$$I_n = \int_0^1 x(1-x)^n \, dx$$

$$= \left[\tfrac{1}{2}x^2(1-x)^n\right]_0^1 - \int_0^1 \tfrac{1}{2}x^2(-n)(1-x)^{n-1} \, dx$$

$$= -\tfrac{n}{2}\int_0^1 \{(1-x)-1\}\, x(1-x)^{n-1} \, dx$$

$$= -\tfrac{n}{2}(I_n - I_{n-1})$$

$$\therefore (n+2)\, I_n = n\, I_{n-1}$$

$$I_n = \frac{n}{n+2} I_{n-1}, \quad n = 1, 2, 3, \ldots$$

ii. For $n = 1, 2, 3, \ldots$

$$I_n = \frac{n}{n+2} I_{n-1}$$

$$= \frac{n(n-1)\ldots 1}{(n+2)(n+1)\ldots 3} I_0$$

$$\therefore I_n = \frac{n!\,2!}{(n+2)!} I_0$$

But $I_0 = \int_0^1 x \, dx = \tfrac{1}{2}\left[x^2\right]_0^1 = \tfrac{1}{2}$

$$\therefore I_n = \frac{1}{2\ {}^{n+2}C_2}, \quad n = 1, 2, 3, \ldots$$

Question 3

a. **Outcomes Assessed:** E3

Marking Guidelines

Criteria	Marks
• applies process to realise denominators	1
• simplifies expression for z to obtain a real number.	1

Answer

$$z = \frac{6-2i}{3+4i} - \frac{6}{5i}$$

$$= \frac{(6-2i)(3-4i)}{9+16} + \frac{6i}{5}$$

$$= \frac{10-30i+30i}{25}$$

$$= \tfrac{2}{5}$$

$\therefore$ z is real.

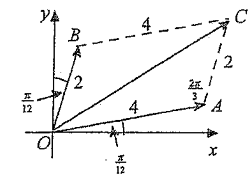
b. **Outcomes Assessed:** i. E3 ii. E3

Marking Guidelines

Criteria	Marks
i. • shows lengths and positions of $\overrightarrow{OA}$; $\overrightarrow{OB}$	1
• completes the parallelogram $OACB$ to show diagonal $\overrightarrow{OC}$	1
ii. • applies cosine rule to find expression for OC^2	1
• evaluates OC in simplest surd form	1

Answer

i.



ii.

$$OC^2 = 4^2 + 2^2 - 2\times 4\times 2\cos\tfrac{2\pi}{3}$$

$$\therefore OC = \sqrt{28} = 2\sqrt{7}$$

$$\therefore |z_1 + z_2| = 2\sqrt{7}$$

c. **Outcomes Assessed:** i. E3 ii. E3

Marking Guidelines

Criteria	Marks
i. • notes equation is polynomial equation with real coefficients	1
• uses the product of the roots to evaluate the modulus of either root	1
ii. • evaluates the sum of the roots directly using modulus and argument.	1
• expresses k in terms of the sum of the roots and evaluates k in surd form.	1

Answer

i. The coefficients of the polynomial equation are real, hence the non-real roots come in complex conjugate pairs and α, β are complex conjugates.

$$\beta = \overline{\alpha} \Rightarrow \alpha\overline{\alpha} = \alpha\beta = 4$$

$$\therefore |\alpha|^2 = 4, \quad |\alpha| \geq 0$$

$$\therefore |\alpha| = |\beta| = 2$$

ii. $\alpha + \overline{\alpha} = 2\,\mathrm{Re}\,\alpha$

$$= 4\cos\tfrac{\pi}{4}$$

$$\therefore -k = 4\cos\tfrac{\pi}{4}$$

$$\therefore k = -2\sqrt{2}$$

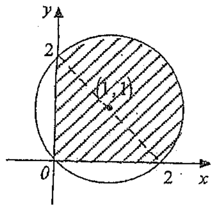
d. **Outcomes Assessed:** i. E3 ii. E3

Marking Guidelines

Criteria	Marks
i. • recognises region lies inside circle of radius $\sqrt{2}$ with centre $(1,1)$	1
• shades part of circle in first quadrant	1
ii. • shows that circle passes through origin with x and y intercepts both 2.	1
• writes down perimeter	1
• writes down area	1

