CSSA
CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NSW

Centre Number

Student Number

2016
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

Morning Session
Friday, 12 August 2016

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- Board-approved calculators may be used
- A Formula Reference Sheet is provided
- In Questions 11 – 14, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and Student Number at the top of this page

Total marks – 70

Section I Pages 2 – 5

10 marks
- Attempt Questions 1 – 10
- Allow 15 minutes for this section

Section II Pages 6 – 12

60 marks
- Attempt Questions 11 – 14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the Multiple-Choice Answer Sheet for Questions 1–10.

1 Simplify $\sin(A+B)\cos A-\cos(A+B)\sin A$.

(A) $\sin B$

(B) $\cos B$

(C) $\sin(2A+B)$

(D) $\cos(2A+B)$

2 The point P divides the interval joining $A(1,-3)$ to $B(3,4)$ externally in the ratio 3:2. What is the x-coordinate of P?

(A) -3

(B) $\frac{8}{5}$

(C) $\frac{11}{5}$

(D) 7

3 Which one of the following is an expression for $\int \frac{3}{\sqrt{1-16x^2}}\,dx$?

(A) $3\sin^{-1}(4x)+C$

(B) $3\cos^{-1}(4x)+C$

(C) $\frac{3}{4}\sin^{-1}(4x)+C$

(D) $\frac{3}{4}\cos^{-1}(4x)+C$

4 What is the size of the acute angle between the lines whose equations are $y=3x-1$ and $x+2y-3=0$?
Give the answer correct to the nearest degree.

(A) $45°$

(B) $54°$

(C) $79°$

(D) $82°$

5 In how many different ways can all the letters in the word COMMONWEALTH be arranged if C occupies the first position and H occupies the last position?

(A) 151 200

(B) 907 200

(C) 119 750 400

(D) 479 001 600

6 $\dfrac{n!}{r!(n-r)!}$ is equivalent to

(A) $\binom{n}{r}+\binom{n-1}{r-1}$

(B) $\binom{n-1}{r}+\binom{n-1}{r-1}$

(C) $\binom{n+1}{r}+\binom{n-1}{r-1}$

(D) $\binom{n}{r+1}+\binom{n-1}{r-1}$

7 A body is moving with simple harmonic motion.
Its velocity, v ms^{-1} at displacement x metres is given by $v^2=36-4x^2$.

Which one of the following statements is correct?

(A) Period $=\dfrac{\pi}{2}$ seconds, amplitude $=3$ metres

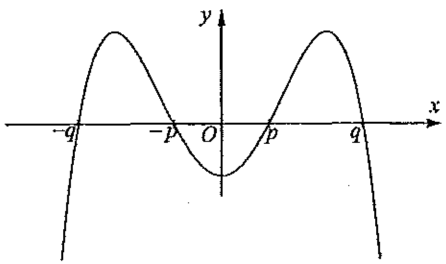
(B) Period $=\dfrac{\pi}{2}$ seconds, amplitude $=9$ metres

(C) Period $=\pi$ seconds, amplitude $=3$ metres

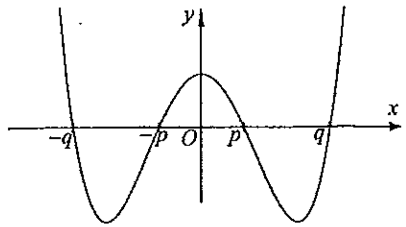
(D) Period $=\pi$ seconds, amplitude $=9$ metres

8 Which one of the following graphs best represents $f(x)=(p^2-x^2)(x-q)^2$ where $q>p$?

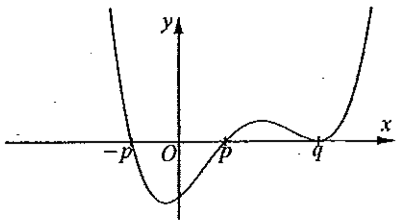
(A)



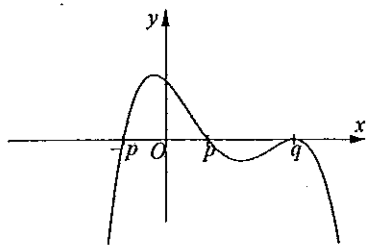
(B)



(C)



(D)



9 How many solutions are there to the equation $\sin 2x = \sin x$ where $0 < x < 2\pi$?

(A) 2

(B) 3

(C) 4

(D) 5

10 When a polynomial $P(x)$ is divided by $(x-2)$ and $(x+1)$, the respective remainders are 5 and 8.

What is the remainder when $P(x)$ is divided by $(x-2)(x+1)$?

(A) $7-x$

(B) $x-7$

(C) $x+7$

(D) $-7-x$

End of Section I

Section II

60 marks
Attempt Questions 11 - 14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11-14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Differentiate $x^2 \cos^{-1}(2x)$. **2**

(b) Solve $\dfrac{x^2+x-20}{x} \geq 2$. **3**

(c) Prove that $\dfrac{2\sin^3\theta + 2\cos^3\theta}{\sin\theta + \cos\theta} = 2 - \sin 2\theta$, where $\sin\theta + \cos\theta \neq 0$. **2**

(d) Find the Cartesian equation of a curve whose parametric equations are $x = 2\cos\theta$ and $y = \sqrt{3}\sin\theta$ for $0 \leq \theta \leq 2\pi$. **2**

(e) Evaluate $\displaystyle\int_0^2 4x\sqrt{1-\frac{x}{2}}\,dx$ using the substitution $u = 1 - \dfrac{x}{2}$. **3**

(f) Find the exact value of $\sin\left[\cos^{-1}\dfrac{2}{3} + \tan^{-1}(-\dfrac{3}{4})\right]$. **3**

End of Question 11

Question 12 (15 marks)

Use a SEPARATE writing booklet.

