Centre Number

Student Number

CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NEW SOUTH WALES

2001
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Morning Session
Friday 17 August 2001

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided separately
- All necessary working should be shown in every question

Total marks (**120**)

- Attempt Questions 1 – 8
- All questions are of equal value

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a>0, \quad -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x>0$

Question 1 **Begin a new page** **Marks**

(a) $P(x)=(x+2)(x-1)(x-3)$

(i) Sketch $y=P(x)$ showing the intercepts on the coordinate axes. **1**

(ii) On separate diagrams, sketch the graphs of $y=|P(x)|$, $y=P(|x|)$, $y=\dfrac{1}{P(x)}$ showing the intercepts on the coordinate axes and the equations of any asymptotes. **4**

(b) (i) $P(x_1, y_1)$ is a point on the curve $y=e^{-x}$. The tangent to the curve at P passes through the origin. Find the coordinates of P. **3**

(ii) Find the set of values of the real number k such that the equation $e^{-x}=kx$ has two real and distinct solutions. **2**

(c) Consider the function $f(x)=\ln(1+\cos x)$, $-2\pi \le x \le 2\pi$, where $x \ne \pi$, $x \ne -\pi$.

(i) Show that the function f is even and the curve $y=f(x)$ is concave down for all values of x in its domain. **3**

(ii) Sketch the graph of the curve $y=f(x)$. **2**

Question 2 **Begin a new page**

(a) Find all the complex numbers $z=a+ib$, a, b real, such that $|z|^2-iz=16-2i$. **3**

(b) (i) Find $\displaystyle\int \frac{e^x+1}{e^x}\,dx$. **1**

(ii) Find $\displaystyle\int \frac{x^2+x+1}{x(x^2+1)}\,dx$. **2**

(c) (i) Use the substitution $t=\tan\frac{x}{2}$ to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x+\sin x}\,dx$. **2**

(ii) Hence use the substitution $u=\frac{\pi}{2}-x$ to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{x}{1+\cos x+\sin x}\,dx$. **2**

(d) (i) If $I_n=\displaystyle\int_0^1 (1+x^2)^n\,dx$, $n=0, 1, 2, \ldots$ show that $(2n+1)\,I_n=2^n+2n\,I_{n-1}$ for $n=1, 2, 3, \ldots$ **3**

(ii) Hence find a reduction formula for $J_m=\displaystyle\int_0^{\frac{\pi}{4}} \sec^{2m} x\,dx$ **2**

Question 3 **Begin a new page** **Marks**

(a) In an Argand Diagram, the point P representing the complex number z moves so that $|z-(1+i)|=1$.

(i) Sketch the locus of P. **1**

(ii) Shade the region where $|z-(1+i)|\le 1$ and $0<\arg(z-i)<\frac{\pi}{4}$ **1**

(b) In an Argand Diagram, a regular hexagon $ABCDEF$, with the vertices taken in anticlockwise order, has its centre at the origin O and vertex A at $z=2$.

(i) Find the set of values of $\operatorname{Im}(z)$ for points z on the hexagon. **1**

(ii) Find the set of values of $|z|$ for points z on the hexagon. **1**

(iii) If the hexagon is rotated in a clockwise direction about the origin through an angle of $45°$, find the value in modulus / argument form of the complex number which is represented by the new position of the vertex C. **1**

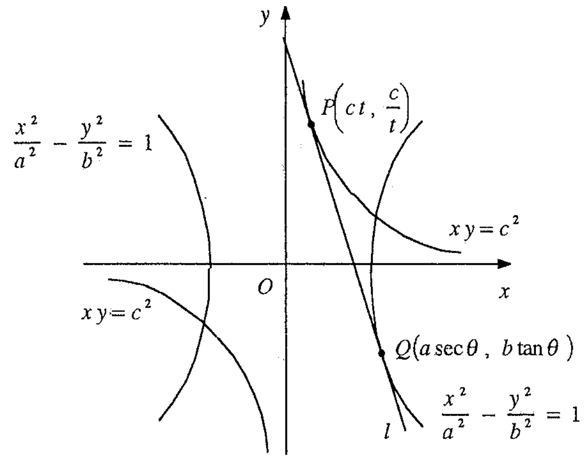
(c) (i) If $z=\cos\theta+i\sin\theta$, show that for positive integers n, $z^n+\dfrac{1}{z^n}=2\cos n\theta$ and $z^n-\dfrac{1}{z^n}=2i\sin n\theta$. Hence expand $\left(z+\dfrac{1}{z}\right)^4+\left(z-\dfrac{1}{z}\right)^4$ to show that $\cos^4\theta+\sin^4\theta=\frac{1}{4}(\cos 4\theta+3)$. **3**