Answer

i.



ii. Perimeter is $4+\pi\sqrt{2}$

Area is $2+\pi$

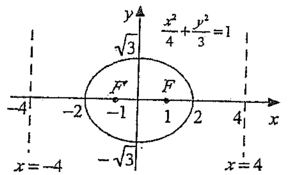
Question 4

a. **Outcomes Assessed:** E3

Marking Guidelines

Criteria	Marks
• shows ellipse with intercepts on axes	1
• calculates the eccentricity	1
• shows foci	1
• shows directrices	1

Answer



$\left.\begin{array}{l} a=2,\quad b=\sqrt{3} \\ b^2=a^2(1-e^2) \end{array}\right\} \Rightarrow \begin{array}{l} 1-e^2=\frac{3}{4} \\ e=\frac{1}{2} \end{array}$

Foci : F, F' at $(\pm ae, 0)=(\pm 1, 0)$

Directrices : $x=\pm\frac{a}{e}=\pm 4$

b. **Outcomes Assessed:** i. E4 ii. E4

Marking Guidelines

Criteria	Marks
i. • writes equation of line in point gradient form	1
ii. • uses simultaneous equations to write equation for x coordinate of intersection point	1
• uses $b^2=a^2(e^2-1)$ to eliminate b	1
• obtains $x=\frac{a}{e}$ as solution of the equation and deduces result.	1

Answer

i. Line through $(ae, 0)$ with gradient $-\frac{a}{b}$ has equation $y=-\frac{a}{b}(x-ae)$. Rearrangement gives $ax+by-a^2e=0$.

ii. At the point of intersection of this line with the asymptote $y=\frac{b}{a}x$

$$ax+b\frac{b}{a}x-a^2e=0$$

$$x\left(1+\frac{b^2}{a^2}\right)-ae=0$$

$$x\left(1+e^2-1\right)-ae=0$$

$$x=\frac{a}{e}.$$

Hence the point of intersection lies on the corresponding directrix.

c. **Outcomes Assessed:** i. E4 ii. E4 iii. E3 , E9

Marking Guidelines

Criteria	Marks
i. • finds gradient of chord	1
• finds equation of chord	1
• substitutes coordinates of A to find required relation between p and q.	1
ii. • establishes coordinates of point of contact of tangent from A to hyperbola.	1
iii. • finds the coordinates of M and shows that M lies on the line $y=1$	1
• sketches one case for chord PQ and deduces corresponding restriction on the domain	1
• sketches second case for PQ and deduces locus of M with appropriate restriction	1

Answer

i. PQ has gradient

$$\frac{\frac{1}{p}-\frac{1}{q}}{p-q}=\frac{q-p}{pq}\times\frac{1}{p-q}$$

$$=-\frac{1}{pq}$$

and equation

$$y-\frac{1}{p}=-\frac{1}{pq}(x-p).$$

Rearrangement gives

$x+pqy-(p+q)=0$

If $A(0, 2)$ lies on PQ, then

$2pq-(p+q)=0$

$\therefore p+q=2pq$

ii. APQ is tangent to the hyperbola if $p=q$. Then

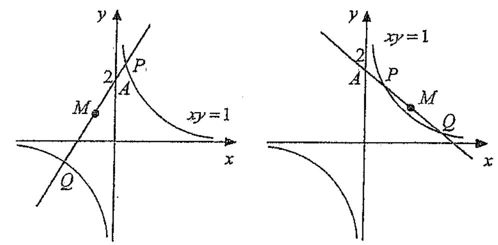
$2p=2p^2$

$0=p(p-1)$

$\therefore p=1$ (since $p\neq 0$)

$\therefore$ Point of contact of tangent from A is $P(1, 1)$.

iii.