(a) Find $\lim_{x \to 0} \frac{\sin \frac{x}{2}}{4x}$. **1**

(b) Prove by mathematical induction that $3^{2n} - 2^n$ is divisible by 7 for all integers $n \geq 1$. **3**

(c) The population of a colony of animals may be modelled by the equation $P = 1000 + Ae^{kt}$ where P is the population after t years and A and k are constants.

Initially, the population was 200 animals.
After 1 year, the population had increased to 300 animals.

(i) Evaluate the constants A and k. **2**

(ii) Determine the population of the animals after 10 years.
Give your answer correct to the nearest 10 animals. **1**

(d) Triangle ABC is inscribed in a circle.
The tangent at A meets the side BC produced at D.
The bisector of the angle BAC meets the chord BC at E.

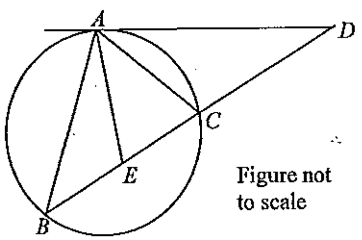


Figure not to scale

(i) Prove that $DA = DE$. **2**

(ii) Given that $BD = 10$ cm and $CD = 6$ cm, find the length of DE. **1**

Question 12 continues on page 8

(e) Four adults and four children are to be seated around a circular table. A particular child cannot sit next to any adult and a particular adult cannot sit next to any child. **2**

Find how many such arrangements are possible.

(f) (i) By using the expansion of $(1+x)^n$, show that **1**

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \dots\dots\dots + n\binom{n}{n}x^{n-1}.$$

(ii) Hence, deduce that **2**

$$n(3^{n-1} - 1) = 4\binom{n}{2} + 12\binom{n}{3} + \dots\dots\dots + n\binom{n}{n}2^{n-1}.$$

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) In a recently conducted poll before an election, 30% of voters intend to vote for Party B. **2**

Ten voters are interviewed at random.

Find the probability that at least two of them intend to vote for Party B. Give your answer correct to two decimal places.

(b) Consider the function $f(x)=2\tan x$ where $-\frac{\pi}{2}<x<\frac{\pi}{2}$.

(i) Sketch the graph of $y=f(x)$. **1**

(ii) Find the inverse function, $f^{-1}(x)$, and state its domain. **2**

(iii) Hence, or otherwise, find the area of the region bounded by the curve $y=f^{-1}(x)$, the x-axis and the lines $x=0$ and $x=2$. Express your answer as an exact value. **3**

(c) The points $P(2ap,ap^2)$ and $Q(2aq,aq^2)$ are two points on the parabola $x^2=4ay$ such that PQ is a focal chord.
The normals at P and Q intersect at R.

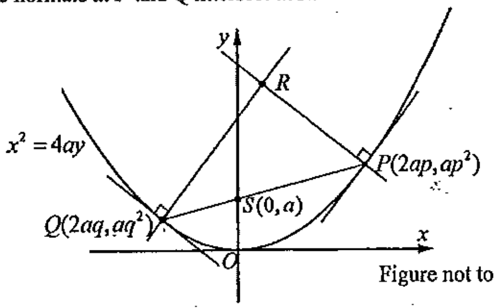


Figure not to scale

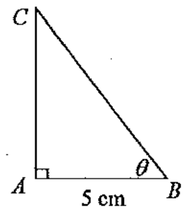
(i) Show that R is the point $(-apq(p+q),a(p^2+q^2+pq+2))$. **2**

(ii) Hence show that the equation of the locus of R as P and Q move on the parabola is given by $x^2=a(y-3a)$. **2**

Question 13 continues on page 10

Question 13 continued

(d) The triangle ABC is right angled at A as shown in the diagram. **3**
Let $AB=5$ cm and $\angle ABC=\theta$ radians.



If θ is increasing at a constant rate of 0.1 radians per second, find the rate at which the perimeter of the triangle is increasing when $\theta=\frac{\pi}{3}$ radians.

Leave your answer in exact form.

End of Question 13

Question 14 (15 marks)

Use a SEPARATE writing booklet.

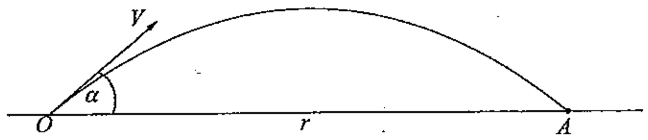
(a) Let $f(x)=x^2-\cos x$.

(i) Show that $f(x)$ has a zero between $x=0.7$ and $x=0.9$. **1**

(ii) Use one application of Newton's method with $x_1=0.8$ to find another approximation to the zero of $f(x)$. Give your answer correct to 2 significant figures. **2**

(b) Find the coefficient of x^6 in the expansion of $(1-4x)^2(2+3x^2)^6$. **2**

(c) A particle is projected from a point O with a velocity V ms^{-1} at an angle α to the horizontal.
It is subject only to the acceleration due to gravity.
The particle lands at a point A, where OA is horizontal.



(Figure not to scale.)

The equations of motion of a projectile fired from the origin with initial velocity V ms^{-1} at angle θ to the horizontal are

$x=Vt\cos\theta$ and $y=Vt\sin\theta-\dfrac{gt^2}{2}$. (Do NOT prove this.)

(i) Show that if $OA=r$, then $r=\dfrac{V^2\sin 2\alpha}{g}$. **2**

(ii) Show that if V is unchanged but the angle of projection is $\left(\dfrac{\pi}{2}-\alpha\right)$ then r remains unchanged. **1**

(iii) It is given that h_1 is the maximum height corresponding to an angle of projection of α and h_2 is the maximum height corresponding to an angle of projection of $\left(\dfrac{\pi}{2}-\alpha\right)$. **3**

If $h_2=3h_1$, find the value of α.