(ii) By letting $x=\cos\theta$, show that the equation $8x^4+8(1-x^2)^2=7$ has roots $\pm\cos\frac{\pi}{12}$, $\pm\cos\frac{5\pi}{12}$. **2**

(iii) Deduce that $\cos\frac{\pi}{12}$, $\cos\frac{5\pi}{12}$ have a product of $\frac{1}{4}$ and a sum of $\sqrt{\frac{3}{2}}$. **3**

(iv) Hence or otherwise find a surd expression for $\cos\frac{\pi}{12}$. **2**

Question 4 **Begin a new page** **Marks**



The line l is a common tangent to the hyperbolas $xy = c^2$, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with points of contact P and Q respectively.

(i) Considering l as a tangent to $xy = c^2$ at $P\left(ct, \frac{c}{t}\right)$, show l has equation $x + t^2 y = 2ct$. **2**

(ii) Considering l as a tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $Q(a\sec\theta, b\tan\theta)$, show l has equation $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$. **2**

(iii) Deduce that $\frac{\sec\theta}{a} = \frac{-\tan\theta}{bt^2} = \frac{1}{2ct}$. **1**

(iv) Write the coordinates of Q in terms of t, a, b and c, and show that $b^2t^4 + 4c^2t^2 - a^2 = 0$. Deduce that there are exactly two such common tangents to the hyperbolas. **3**

(v) Copy the diagram and use the symmetry in the graphs to draw in the second common tangent with points of contact R on $xy = c^2$ and S on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Write the coordinates of R and S in terms of t, a, b and c. **2**

(vi) Show that if $PQRS$ is a rhombus, then $b^2 = a^2$ and deduce that $t^2 < 1$. **2**

(vii) Show that if $PQRS$ is a square, then $t^4 + 2t^2 - 1 = 0$ and deduce that $2c^2 = a^2$. What is the relationship between the two hyperbolas if $PQRS$ is a square? **3**

Question 5 **Begin a new page** **Marks**

(a) $I = \int_0^{\pi} xe^x \cos x \, dx$ and $J = \int_0^{\pi} e^x \cos x \, dx$

(i) Use integration by parts to show that $I - J = -\int_0^{\pi} xe^x \sin x \, dx$. **2**

(ii) Differentiate xe^x and hence find $\int (x+1)e^x \, dx$. Hence or otherwise show that **2**

$$I + J = -\pi e^{\pi} + \int_0^{\pi} xe^x \sin x \, dx.$$

(iii) Evaluate I. **1**

(b) (i) On the same diagram and without using calculus, sketch the graphs of $y = e^{-x}$, $y = -e^{-x}$ and $y = e^{-x}\cos x$, $0 \le x \le 2\pi$. Shade the region bounded by $y = e^{-x}$, $y = e^{-x}\cos x$ and $x = \pi$ for $x \ge 0$. **3**

(ii) The region shaded in (i) is rotated through one revolution about the line $x = \pi$. Use the method of cylindrical shells to show that the volume of the solid of revolution is given by **2**

$$V = 2\pi \int_0^{\pi} (\pi - x)e^{-x}(1 - \cos x) \, dx.$$

(iii) Use the substitution $u = \pi - x$ to show $V = 2\pi e^{-\pi}\left\{\int_0^{\pi} ue^u \, du + I\right\}$, where I is as defined in (a). **2**

(iv) Hence find the volume of the solid. **3**

Question 6 **Begin a new page**

An object of mass m kg is dropped from rest from the top of a cliff 40 m above the water. Before the object reaches the water, the resistance to its motion has magnitude $\frac{1}{10}mv$ when the object has speed v ms^{-1}. After the object enters the water, the resistance to its motion has magnitude $\frac{1}{10}mv^2$. Take $g = 10$ms^{-2}.