M has coordinates $\left(\frac{p+q}{2}, \frac{p+q}{2pq}\right)=(pq, 1)$

$\therefore M$ lies on the line $y=1$.

If P, Q lie on different branches of the hyperbola, (as in the diagram on the left), $x=pq<0$ at M.

If P, Q lie on the branch in the first quadrant, since tangent from A touches hyperbola at $(1, 1)$, by inspection of the diagram on the right, $x\geq 1$ at M.

Hence the locus of M consists of the two parts of the line $y=1$ where either $x<0$ or $x\geq 1$.

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Question 5

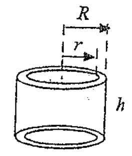
a. Outcomes Assessed: i. E7 ii. E8

Marking Guidelines

Criteria	Marks
i. • finds height of cylindrical shell	1
• finds expression for volume of cylindrical shell	1
• expresses V as limiting sum of shell volumes and hence as definite integral	1
ii. • applies process of integration by parts	1
• evaluates definite integral in terms of X and e^{-X}	1
• deduces that $e^{-X}=X$ and hence evaluates V in terms of X	1

Answer

i.



$R=x+\delta x$

$r=x$

$h=x-(x-e^{-x})$

$\quad =e^{-x}$

Volume of cylindrical shell is

$$\delta V=\pi\left(R^2-r^2\right)h$$
$$=\pi\left(R+r\right)\left(R-r\right)h$$
$$=\pi\left(2x+\delta x\right)\left(\delta x\right)e^{-x}$$

Ignoring second order terms,

$$\delta V=2\pi\, x e^{-x}\,\delta x$$
$$\therefore V=\lim_{\delta x\to 0}\sum_{x=0}^{X}2\pi\, x e^{-x}\,\delta x$$
$$=2\pi\int_0^X x e^{-x}\,dx$$

ii.

$$\int_0^X x e^{-x}\,dx=\left[-xe^{-x}\right]_0^X+\int_0^X e^{-x}\,dx$$
$$=-Xe^{-X}-\left[e^{-x}\right]_0^X$$
$$=-Xe^{-X}-\left(e^{-X}-1\right)$$
$$=1-e^{-X}-Xe^{-X}$$

But $0=X-e^{-X}\Rightarrow e^{-X}=X$

$$\therefore V=2\pi\left(1-X-X^2\right)$$

b. Outcomes Assessed: i. E3 ii. E3 iii. H5

Marking Guidelines

Criteria	Marks
i. • writes $1+\cos\theta,\ \sin\theta$ in terms of $\cos\frac{\theta}{2},\ \sin\frac{\theta}{2}$	1
• expresses $1+z$ in modulus / argument form	1
• uses De Moivre's theorem to obtain required result	1
ii. • applies Binomial theorem expansion in terms of powers of z	1
• equates real parts, applying De Moivre's theorem, to obtain required result	1
• similarly obtains second expression by equating imaginary parts.	1
iii.• obtains first expression for $\tan 2\theta$ by division	1
• expresses denominator of second expression as multiple of $\cos 2\theta$	1
• expresses numerator as multiple of $\sin 2\theta$ then divides to obtain $\tan 2\theta$	1

Answer

i.

$$1+z=1+\cos\theta+i\sin\theta$$
$$=2\cos^2\tfrac{\theta}{2}+i\left(2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\right)$$
$$=2\cos\tfrac{\theta}{2}\left(\cos\tfrac{\theta}{2}+i\sin\tfrac{\theta}{2}\right)$$

Using De Moivre's theorem :

$$\left(1+z\right)^4=16\cos^4\tfrac{\theta}{2}\left(\cos 4\tfrac{\theta}{2}+i\sin 4\tfrac{\theta}{2}\right)$$
$$=16\cos^4\tfrac{\theta}{2}\left(\cos 2\theta+i\sin 2\theta\right)$$

ii. $\left(1+z\right)^4=1+4z+6z^2+4z^3+z^4$

Using De Moivre's Theorem and equating real parts of the two expressions for $\left(1+z\right)^4$:

$$1+4\cos\theta+6\cos 2\theta+4\cos 3\theta+\cos 4\theta=16\cos^4\tfrac{\theta}{2}\cos 2\theta$$