Question 14 continues on page 12

Question 14 continued

(d) The diagram below shows a vertical pole of height h metres.
The angle of elevation of P from A is $\dfrac{\pi}{4}$ radians.
B is a variable point so that $\angle AOB$ is $\dfrac{\pi}{3}$ radians.
The angle of elevation of P from B is α radians and $\angle APB$ is θ radians.

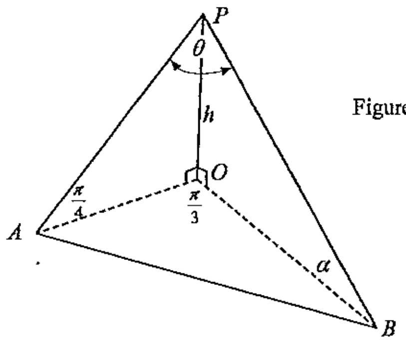


Figure not to scale

(i) By considering the triangle PAB, show that
$AB^2=2h^2+h^2\operatorname{cosec}^2\alpha-2\sqrt{2}\,h^2\operatorname{cosec}\alpha\cos\theta$. **2**

(ii) Hence, show that $\cos\theta=\dfrac{\sqrt{2}}{4}(2\sin\alpha+\cos\alpha)$. **2**

End of Paper

EXAMINERS

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CATHOLIC SECONDARY SCHOOLS ASSOCIATION OF NSW
2016 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION
MATHEMATICS EXTENSION 1 - MARKING GUIDELINES

Section I

10 marks
Questions 1-10 (1 mark each)

Question 1 (1 mark)
Outcomes Assessed: PE 2
Targeted Performance Bands: E2

Solution	Mark
$\sin(A+B)\cos A-\cos(A+B)\sin A$ $=\sin(A+B-A)$ $=\sin B$ Hence (A)	1

Question 2 (1 mark)
Outcomes Assessed: PE2
Targeted Performance Bands: E2

Solution	Mark
$x_1=1 \quad x_2=3 \quad m=3 \quad n=-2$ Using $x=\dfrac{mx_2+nx_1}{m+n}$ $=\dfrac{3\times3-2\times1}{3-2}$ $=7$ Hence (D)	1

6300-2

Question 3 (1 mark)
Outcomes Assessed: HE4
Targeted Performance Bands: E2-E3

Solution	Mark
$\displaystyle\int\frac{3}{\sqrt{1-16x^2}}\,dx$ $\displaystyle=3\int\frac{1}{\sqrt{16(\frac{1}{16}-x^2)}}\,dx$ $=\dfrac{3}{4}\sin^{-1}(4x)+C$ Hence (C)	1

Question 4 (1 mark)
Outcomes Assessed: PE3
Targeted Performance Bands: E2

Solution	Mark
$m_1=3, m_2=-\dfrac{1}{2}$ $\tan\theta=\left\|\dfrac{m_1-m_2}{1+m_1m_2}\right\|$ $=\left\|\dfrac{3-(-\frac{1}{2})}{1+3\times(-\frac{1}{2})}\right\|$ $=7$ $\theta=81.86^\circ$ $\approx 82^\circ$ Hence (D)	1

Question 5 (1 mark)
Outcomes Assessed: PE3
Targeted Performance Bands: E2

Solution	Mark
COMMONWEALTH C and H do not move so ignore them. Of the remaining 10 letters, M and O both appear twice. $\therefore$ No of different arrangements $= \dfrac{10!}{2!2!}$ $= 907\,200$ Hence (B)	1

Question 6 (1 mark)
Outcomes Assessed: HE3
Targeted Performance Bands: E2-E3

Solution	Mark
By inspection, we are looking at the Pascal triangle relationship between coefficients. Hence (B)	1

Question 7 (1 mark)
Outcomes Assessed: HE3
Targeted Performance Bands: E2-E3

Solution	Mark
$v^2 = 36 - 4x^2$ $= 4(9 - x^2)$ This is in the form $v^2 = n^2(a^2 - x^2)$ Hence $n = 2$ and $a = 3$ Period $= \dfrac{2\pi}{n}$ $= \pi$ seconds Amplitude $= a$ $= 3$ metres Hence (C)	1

Question 8 (1 mark)
Outcomes Assessed: HE3
Targeted Performance Bands: E2-E3

Solution	Mark
f(x) has one double root and two single roots and its leading term is $-x^4$. Hence (D)	1

Question 9 (1 mark)
Outcomes Assessed: PE3
Targeted Performance Bands: E3

Solution	Mark
$\sin 2x = \sin x$ $2\sin x\cos x - \sin x = 0$ $\sin x(2\cos x - 1) = 0$ $\sin x = 0, \cos x = \dfrac{1}{2}$ For $0 < x < 2\pi$, $\sin x = 0$ when $x = \pi$ $\cos x = \dfrac{1}{2}$ when $x = \dfrac{\pi}{3}$ and $x = \dfrac{5\pi}{3}$ Hence (B)	1

Question 10 (1 mark)
Outcomes Assessed: PE3
Targeted Performance Bands: E3-E4

Solution	Mark
Let $P(x) = (x-2)(x+1)Q(x) + Ax + B$ $P(2) = 5 \quad 2A + B = 5 \quad (1)$ $P(-1) = 8 \quad -A + B = 8 \quad (2)$ $(1)-(2) \quad 3A = -3$ $A = -1 \quad (3)$ Sub (3) in (1) $\quad -2 + B = 5$ $B = 7$ Hence (A)	1

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Section II

60 marks

Question 11 (15 marks)