(a) (i) Write an expression for $\ddot{x}$ before the object enters the water, where x metres is the distance the object has fallen in t seconds. **1**

(ii) Show $10\frac{dv}{dx} = \frac{100 - v}{v}$, and show that the speed of the object as it enters the water is V ms^{-1} where V satisfies $\frac{V}{100} + \ln\left(1 - \frac{V}{100}\right) + 0{\cdot}04 = 0$. **3**

(iii) Show this equation has a solution for V between 20 and 30, and taking 25 as a first approximation, use Newton's Method to show that $V \approx 25{\cdot}7$ to one decimal place. **3**

Question 6 continued **Marks**

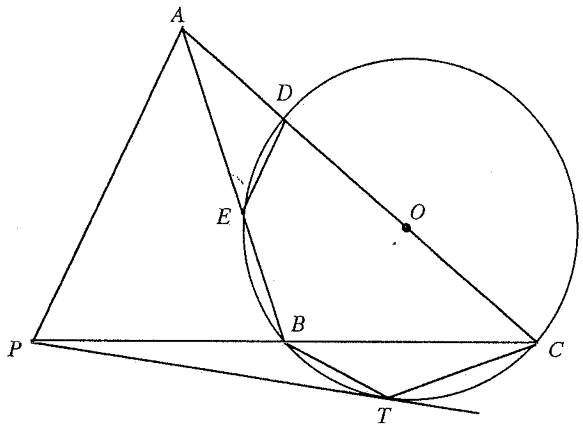
(b) (i) Write an expression for $\ddot{x}$ after the object enters the water. Deduce the object slows on entry to the water, and find its terminal velocity in the water. 3

(ii) Show that t seconds after entering the water $10\frac{dv}{dt} = 100 - v^2$, and the velocity v ms^{-1} of the object is given by $2t = \ln\left\{\frac{(v+10)(V-10)}{(v-10)(V+10)}\right\}$, where V is the velocity on entry to the water calculated in (a). 3

(iii) How long after it enters the water will the body slow to 105% of its terminal velocity? 2

Question 7 **Begin a new page**

(a)



A is a point outside a circle with centre O. P is a second point outside the circle such that $PT = PA$ where PT is a tangent to the circle at T. AO cuts the circle at D and C. PC cuts the circle at B. AB cuts the circle at E.

(i) Copy the diagram.

(ii) Show that ΔPBT ||| ΔPTC. 2

(iii) Show that ΔAPB ||| ΔCPA. 3

(iv) Hence show that DE is parallel to AP. 3

(b) A sequence $u_1, u_2, u_3, \ldots$ is defined by $u_1 = 2$, $u_2 = 12$ and $u_n = 6u_{n-1} - 8u_{n-2}$ for $n \geq 3$.

(i) Use Mathematical Induction to show that $u_n = 4^n - 2^n$ for $n \geq 1$. 4

(ii) If $S_n = u_1 + u_2 + u_3 + \ldots + u_n$, find an expression for S_n in the form $S_n = a\,2^{2n+2} + b\,2^{n+1} + c$ where a, b, c are numerical constants. 3

Marks

Question 8 **Begin a new page**

(a) (i) Given that $y = x - \ln(\sec x + \tan x)$, $0 \leq x < \frac{\pi}{2}$, show that $\frac{dy}{dx} = 1 - \sec x$. 2

(ii) Hence show that $x < \ln(\sec x + \tan x)$ for $0 < x < \frac{\pi}{2}$. 3

(b) (i) Show that $\frac{\sin(A+B) - \sin(A-B)}{2\sin B} = \cos A$. 1

(ii) Hence show that 2

$$\cos x + \cos 3x + \cos 5x + \ldots + \cos(2n-3)x + \cos(2n-1)x = \frac{\sin 2nx}{2\sin x}$$

(iii) Hence evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin 8x}{\sin x}\,dx$. 2

(c) (i) Find the values of the constants A and B such that 2

$$4x^4 + 1 = (2x^2 + Ax + 1)(2x^2 + Bx + 1)$$

(ii) Hence find the prime factors of the integer $2^{14} + 1$. 3

CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NEW SOUTH WALES

2001
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2
Marking guidelines/ solutions