Equating imaginary parts of the two expressions:

$$4\sin\theta+6\sin 2\theta+4\sin 3\theta+\sin 4\theta=16\cos^4\tfrac{\theta}{2}\sin 2\theta$$

iii. $$\frac{4\sin\theta+6\sin 2\theta+4\sin 3\theta+\sin 4\theta}{1+4\cos\theta+6\cos 2\theta+4\cos 3\theta+\cos 4\theta}=\frac{16\cos^4\frac{\theta}{2}\sin 2\theta}{16\cos^4\frac{\theta}{2}\cos 2\theta}=\tan 2\theta$$

Also

$$\frac{4\sin\theta+4\sin 3\theta+\sin 4\theta}{1+4\cos\theta+4\cos 3\theta+\cos 4\theta}=\frac{\left(16\cos^4\frac{\theta}{2}-6\right)\sin 2\theta}{\left(16\cos^4\frac{\theta}{2}-6\right)\cos 2\theta}=\tan 2\theta$$

Question 6

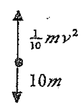
a. Outcomes Assessed: i. E5 ii. E5, E8 iii. E5, E8 iv. E5

Marking Guidelines

Criteria	Marks
i. • shows the forces on the particle	1
• applies Newton's second law to obtain the expression for a.	1
ii. • finds $\frac{dt}{dv}$ in partial fraction form	1
• integrates to find required expression for t	1
iii.• rearranges to express v in terms of t	1
• finds the primitive function for $\frac{dx}{dt}$	1
• uses initial conditions to evaluate the constant of integration and find x in terms of t	1
iv.• explains why the terminal velocity is $10\,\text{ms}^{-1}$	1
• finds the exact time taken to attain 80% of terminal velocity	1
• finds the exact distance fallen	1

Answer

i.



By Newton's second law:

$$ma=10m-\tfrac{1}{10}mv^2$$
$$a=\frac{100-v^2}{10}$$

ii.

$$\frac{dv}{dt}=\frac{100-v^2}{10}$$
$$\frac{dt}{dv}=\frac{10}{100-v^2}$$
$$\frac{dt}{dv}=\frac{1}{2}\left(\frac{1}{10+v}+\frac{1}{10-v}\right)$$
$$t=\tfrac{1}{2}\left\{\ln\left(10+v\right)-\ln\left(10-v\right)\right\}+c$$
$$t=0,\ v=0\Rightarrow c=0$$
$$\therefore t=\tfrac{1}{2}\ln\left(\frac{10+v}{10-v}\right)$$

iii.

$$\frac{10+v}{10-v}=e^{2t}$$
$$10+v=10e^{2t}-ve^{2t}$$
$$v\left(1+e^{2t}\right)=10\left(e^{2t}-1\right)$$
$$v=10\left(\frac{e^{2t}-1}{e^{2t}+1}\right)$$

$$\frac{dx}{dt}=10\left(\frac{e^{t}-e^{-t}}{e^{t}+e^{-t}}\right)$$
$$x=10\ln A\left(e^{t}+e^{-t}\right)$$
$$t=0, x=0 \Rightarrow A=\tfrac{1}{2}$$
$$x=10\ln\tfrac{1}{2}\left(e^{t}+e^{-t}\right)$$

iv. $a\to 0$ as $v\to 10$

$\therefore$ terminal velocity is $10\,\text{ms}^{-1}$

$$v=8\Rightarrow\begin{cases} t=\tfrac{1}{2}\ln 9=\ln 3 \\ x=10\ln\tfrac{1}{2}(3+\tfrac{1}{3})=10\ln\tfrac{5}{3}\end{cases}$$

Hence 80% of terminal velocity is attained after $\ln 3$ seconds when particle has fallen $10\ln\frac{5}{3}$ metres.