(a) (2 marks)
Outcomes Assessed: HE4
Targeted Performance Bands: E2-E3

Criteria	Marks
* Correct solution	2
* Shows some correct derivative	1

Sample Answer:

$$\frac{d}{dx}(x^2\cos^{-1}(2x))$$

$$=\cos^{-1}(2x)\times 2x + x^2\times\frac{-1}{\sqrt{1-(2x)^2}}\times 2$$

$$=2x\cos^{-1}(2x)-\frac{2x^2}{\sqrt{1-4x^2}}$$

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(b) (3 marks)
Outcomes Assessed: PE3
Targeted Performance Bands: E2-E3

Criteria	Marks
* Correct solution	3
* Obtains three correct zeros	2
* Shows some correct working	1

Sample Answer:

$$\frac{x^2+x-20}{x}\geq 2 \qquad x\neq 0$$

$$\frac{x^2(x^2+x-20)}{x}\geq 2x^2$$

$$x(x^2+x-20)\geq 2x^2$$

$$x^3+x^2-20x\geq 2x^2$$

$$x^3-x^2-20x\geq 0$$

$$x(x^2-x-20)\geq 0$$

$$x(x-5)(x+4)\geq 0$$

Zeros are $-4,0,5$

Test $x=1$: $\frac{1^2+1-20}{1}=-18 \rightarrow -18\geq 2$ is false

$\therefore$ Solutions are $-4\leq x<0$ and $x\geq 5$.

(c) (2 marks)
Outcomes Assessed: HE7
Targeted Performance Bands: E2-E3

Criteria	Marks
* Correct solution	2
* Factorises $\sin^3\theta+\cos^3\theta$	1

Sample Answer:

$$\frac{2\sin^3\theta+2\cos^3\theta}{\sin\theta+\cos\theta}$$

$$=\frac{2(\sin\theta+\cos\theta)(\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta)}{\sin\theta+\cos\theta}$$

$$=2(1-\sin\theta\cos\theta)$$

$$=2-2\sin\theta\cos\theta$$

$$=2-\sin 2\theta\text{, as required.}$$

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(d) (2 marks)
Outcomes assessed: PE4
Targeted Performance Bands: E2

Criteria	Marks
* Correct solution	2
* Establishes correct expressions for sin θ and cos θ	1

Sample Answer:

$x = 2\cos\theta \quad \rightarrow \frac{x}{2} = \cos\theta \quad (1)$

$y = \sqrt{3}\sin\theta \quad \rightarrow \frac{y}{\sqrt{3}} = \sin\theta \quad (2)$

Square and add (1) and (2)

$$\frac{x^2}{4} + \frac{y^2}{3} = \cos^2\theta + \sin^2\theta$$

$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

(e) (3 marks)
Outcomes assessed: HE6
Targeted Performance Bands: E3

Criteria	Marks
* Correct solution	3
* Demonstrates significant progress towards answer	2
* Obtains limit values for u	1

Sample Answer:

$$I = \int_0^2 4x\sqrt{1-\frac{x}{2}}\,dx$$

$$= -8\int_1^0 (2-2u)\sqrt{u}\,du$$

$$= -16\int_1^0 (1-u)\sqrt{u}\,du$$

$$= 16\int_0^1 (u^{\frac{1}{2}} - u^{\frac{3}{2}})\,du$$

$$= 16\left[\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right]_0^1$$

$$= 16(\frac{2}{3} - \frac{2}{5})$$

$$= 16 \times \frac{4}{15}$$

$$= \frac{64}{15}$$

$u = 1 - \frac{x}{2}$

$\frac{du}{dx} = -\frac{1}{2}$

$dx = -2\,du$

$x = 2, u = 0$

$x = 0, u = 1$

$x = 2 - 2u$

(f) (3 marks)
Outcomes assessed: HE4
Targeted Performance Bands: E3-E4

Criteria	Marks
* Correct solution	3
* Expands sin $(A-B)$ and substitutes correctly	2
* Finds correct values for $\sin A$, $\sin B$ and $\cos B$	1

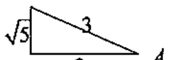
Sample Answer:

Let $E=\sin\left[\cos^{-1}\frac{2}{3}+\tan^{-1}(-\frac{3}{4})\right]$

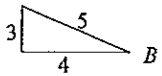
$=\sin\left[\cos^{-1}\frac{2}{3}-\tan^{-1}\frac{3}{4}\right]$

Let $A=\cos^{-1}\frac{2}{3}$ and $B=\tan^{-1}\frac{3}{4}$

$\cos A=\frac{2}{3}$ and $\tan B=\frac{3}{4}$



Missing side is $\sqrt{5}$



Missing side is 5

$E=\sin(A-B)$

$=\sin A\cos B-\cos A\sin B$

$\sin A=\frac{\sqrt{5}}{3}, \cos A=\frac{2}{3}$

$\sin B=\frac{3}{5}, \cos B=\frac{4}{5}$

$\therefore E=\frac{\sqrt{5}}{3}\times\frac{4}{5}-\frac{2}{3}\times\frac{3}{5}$

$=\frac{4\sqrt{5}-6}{15}$

Question 12 (15 marks)

(a) (1 mark)
Outcomes Assessed: HE4
Targeted Performance Bands: E2-E3

Criteria	Mark
* Correct answer	1

Sample Answer:

$$\lim_{x\to 0}\frac{\sin\frac{x}{2}}{4x}$$

$$=\lim_{x\to 0}\frac{\sin\frac{x}{2}}{4\times 2\times\frac{x}{2}}$$

$$=\frac{1}{8}\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}$$

$$=\frac{1}{8},\ \text{since}\ \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$$

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(b) (3 marks)
Outcomes Assessed: HE2
Targeted Performance Bands: E3-E4

Criteria	Marks
* Correct solution	3
* Demonstrates significant progress towards solution	2
* Shows $n = 1$ is true.	1

Sample Answer:

Let $P(n)$ be the proposition that $3^{2n} - 2^n$ is divisible by 7.