2604 - 2

Question 1

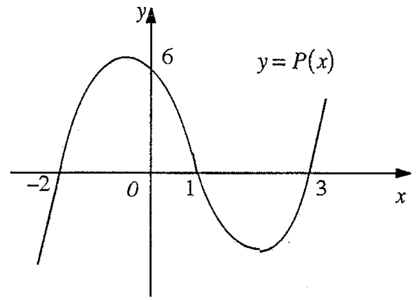
(a) **Outcomes Assessed:** (i) **PE3** (ii) **E6**

Marking Guidelines

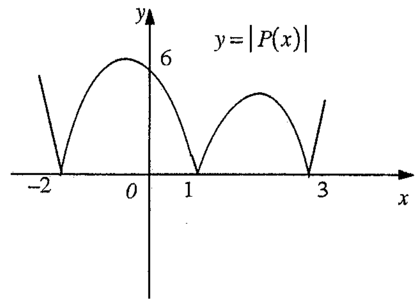
Criteria	Marks
(i) • one mark for graph of $y=P(x)$	1
(ii) • one mark for graph of $y=\lvert P(x)\rvert$ • one mark for graph of $y=P(\lvert x\rvert)$ • one mark for asymptotes and intercepts of graph of $y=\dfrac{1}{P(x)}$ • one mark for graph of $y=\dfrac{1}{P(x)}$	4

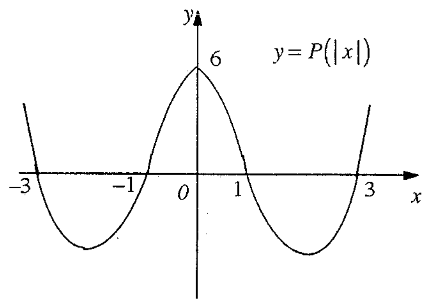
Answer $P(x)=(x+2)(x-1)(x-3)$

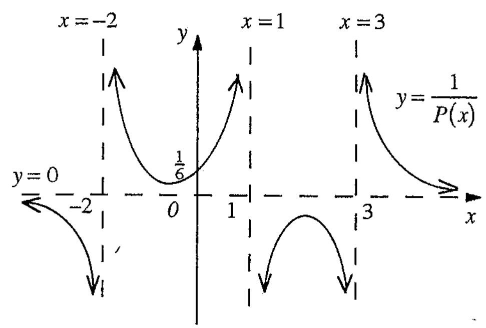
(i)



(ii)





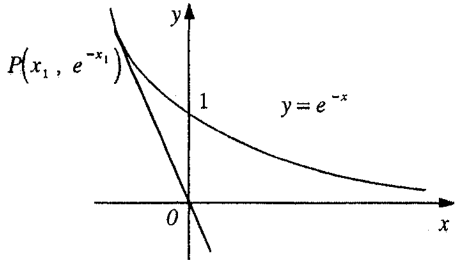


(b) **Outcomes Assessed:** (i) **E6** (ii) **E6**

Marking Guidelines

Criteria	Marks
(i) • one mark for gradient $OP=\dfrac{e^{-x_1}}{x_1}$ • one mark for gradient $OP=-e^{-x_1}$ • one mark for coordinates of P	3
(ii) • one mark for gradient of tangent $=-e$ • one mark for set of values of k	2

Answer



(i)

$$P(x_1,\ e^{-x_1})$$

$$y = e^{-x} \qquad grad.\ OP = \frac{e^{-x_1}}{x_1}$$

$$\frac{dy}{dx} = -e^{-x} \qquad grad.\ tangent\ at\ P = -e^{-x_1}$$

Since OP is tangent at P, $\dfrac{e^{-x_1}}{x_1} = -e^{-x_1}$

$$\therefore (x_1+1)e^{-x_1} = 0$$

$$\therefore x_1 = -1,\quad P(-1,\ e)$$

(ii) $y = -e\,x$ is tangent to the curve $y = e^{-x}$ at $P(-1,\ e)$, and intersects the curve at no other point.

By inspection of the graph, for $-e < k \le 0$, $y = kx$ has no points of intersection with the curve.

for $k > 0$, $y = kx$ has exactly one point of intersection with the curve.