b. **Outcomes Assessed:** i. E4 ii. E4

Marking Guidelines

Criteria	Marks
i. • replaces x by $x^{\frac{1}{2}}$ to find an equation satisfied by all three squares	1
• rearranges to create a monic cubic equation with required roots	1
ii. • expresses $\frac{1}{\alpha}+\frac{1}{\beta}$ in terms of γ^2	1
• writes similar expressions for $\frac{1}{\beta}+\frac{1}{\gamma}$, $\frac{1}{\gamma}+\frac{1}{\alpha}$	1
• uses the equation with roots α^2, β^2, γ^2 to form required cubic equation	1

Answer

i. α^2, β^2, γ^2 satisfy

$$\left(x^{\frac{1}{2}}\right)^3+px^{\frac{1}{2}}+q=0$$
$$\left(x^{\frac{3}{2}}+px^{\frac{1}{2}}\right)^2=(-q)^2$$
$$x^3+2px^2+p^2x=q^2$$
$$x^3+2px^2+p^2x-q^2=0$$

ii.

$$\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}$$
$$=\frac{(\alpha+\beta+\gamma)-\gamma}{\alpha\beta}$$
$$=\frac{(0-\gamma)\gamma}{\alpha\beta\gamma}$$
$$=\frac{\gamma^2}{q}$$

Similarly $\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\alpha^2}{q}$ and $\frac{1}{\gamma}+\frac{1}{\alpha}=\frac{\beta^2}{q}$.

But $\frac{\gamma^2}{q}$, $\frac{\alpha^2}{q}$, $\frac{\beta^2}{q}$ satisfy

$$(qx)^3+2p(qx)^2+p^2(qx)-q^2=0$$

Hence required equation is

$$q^3x^3+2pq^2x^2+p^2qx-q^2=0$$

Question 7

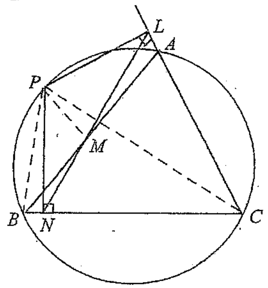
a. **Outcomes Assessed:** ii. PE3 iii. PE2, PE3 iv. PE2, PE3

Marking Guidelines

Criteria	Marks
i. • copies diagram	0
ii. • notes that opposite angles of $PNCL$ at N and L are supplementary	1
iii. • uses circle $BPAC$ to show $\angle PBA=\angle PCA$	1
• uses circle $PNCL$ to show $\angle PCL=\angle PNL$	1
• uses these results to deduce that $\angle PBM=\angle PNM$	1
iv. • deduces that $PBNM$ is a cyclic quadrilateral	1
• uses circle $PBNM$ to show $\angle PMB=\angle PNB$	1
• deduces $PM\perp AB$	1

Answer

i.



ii. Opposite angles at N and L are supplementary.
Hence $PNCL$ is a cyclic quadrilateral

iii. Construct PB and PC

$\angle PBM=\angle PBA$ (B, M, A are collinear)

$\angle PBA=\angle PCA$ ($\angle$'s at circumference of circle $BPAC$ standing on same arc PA are equal)

$\angle PCA=\angle PCL$ (C, A, L are collinear)

But $PNCL$ is a cyclic quadrilateral

$\therefore \angle PCL=\angle PNL$ ($\angle$'s at circumference of circle $PNCL$ standing on same arc PL are equal)

Also $\angle PNL=\angle PNM$ (N, M, L are collinear)

$\therefore \angle PBM=\angle PNM$

iv. Construct PM. Then $PBNM$ is a cyclic quadrilateral (equal $\angle$'s subtended by PM at B, N on same side of PM)

$\therefore \angle PMB=\angle PNB$ ($\angle$'s at circumference of circle $PBNM$ standing on same arc PB are equal)

$\therefore \angle PMB=90°$ and $PM\perp AB$.