$P(1) = 3^2 - 2^1 = 7$

$\therefore P(1)$ is true.

Assume that $P(k)$ is true.

ie Assume that $3^{2k} - 2^k = 7Q$ where Q is an integer.

Need to show that $P(k+1)$ is true.

ie Need to show that $3^{2(k+1)} - 2^{k+1} = 7R$ where R is an integer.

$$\begin{aligned}3^{2(k+1)} - 2^{k+1} &= 3^{2k}3^2 - 2^k 2\\ &= 9(2^k + 7Q) - 2^k 2\\ &= 7\times 2^k + 7\times 9Q\\ &= 7(2^k + 9Q)\end{aligned}$$

$\therefore$ If $P(k)$ is true then $P(k+1)$ is true.

$\therefore$ By mathematical induction $P(n)$ is true for all integers $n \ge 1$.

(c) (i) (2 marks)
Outcomes Assessed: HE3
Targeted Performance Bands: E3

Criteria	Marks
* Correct solution	2
* Finds correctly either A or k	1

Sample Answer:

$P = 1000 + Ae^{kt}$

When $t = 0$, $P = 200$.

$\therefore 200 = 1000 + A$

$A = -800$

When $t = 1$, $P = 300$

$\therefore 300 = 1000 - 800e^k$

$800e^k = 700$

$e^k = \dfrac{7}{8}$

$k = \log_e \dfrac{7}{8}$

(c) (ii) (1 mark)
Outcomes Assessed: HE3
Targeted Performance Bands: E2

Criteria	Mark
* Correct answer	1

Sample Answer:

$$\begin{aligned}\text{When } t = 10,\ P &= 1000 - 800e^{10\log_e \frac{7}{8}}\\ &= 789.54\\ &\approx 790 \quad \text{(nearest 10 animals)}\end{aligned}$$

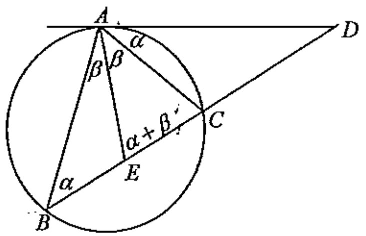
(d) (i) (2 marks)

Outcomes Assessed: PE2

Targeted Performance Bands: E2-E3

Criteria	Marks
* Correct solution	2
* Shows some correct working	1

Sample Answer:



$\angle DAC = \angle ABC \ (= \alpha)$ (Angle between a tangent and chord is equal to the angle in the alternate segment.)

$\angle BAE = \angle CAE \ (= \beta)$ (Given AE bisects $\angle BAC$.)

$\angle AEC = \angle ABE + \angle BAE \ (= \alpha + \beta)$ (Exterior angle of a triangle is equal to the sum of the opposite interior angles.)

$\angle DAE = \angle DAC + \angle CAE \ (= \alpha + \beta)$

$\therefore \angle DAE = \angle DEA$ and hence ΔDAE is isosceles.

$\therefore DA = DE$ (Corresponding sides in an isosceles triangle)

(d) (ii) (1 mark)

Outcomes Assessed: PE2

Targeted Performance Bands: E2-E3

Criteria	Mark
* Correct answer	1

Sample Answer:

$AD^2 = BD \times DC$ (The square of the length of a tangent from an external point is equal to the product of intercepts of the secant passing throught the point.)

$\therefore AD^2 = 10 \times 6$

$AD = \sqrt{60}$

$\quad = 2\sqrt{15}$

From (i), $DE = DA$

$\therefore DE = 2\sqrt{15}$ cm.

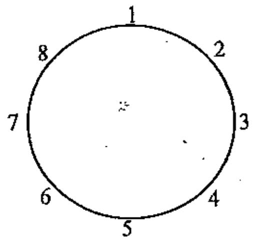
(e) (2 marks)

Outcomes Assessed: PE3

Targeted Performance Bands: E3-E4

Criteria	Marks
* Correct solution	2
* Demonstrates significant progress towards answer	1

Sample Answer:



Seat the particular child anywhere (say seat 1)

Then fill positions 2 and 8 with children in 3×2 ways

Then seat the particular adult in any 1 of 3 positions (4,5 or 6) in 3 ways

Fill the seats on either side of the particular adult with adults in 3×2 ways

Finally, fill the remaining 2 seats in 2×1 ways.

$\therefore$ Total number of arrangements $= 3\times2\times3\times3\times2\times2$

$= 216$

(f) (i) (1 mark)

Outcomes Assessed: HE2

Targeted Performance Bands: E3

Criteria	Mark
* Differentiates both sides correctly	1

Sample Answer:

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots\ldots\ldots + \binom{n}{n}x^n$$

Differentiating both sides

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \ldots\ldots\ldots + n\binom{n}{n}x^{n-1}, \text{ as required.}$$

(f) (ii) (2 marks)
Outcomes Assessed: HE2
Targeted Performance Bands: E3

Criteria	Marks
* Correct solution	2
* Substitutes $x = 2$ into both sides of identity	1

Sample Answer:

From (i)

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \dots\dots\dots + n\binom{n}{n}x^{n-1}$$

Let $x = 2$.

$$n(3)^{n-1} = \binom{n}{1} + 2\binom{n}{2}2 + 3\binom{n}{3}4 + \dots\dots\dots + n\binom{n}{n}2^{n-1}$$

$$n(3)^{n-1} - \binom{n}{1} = 4\binom{n}{2} + 12\binom{n}{3} + \dots\dots\dots + n\binom{n}{n}2^{n-1}$$

$$n(3)^{n-1} - n = 4\binom{n}{2} + 12\binom{n}{3} + \dots\dots\dots + n\binom{n}{n}2^{n-1}$$

$$n(3^{n-1} - 1) = 4\binom{n}{2} + 12\binom{n}{3} + \dots\dots\dots + n\binom{n}{n}2^{n-1}, \text{as required.}$$