Since $y = e^{-x}$ is steeper than any linear function of x as $x \to -\infty$, lines $y = kx$, $k < -e$, will intersect the curve in two distinct points.

Hence $e^{-x} = kx$ has two real and distinct solutions for $\{k:\ k < -e\}$.

(c) **Outcomes Assessed:** (i) **P5, H5** (ii) **E6**

Marking Guidelines

Criteria	Marks
(i) • one mark for showing $f(-x) = f(x)$ • one mark for finding $f''(x)$ • one mark for showing $f''(x) < 0$	3
(ii) • one mark for asymptotes, endpoints and intercepts of graph $y = f(x)$ • one mark for graph $y = f(x)$	2

Answer

(i)

$f(x) = \ln(1+\cos x)$

$f(-x) = \ln\{1+\cos(-x)\} = \ln(1+\cos x) = f(x)$

Hence f is an even function.

$$f'(x) = \frac{-\sin x}{1+\cos x}$$

$$f''(x) = -\frac{\cos x\,(1+\cos x) - \sin x(-\sin x)}{(1+\cos x)^2}$$

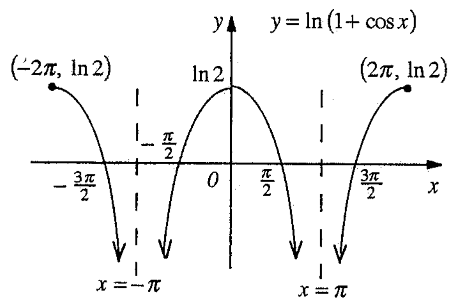
$$= -\frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$$

$$= -\frac{\cos x + 1}{(1+\cos x)^2}$$

$\therefore f''(x) = \dfrac{-1}{1+\cos x} < 0$ (since $1+\cos x > 0$, $x \ne \pm\pi$)

Hence curve is concave down throughout its domain.

(ii)



Question 2

(a) **Outcomes Assessed: E3**

Marking Guidelines

Criteria	Marks
• one mark for equating imaginary parts to evaluate a • one mark for equating real parts to get equation in b • one mark for values of z	3

Answer

$z = a + ib$, a, b real.

$|z|^2 - iz = a^2 + b^2 - ia + b$

$\therefore 16 - 2i = (a^2 + b^2 + b) - ia$

Equating real and imaginary parts,

$$\left.\begin{array}{l} a = 2 \\ a^2 + b^2 + b = 16 \end{array}\right\} \Rightarrow \begin{array}{l} b^2 + b - 12 = 0 \\ (b+4)(b-3) = 0 \end{array}$$

$\therefore a = 2,\ b = -4$ or $a = 2,\ b = 3$

Hence $z = 2 - 4i$ or $z = 2 + 3i$

(b) **Outcomes Assessed:** (i) **H5** (ii) **E8**

Marking Guidelines

Criteria	Marks
(i) • one mark for integration	1
(ii) • one mark for partial fractions • one mark for integration	2

Answer

(i) $\displaystyle\int \frac{e^x+1}{e^x}\,dx = \int (1+e^{-x})\,dx = x - e^{-x} + c$

(ii) $\displaystyle\int \frac{x^2+x+1}{x(x^2+1)}\,dx = \int \frac{(x^2+1)+x}{x(x^2+1)}\,dx = \int \left(\frac{1}{x} + \frac{1}{x^2+1}\right) dx = \ln|x| + \tan^{-1} x + c$

(c) **Outcomes Assessed:** (i) **E8** (ii) **E8**

Marking Guidelines

Criteria	Marks
(i) • one mark for integral in terms of t • one mark for evaluation of integral	2
(ii) • one mark for integral in terms of u • one mark for evaluation of integral	2

Answer

(i)

$t = \tan\frac{x}{2}$

$dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

$2\,dt = \left(1+\tan^2\frac{x}{2}\right)dx$

$dx = \dfrac{2}{1+t^2}\,dt$

$1 + \cos x + \sin x = 1 + \dfrac{1-t^2}{1+t^2} + \dfrac{2t}{1+t^2} = \dfrac{2+2t}{1+t^2}$