b. **Outcomes Assessed:** i. PE3 ii. PE3 iii. HE2, E9

Marking Guidelines

Criteria	Marks
i. • reads $\alpha+\beta$ from the coefficients of the quadratic equation to evaluate T_1	1
• expresses $\alpha^2+\beta^2$ in terms of the sum and product of the roots to evaluate T_2	1
ii. • expresses sum of nth powers of α, β in terms of sums of lower powers	1
• uses values of sum and product of the roots to deduce required result	1
iii. • defines the sequence of induction statements and shows both S_1, S_2 are true	1
• writes T_{k+1} $(k\geq 2)$ as a sum of cosines given the truth of S_n, $n\leq k$	1
• uses cosine expansion to express T_{k+1} in terms of $\cos\frac{2k\pi}{3}$ and $\sin\frac{2k\pi}{3}$	1
• expresses T_{k+1} as a single cosine and completes the mathematical induction process.	1

Answer

i. $\alpha+\beta=-1 \Rightarrow T_1=-1$

$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$\alpha^2+\beta^2=1-2=-1 \Rightarrow T_2=-1$

ii. For $n=3,4,5,\ldots$

$\alpha^n+\beta^n=(\alpha+\beta)(\alpha^{n-1}+\beta^{n-1})-\alpha\beta(\alpha^{n-2}+\beta^{n-2})$

$\therefore T_n=-T_{n-1}-T_{n-2}$

iii. Let S_n be the statement $T_n=2\cos\frac{2n\pi}{3}$, $n=1,2,3,\ldots$

Consider S_1, S_2 : $2\cos\frac{2\pi}{3}=2\times\left(-\frac{1}{2}\right)=-1=T_1$. Hence S_1 is true.

$2\cos\frac{4\pi}{3}=2\times\left(-\frac{1}{2}\right)=-1=T_2$. Hence S_2 is true.

If S_n *is true for* $n\le k$: $T_n=2\cos\frac{2n\pi}{3}$, $n=1,2,3,\ldots,k$

Consider S_{k+1} (*for some* $k\ge 2$) :

$$\begin{aligned}
T_{k+1}&=-T_k-T_{k-1}\\
&=-2\left(\cos\tfrac{2k\pi}{3}+\cos\tfrac{2(k-1)\pi}{3}\right) \qquad \textit{if } S_n \textit{ is true for } n\le k\\
&=-2\left\{\cos\tfrac{2k\pi}{3}+\cos\left(\tfrac{2k\pi}{3}-\tfrac{2\pi}{3}\right)\right\}\\
&=-2\left(\cos\tfrac{2k\pi}{3}+\cos\tfrac{2k\pi}{3}\cos\tfrac{2\pi}{3}+\sin\tfrac{2k\pi}{3}\sin\tfrac{2\pi}{3}\right)\\
&=-2\left(\cos\tfrac{2k\pi}{3}-\tfrac{1}{2}\cos\tfrac{2k\pi}{3}+\tfrac{\sqrt{3}}{2}\sin\tfrac{2k\pi}{3}\right)\\
&=2\left(-\tfrac{1}{2}\cos\tfrac{2k\pi}{3}-\tfrac{\sqrt{3}}{2}\sin\tfrac{2k\pi}{3}\right)\\
&=2\left(\cos\tfrac{2k\pi}{3}\cos\tfrac{2\pi}{3}-\sin\tfrac{2k\pi}{3}\sin\tfrac{2\pi}{3}\right)\\
&=2\cos\left(\tfrac{2k\pi}{3}+\tfrac{2\pi}{3}\right)\\
&=2\cos\tfrac{2(k+1)\pi}{3}
\end{aligned}$$

Hence if S_n is true for $n\le k$ (where $k\ge 2$), then S_{k+1} is true. But S_1 and S_2 are true, hence S_3 is true, and then S_4 is true, and so on. Hence by Mathematical Induction, S_n is true for all positive integers n.
Hence $T_n=2\cos\frac{2n\pi}{3}$, $n=1,2,3,\ldots$

Question 8

a. Outcomes Assessed: PE3

Marking Guidelines

Criteria	Marks
• expresses both probabilities in terms of binomial coefficients and p	1
• writes an equation for p including numerical expressions for the binomial coefficients	1
• solves the equation for p	1