Question 13 (15 marks)

(a) (2 marks)
Outcomes Assessed: HE3
Targeted Performance Bands: E2-E3

Criteria	Marks
* Correct solution	2
* Obtains expression for probability of at least two votes for Party B	1

Sample Answer:

$P(\text{Vote for Party B}) = 0.3$

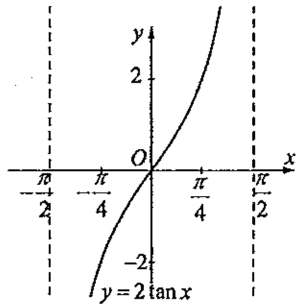
$P(\text{Not vote for Party B}) = 0.7$

$$P(\text{At least two votes for Party B}) = 1 - P(\text{None or one vote for Party B})$$
$$= 1 - (0.7^{10} + {}^{10}C_1 \times 0.7^9 \times 0.3)$$
$$= 0.85$$

(b) (i) (1 mark)
Outcomes Assessed: HE4
Targeted Performance Bands: E2

Criteria	Mark
* Correctly labelled diagram	1

Sample Answer:



(b) (ii) (2 marks)
Outcomes Assessed: HE4
Targeted Performance Bands: E3

Criteria	Marks
* Correct solution	2
* Finds the inverse function	1

Sample Answer:

Inverse is $x = 2\tan y \rightarrow \tan y = \frac{x}{2}$

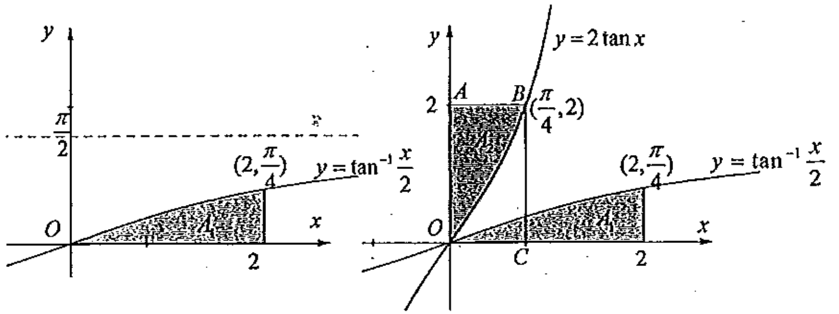
$\therefore f^{-1}(x) = \tan^{-1}(\frac{x}{2})$

Domain of $f^{-1}(x)$ is all real numbers.

(b) (iii) (3 marks)
Outcomes Assessed: HE4
Targeted Performance Bands: E3-E4

Criteria	Marks
* Correct solution	3
* Demonstrates significant progress towards solution	2
* Recognises area bounded by $y = 2\tan x$ and its corresponding inverse area are equal	1

Sample Answer:



Let A_1 be the area bounded by $y = \tan^{-1}\frac{x}{2}$, the x-axis, and the lines $x = 0$ and $x = 2$.

Let A_2 be the area bounded by $y = 2\tan x$, the y-axis and the lines $y = 0$ and $y = 2$.

Hence $A_1 = A_2$.

A_2 = Area of rectangle $OABC$ − area bounded by $y = 2\tan x$, the x-axis, $x = 0$ and $x = \frac{\pi}{4}$

$$= \frac{\pi}{4} \times 2 - \int_0^{\frac{\pi}{4}} 2\tan x \, dx$$

$$= \frac{\pi}{2} - 2\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} \, dx$$

$$= \frac{\pi}{2} + 2\left[\log_e(\cos x)\right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{2} + 2\left[\log_e(\frac{1}{\sqrt{2}}) - \log_e(1)\right]$$

$$= \frac{\pi}{2} + 2\left[\log_e(2^{-\frac{1}{2}})\right]$$

$$= \frac{\pi}{2} - \log_e 2$$

(c) (i) (2 marks)
Outcomes Assessed: PE3
Targeted Performance Bands: E3

Criteria	Marks
* Correct solution	2
* Find correct expression for either x or y	1

Sample Answer:

(i) Normal at P : $x+py=ap^3+2ap$ (1)

Normal at Q : $x+qy=aq^3+2aq$ (2)

(1)−(2) $(p-q)y=a(p^3-q^3)+2a(p-q)$

$=a(p-q)(p^2+pq+q^2)+2a(p-q)$

$y=a(p^2+pq+q^2)+2a$

$=a(p^2+q^2+pq+2)$ (3)

Sub (3) in (1) $x+pa(p^2+q^2+pq+2)=ap^3+2ap$

$x+ap^3+apq^2+ap^2q+2ap=ap^3+2ap$

$x=-ap^2q-apq^2$

$=-apq(p+q)$

$\therefore R$ is $(-apq(p+q),a(p^2+q^2+pq+2))$ as required.

(c) (ii) (2 marks)
Outcomes Assessed: PE3
Targeted Performance Bands: E4

Criteria	Marks
* Correct solution	2
* finds correct expressions for $\frac{x}{a}$ or $\frac{y}{a}$	1

Sample Answer:

At R, $x=-apq(p+q)$

$y=a(p^2+q^2+pq+2)$

PQ is a focal chord so $pq=-1$

$\therefore$ At R, $x=a(p+q)$

$y=a(p^2+q^2+1)$

At R, $\frac{x}{a}=p+q$ (1)

$\frac{y}{a}=p^2+q^2+1$ (2)

Squaring (1) $\frac{x^2}{a^2}=p^2+q^2+2pq$

$\frac{x^2}{a^2}=p^2+q^2-2$ (3)

(3)−(2) $\frac{x^2}{a^2}-\frac{y}{a}=-3$

$x^2-ay=-3a^2$

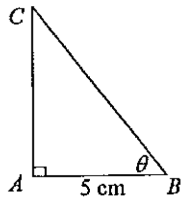
$x^2=ay-3a^2$

$x^2=a(y-3a)$, as required.