$x = 0 \Rightarrow t = 0$

$x = \frac{\pi}{2} \Rightarrow t = 1$

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x+\sin x}\,dx = \int_0^1 \frac{1+t^2}{2(1+t)}\cdot\frac{2}{1+t^2}\,dt$$

$$= \int_0^1 \frac{1}{1+t}\,dt$$

$$= \Big[\ln|1+t|\Big]_0^1$$

$$= \ln 2$$

(ii) Let $I=\int_0^{\frac{\pi}{2}} \frac{x}{1+\cos x+\sin x}\, dx$

$u=\frac{\pi}{2}-x$

$du=-dx$

$x=0 \Rightarrow u=\frac{\pi}{2}$

$x=\frac{\pi}{2} \Rightarrow u=0$

$x=\frac{\pi}{2}-u$

$\cos x+\sin x=\sin u+\cos u$

$$I = \int_{\frac{\pi}{2}}^{0} \frac{\frac{\pi}{2}-u}{1+\sin u+\cos u} \cdot -du = \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2}-u}{1+\cos u+\sin u}\, du$$

$$\therefore I = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos u+\sin u}\, du - \int_0^{\frac{\pi}{2}} \frac{u}{1+\cos u+\sin u}\, du$$

$$I = \frac{\pi}{2}\ln 2 - I$$

$$2I = \frac{\pi}{2}\ln 2$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{x}{1+\cos x+\sin x}\, dx = \frac{\pi}{4}\ln 2$$

(d) **Outcomes Assessed:** (i) **E8** (ii) **E8**

Marking Guidelines

Criteria	Marks
(i) • one mark for integration by parts • one mark for use of $x^2=(1+x^2)-1$ • one mark for obtaining recurrence relation	3
(ii) • one mark for integral in terms of $u=\tan x$ • one mark for recurrence relation	2

Answer

(i)

$$I_n=\int_0^1 (1+x^2)^n\, dx$$

$$=\left[x(1+x^2)^n\right]_0^1-\int_0^1 x.n(1+x^2)^{n-1}.2x\, dx$$

$$=2^n - 2n\int_0^1 x^2(1+x^2)^{n-1}\, dx$$

$$=2^n-2n\int_0^1 (1+x^2-1)(1+x^2)^{n-1}\, dx$$

$$=2^n-2n\left\{\int_0^1 (1+x^2)^n\, dx - \int_0^1 (1+x^2)^{n-1}\, dx\right\}$$

$$I_n=2^n-2n\,I_n + 2n\,I_{n-1}$$

$$\therefore (2n+1)I_n=2^n+2n\,I_{n-1}, \quad n=1,2,3,\ldots$$

(ii)

$u=\tan x \qquad x=0 \Rightarrow u=0$

$du=\sec^2 x\, dx \qquad x=\frac{\pi}{4}\Rightarrow u=1$

$$J_m=\int_0^{\frac{\pi}{4}} \sec^{2m} x\, dx$$

$$=\int_0^{\frac{\pi}{4}} (\sec^2 x)^{m-1}.\sec^2 x\, dx$$

$$=\int_0^1 (1+u^2)^{m-1}\, du$$

$$\therefore J_m=I_{m-1}, \quad m=1,2,3,\ldots$$

$$\{2(m-1)+1\}J_m=2^{m-1}+2(m-1)I_{m-2}$$

$$\therefore (2m-1)J_m=2^{m-1}+2(m-1)J_{m-1}$$

$m=2, 3, 4,\ldots$

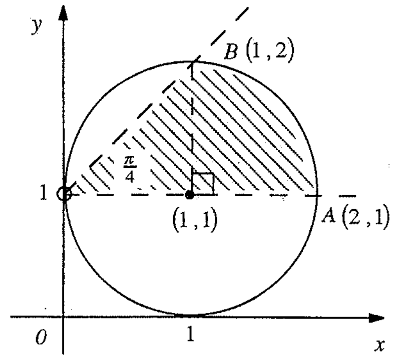
Question 3

(a) **Outcomes Assessed:** (i)**E3** (ii) **E3**

Marking Guidelines

Criteria	Marks
(i) • one mark for sketch	1
(ii) • one mark for shading region	1

Answer (i), (ii) Locus of P is the circle centred on $(1,1)$ with radius 1 unit.