Answer

$^{12}C_4\, p^4(1-p)^8=3\;{}^{12}C_3\, p^3(1-p)^9$

$\frac{12!}{4!\,.\,8!}p^4(1-p)^8=3\times\frac{12!}{3!\,.\,9!}p^3(1-p)^9$

$\therefore p=0,\ p=1,$ or $\quad 9p=12(1-p)$

$21p=12$

$p=\frac{4}{7}$

b. Outcomes Assessed: i. PE3 ii. PE3

Marking Guidelines

Criteria	Marks
i. • finds the coordinates of M	1
• uses the position of M relative to the curve to deduce the required result	1
ii. • uses this result for a and b, then for c and d, adding the inequalities	1
• uses the result a further time for $\frac{1}{2}(a+b)$, $\frac{1}{2}(c+d)$ to obtain the required inequality	1

Answer

i. M has coordinates

$\left(\frac{1}{2}(a+b)\ ,\ \frac{1}{2}(e^a+e^b)\right),\quad a>b.$

The y coordinate of M exceeds the y coordinate of the point on the curve vertically below M.

Hence $\frac{1}{2}\left(e^a+e^b\right)>e^{\frac{1}{2}(a+b)}$

$e^a+e^b>2\,e^{\frac{1}{2}(a+b)}$

ii. $a>b,\ c>d,$ and $a+b>c+d$. Hence

$$\begin{aligned}
\left(e^a+e^b\right)+\left(e^c+e^d\right)&>2\,e^{\frac{1}{2}(a+b)}+2\,e^{\frac{1}{2}(c+d)}\\
&=2\left(e^{\frac{1}{2}(a+b)}+e^{\frac{1}{2}(c+d)}\right)\\
&>2\times 2e^{\frac{1}{2}\left(\frac{1}{2}(a+b)+\frac{1}{2}(c+d)\right)}\\
&=4\,e^{\frac{1}{4}(a+b+c+d)}
\end{aligned}$$

$\therefore e^a+e^b+e^c+e^d>4e^{\frac{1}{4}(a+b+c+d)}$

c. Outcomes Assessed: i. P4 ii. H5 , E1

Marking Guidelines

Criteria	Marks
i. • expresses A in terms of r and the slant height l, and V in terms of r and h	1
• uses the Pythagorean relationship to consider V, A in terms of only two of r, h, l.	1
• obtains the required result by substitution, rearrangement and simplification	1
ii. • differentiates (implicitly or otherwise) to get an expression for $\frac{dV}{dr}$	2
• finds either r or r^2 to make derivative equal to zero.	1
• checks sign of first derivative near positive r solution to determine V is a maximum	1
• substitutes for r^2 and simplifies to find maximum value of V	1

Answer

i. $V=\frac{1}{3}\pi r^2 h,\quad A=\pi r^2+\pi r l=\pi r(r+l)$

where $l^2=r^2+h^2$

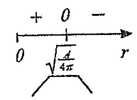
$$\begin{aligned}
\therefore 9V^2&=\pi^2 r^4(l^2-r^2)\\
&=\pi^2 r^4\left\{(l+r)^2-2r(l+r)\right\}\\
&=r^2\left\{\pi^2 r^2(l+r)^2\right\}-2\pi r^4\left\{\pi r(l+r)\right\}\\
&=r^2A^2-2\pi r^4 A
\end{aligned}$$

ii.

$$\begin{aligned}
18V\frac{dV}{dr}&=2rA^2-8\pi r^3 A\\
&=8\pi A r\left(\frac{A}{4\pi}-r^2\right)
\end{aligned}$$

$\frac{dV}{dr}=0 \Rightarrow r^2=\frac{A}{4\pi}$, where r, V, A all positive.

sign of $\frac{dV}{dr}$



Hence V takes a maximum value when $r^2=\frac{A}{4\pi}$,

$9V^2=\frac{A}{4\pi}.\,A^2-2\pi\frac{A^2}{16\pi^2}.\,A=\frac{A^3}{8\pi}$

$\therefore$ the maximum value of V is $\sqrt{\frac{A^3}{72\pi}}$.