(d) (3 marks)
Outcomes Assessed: HE3
Targeted Performance Bands: E3-E4

Criteria	Marks
* Correct solution	3
* Finds $\frac{dP}{d\theta}$	2
* Finds correct perimeter expression	1

Sample Answer:



Let P represent the perimeter.

$P = AB + AC + BC$

$\tan\theta = \frac{AC}{5} \quad \rightarrow \quad AC = 5\tan\theta$

$\cos\theta = \frac{5}{BC} \quad \rightarrow BC = \frac{5}{\cos\theta} = 5\sec\theta$

$\therefore P = 5 + 5\tan\theta + 5\sec\theta$

$\quad = 5 + 5\tan\theta + 5(\cos\theta)^{-1})$

$\frac{dP}{d\theta} = 5\sec^2\theta - 5(-1)(\cos\theta)^{-2}(-\sin\theta)$

$\quad = 5\sec^2\theta + 5\frac{\sin\theta}{\cos^2\theta}$

$\quad = 5\sec^2\theta + 5\tan\theta\sec\theta$

$\frac{d\theta}{dt} = 0.1$

$\frac{dP}{dt} = \frac{d\theta}{dt} \times \frac{dP}{d\theta}$

When $\theta = \frac{\pi}{3}, \frac{dP}{dt} = 0.1 \times 5\sec^2\frac{\pi}{3} + 5\tan\frac{\pi}{3}\sec\frac{\pi}{3}$

$\sec\frac{\pi}{3} = 2, \tan\frac{\pi}{3} = \sqrt{3}$

$\therefore \frac{dP}{dt} = \frac{1}{10} \times (20 + 10\sqrt{3})$

$\quad = 2 + \sqrt{3}$ cm per second.

Question 14 (15 marks)

(a) (i) (1 mark)
Outcomes Assessed: HE3
Targeted Performance Bands: E2

Criteria	Mark
* Correct solution	1

Sample Answer:

$f(x) = x^2 - \cos(x)$

$f(0.7) = 0.7^2 - \cos(0.7)$

$\quad = -0.274$

$f(0.9) = 0.9^2 - \cos(0.9)$

$\quad = 0.188$

$f(x)$ is continuous for $0.7 \le x \le 0.9$ and has changed sign, then $f(x)$ must have a zero between 0.7 and 0.9

(a) (ii) (2 marks)
Outcomes Assessed: HE3
Targeted Performance Bands: E2-E3

Criteria	Marks
* Correct solution	2
* Demonstrates some correct working	1

Sample Answer:

By Newton's method,

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$\quad = x_1 - \frac{x^2 - \cos x}{2x + \sin x}$

$\quad = 0.8 - \frac{0.8^2 - \cos 0.8}{2 \times 0.8 + \sin 0.8}$

$\quad = 0.824....$

$\therefore$ Next approximation is 0.82, to two significant figures.

(b) (2 marks)
Outcomes Assessed: HE2
Targeted Performance Bands: E3-E4

Criteria	Marks
* Correct solution	2
* Expands correctly $(2+3x^2)^6$	1

Sample Answer:

$(1-4x)^2(2+3x^2)^6$

$=(1-8x+16x^2)(2^6+\binom{6}{1}2^5(3x^2)+\binom{6}{2}2^4(3x^2)^2+\binom{6}{3}2^3(3x^2)^3+.....+(3x^2)^6)$

Term in $x^6 \ = 16x^2\times\binom{6}{2}2^4(3x^2)^2+1\times\binom{6}{3}2^3(3x^2)^3$

$=(16\times\binom{6}{2}2^4\times3^2+\binom{6}{3}2^3(3)^3)x^6$

Coefficient of $x^6=16\times15\times16\times9+20\times8\times27$

$=38\ 880$

(c) (i) (2 marks)
Outcomes Assessed: HE3
Targeted Performance Bands: E3-E4

Criteria	Marks
* Correct solution	2
* Finds correct expression for t for time of flight	1

Sample Answer:

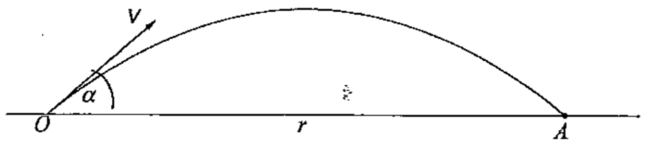


Figure not to scale.

$y=Vt\sin\theta-\frac{1}{2}gt^2$

At end of flight, $y=0$.

$\therefore Vt\sin\theta-\frac{1}{2}gt^2=0$

$t(V\sin\theta-\frac{1}{2}gt)=0$

$\therefore$ At end of flight, $t=\frac{2V\sin\theta}{g}$.

$x=Vt\cos\theta$

At end of flight, $x=V(\frac{2V\sin\theta}{g})\cos\theta$

$=\frac{2V^2\sin\theta\cos\theta}{g}$

$=\frac{V^2\sin2\theta}{g}$

If angle of projection is α, then range $r\ =\ \frac{V^2\sin2\alpha}{g}$, as required.

(c) (ii) (1 mark)
Outcomes Assessed: HE3
Targeted Performance Bands: E3-E4

Criteria	Mark
* Correct solution	1

Sample Answer:

From (i), Range $= \dfrac{V^2 \sin 2\theta}{g}$, where θ is the angle of projection.

$$\text{If } \theta = \frac{\pi}{2} - \alpha, \text{ Range} = \frac{V^2 \sin 2(\frac{\pi}{2} - \alpha)}{g}$$

$$= \frac{V^2 \sin(\pi - 2\alpha)}{g}$$

$$= \frac{V^2 \sin 2\alpha}{g}$$

$= r$, as required.