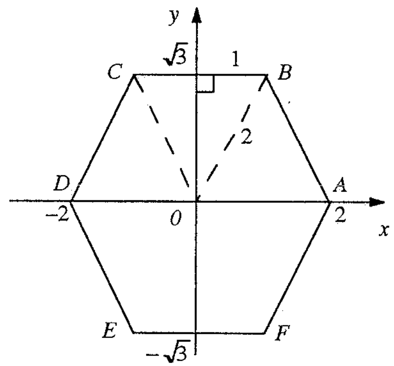


(b) **Outcomes Assessed:** (i) **E3** (ii) **E3** (iii) **E3**

Marking Guidelines

Criteria	Marks
(i) • one mark for set of values of $\text{Im}(z)$	1
(ii) • one mark for set of values of $\lvert z\rvert$	1
(iii) • one mark for value of complex number	1

Answer



(i) $-\sqrt{3}\le \text{Im}(z)\le\sqrt{3}$

(ii) $\sqrt{3}\le\lvert z\rvert\le 2$

(iii) Each of the triangles ΔAOB, ΔBOC, ... is equilateral with side 2 units.

$\therefore A\hat{O}C=2\times 60°=120°$

After rotation clockwise through $45°$, OC will make an angle $75°$, or $\frac{5\pi}{12}$ radians, with the positive x axis. Hence C will then represent the complex number $2\left(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}\right)$.

(c) **Outcomes Assessed:** (i) **E2, E3** (ii) **E2, E3** (iii) **E4** (iv) **E4**

Marking Guidelines

Criteria	Marks
(i) • one mark for use of De Moivre's Theorem to obtain expressions for $z^n \pm \frac{1}{z^n}$ • one mark for expansion of $\left(z+\frac{1}{z}\right)^4+\left(z-\frac{1}{z}\right)^4$ in terms of z • one mark for obtaining expression for $\cos^4\theta+\sin^4\theta$ in terms of $\cos 4\theta$	3
(ii) • one mark for showing equation reduces to $\cos 4\theta=\frac{1}{2}$ • one mark for solving this equation to obtain values of x	2
(iii) • one mark for using product of roots in terms of coefficients to evaluate $\cos\frac{\pi}{12}\cos\frac{5\pi}{12}$ • one mark for using sum of products of roots taken two at a time in terms of coefficients to evaluate $\cos^2\frac{\pi}{12}+\cos^2\frac{5\pi}{12}$ • one mark for evaluating $\cos\frac{\pi}{12}+\cos\frac{5\pi}{12}$	3
(iv) • one mark for forming quadratic equation with roots $\cos\frac{\pi}{12}$, $\cos\frac{5\pi}{12}$ • one mark for value of $\cos\frac{\pi}{12}$	2

Answer

(i) Using De Moivre's Theorem,

$z = \cos\theta + i\sin\theta$

$z^n = \cos n\theta + i\sin n\theta$

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$

$\therefore\ z^n + \frac{1}{z^n} = 2\cos n\theta,\quad z^n - \frac{1}{z^n} = 2i\sin n\theta$

$$\left(z+\frac{1}{z}\right)^4 + \left(z-\frac{1}{z}\right)^4 = 2\left(z^4 + 6z^2\cdot\frac{1}{z^2} + \frac{1}{z^4}\right)$$

$$= 2\left(z^4 + \frac{1}{z^4}\right) + 12$$

$$(2\cos\theta)^4 + (2i\sin\theta)^4 = 2(2\cos 4\theta) + 12$$

$$16\left(\cos^4\theta + \sin^4\theta\right) = 4(\cos 4\theta + 3)$$

$$\therefore\quad \cos^4\theta + \sin^4\theta = \tfrac{1}{4}(\cos 4\theta + 3)$$

(ii)

$x = \cos\theta,\qquad 8x^4 + 8(1-x^2)^2 = 7$

$1 - x^2 = \sin^2\theta \Rightarrow 8(\cos^4\theta + \sin^4\theta) = 7$

$2(\cos 4\theta + 3) = 7$

Hence equation becomes

$x = \cos\theta,\qquad \cos 4\theta = \frac{1}{2}$

$4\theta = 2n\pi \pm \frac{\pi}{3} \Rightarrow \theta = \frac{(6n\pm 1)\pi}{12}$

$n = 0, \pm 1, \pm 2, \ldots$

$x = \cos\frac{\pi}{12},\ \cos\frac{5\pi}{12},\ \cos\frac{7\pi}{12},\ \cos\frac{11\pi}{12}$