(c) (iii) (3 marks)
Outcomes Assessed: HE3
Targeted Performance Bands: E4

Criteria	Marks
* Correct solution	3
* Demonstrates significant progress towards answer	2
* Finds a maximum height	1

Sample Answer:

$$y = Vt\sin\theta - \frac{1}{2}gt^2$$

$$\dot{y} = V\sin\theta - gt$$

At maximum height, $\dot{y} = 0$

$\therefore$ At maximum height, $t = \dfrac{V\sin\theta}{g}$

Maximum height is $y = V\left(\dfrac{V\sin\theta}{g}\right)\sin\theta - \dfrac{1}{2}g\left(\dfrac{V\sin\theta}{g}\right)^2$

$$= \frac{V^2\sin^2\theta}{2g}$$

h_1 is maximum height when angle of projection is α

$$\therefore h_1 = \frac{V^2\sin^2\alpha}{2g}$$

h_2 is maximum height when angle of projection is $(\frac{\pi}{2} - \alpha)$

$$\therefore h_2 = \frac{V^2\sin^2(\frac{\pi}{2} - \alpha)}{2g}$$

$$= \frac{V^2\cos^2\alpha}{2g}$$

$$h_2 = 3h_1$$

$$\therefore \frac{V^2\cos^2\alpha}{2g} = 3\frac{V^2\sin^2\alpha}{2g}$$

$$\frac{\sin^2\alpha}{\cos^2\alpha} = \frac{1}{3}$$

$$\tan\alpha = \frac{1}{\sqrt{3}}$$

$$\alpha = \frac{\pi}{6}$$

(d) (i) (2 marks)
Outcomes Assessed: PE3
Targeted Performance Bands: E4

Criteria	Marks
* Correct solution	2
* Obtains correct expressions for AP and BP.	1

Sample Answer:

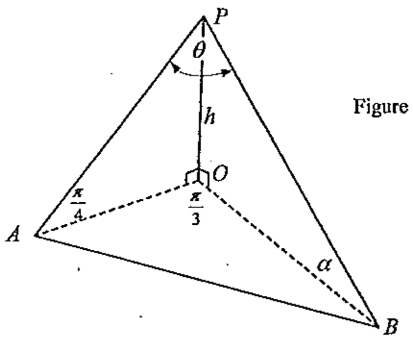


Figure not to scale

In ΔPAB, $AB^2 = AP^2 + BP^2 - 2(AP)(BP)\cos\theta$ (cosine rule)

Now, in ΔPAO, $\sin\frac{\pi}{4} = \frac{h}{AP}$

$$h = AP \times \frac{1}{\sqrt{2}}$$

$$\therefore AP = h\sqrt{2}$$

In ΔPOB, $\sin\alpha = \frac{h}{BP}$

$$h = BP\sin\alpha$$

$$BP = h\operatorname{cosec}\alpha$$

$$\therefore AB^2 = (h\sqrt{2})^2 + (h\operatorname{cosec}\alpha)^2 - 2\times h\sqrt{2}\times h\operatorname{cosec}\alpha\times\cos\theta$$

$$= 2h^2 + h^2\operatorname{cosec}^2\alpha - 2\sqrt{2}\,h^2\operatorname{cosec}\alpha\cos\theta, \text{ as required.}$$

(d) (ii) (2 marks)
Outcomes Assessed: PE3
Targeted Performance Bands: E4

Criteria	Marks
* Correct solution	2
* Obtains correct expression for AB^2	1

Sample Answer:

In ΔAOP, $\tan\frac{\pi}{4} = \frac{h}{AO}$, $AO = h$

In ΔBOP, $\tan\alpha = \frac{h}{BO}$, $BO = \frac{h}{\tan\alpha} = h\cot\alpha$

In ΔABO, $AB^2 = AO^2 + BO^2 - 2(AO)(BO)\cos\frac{\pi}{3}$

$$AB^2 = h^2 + h^2\cot^2\alpha - 2(h)(h\cot\alpha)\frac{1}{2}$$

$$AB^2 = h^2 + h^2\cot^2\alpha - h^2\cot\alpha$$

Hence from (i)

$$2h^2 + h^2\operatorname{cosec}^2\alpha - 2\sqrt{2}\,h^2\operatorname{cosec}\alpha\cos\theta = h^2 + h^2\cot^2\alpha - h^2\cot\alpha$$

$$2h^2 + h^2(1+\cot^2\alpha) - 2\sqrt{2}\,h^2\operatorname{cosec}\alpha\cos\theta = h^2(1+\cot^2\alpha) - h^2\cot\alpha$$

$$-h^2\cot\alpha = 2h^2 - 2\sqrt{2}\,h^2\operatorname{cosec}\alpha\cos\theta$$

$$-\cot\alpha = 2 - 2\sqrt{2}\operatorname{cosec}\alpha\cos\theta$$

$$2\sqrt{2}\operatorname{cosec}\alpha\cos\theta = 2 + \cot\alpha$$

$$2\sqrt{2}\cos\theta = 2\sin\alpha + \cos\alpha$$

$$\cos\theta = \frac{1}{\sqrt{2}}\sin\alpha + \frac{1}{2\sqrt{2}}\cos\alpha$$

$$\cos\theta = \frac{\sqrt{2}}{2}\sin\alpha + \frac{\sqrt{2}}{4}\cos\alpha$$

$$\cos\theta = \frac{\sqrt{2}}{4}(2\sin\alpha + \cos\alpha), \text{ as required.}$$