$x = \cos\frac{\pi}{12},\ \cos\frac{5\pi}{12},\ \cos\left(\pi - \frac{5\pi}{12}\right),\ \cos\left(\pi - \frac{\pi}{12}\right)$

$\therefore\ x = \pm\cos\frac{\pi}{12},\ \pm\cos\frac{5\pi}{12}$

(iii) $8x^4 + 8(1-x^2)^2 = 7$ simplifies to give $16x^4 - 16x^2 + 1 = 0$, with roots $\cos\frac{\pi}{12}, -\cos\frac{\pi}{12}, \cos\frac{5\pi}{12}, -\cos\frac{5\pi}{12}$.

Then $\alpha\beta\gamma\delta = \cos^2\frac{\pi}{12}\cos^2\frac{5\pi}{12} = \frac{1}{16}$

$\sum\alpha\beta = -\cos^2\frac{\pi}{12} - \cos^2\frac{5\pi}{12} = -1$

where $0 < \frac{\pi}{12} < \frac{5\pi}{12} < \frac{\pi}{2}$.

Then $\cos\frac{\pi}{12}\cos\frac{5\pi}{12} = +\sqrt{\frac{1}{16}} = \frac{1}{4}$, and

$\cos^2\frac{\pi}{12} + \cos^2\frac{5\pi}{12} + 2\cos\frac{\pi}{12}\cos\frac{5\pi}{12} = 1 + \frac{1}{2}$

$\therefore\quad \left(\cos\frac{\pi}{12} + \cos\frac{5\pi}{12}\right)^2 = \frac{3}{2}$

$\cos\frac{\pi}{12} + \cos\frac{5\pi}{12} = \sqrt{\frac{3}{2}}$

(iv) $\cos\frac{\pi}{12},\ \cos\frac{5\pi}{12}$ are roots of the quadratic equation $x^2 - \sqrt{\frac{3}{2}}\,x + \frac{1}{4} = 0$.

$$x = \frac{\sqrt{\frac{3}{2}} \pm \sqrt{\frac{3}{2} - 1}}{2} = \frac{\sqrt{3}\pm 1}{2\sqrt{2}}$$

$$\cos\frac{\pi}{12} > \cos\frac{5\pi}{12} \Rightarrow \cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Question 4

Outcomes Assessed: (i) **E2, E3, E4** (ii) **E2, E3, E4** (iii) **E2, E4** (iv) **E2, E4** (v) **E4, E6** (vi) **E2, E4, E9** (vii) **E2, E4, E9**

Marking Guidelines

Criteria	Marks
(i) • one mark for finding gradient of tangent in terms of t • one mark for obtaining equation of tangent	2
(ii) • one mark for finding gradient of tangent in terms of θ • one mark for finding equation of tangent	2
(iii) • one mark for comparing coefficients to obtain result	1
(iv) • one mark for coordinates of Q in terms of t • one mark for obtaining quartic equation in t • one mark for using this equation to deduce there are exactly two common tangents	3
(v) • one mark for diagram showing second common tangent • one mark for coordinates of R and S	2
(vi) • one mark for using geometrical properties of a rhombus to show $b^2 = a^2$ • one mark for deducing $t^2 < 1$	2
(vii) • one mark for using geometrical properties of a square to obtain equation in t • one mark for deducing that $2c^2 = a^2$ • one mark for recognising the relationship between the rectangular hyperbolas	3

Answer

(i)

$x = ct \Rightarrow \frac{dx}{dt} = c$

$y = \frac{c}{t} \Rightarrow \frac{dy}{dt} = \frac{-c}{t^2}$

$\therefore\ \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = -\frac{1}{t^2}$

Hence tangent l has gradient $-\frac{1}{t^2}$ and equation $x + t^2y = k$, k constant, where $P\left(ct, \frac{c}{t}\right)$ lies on $l \Rightarrow ct + ct = k$. Hence l has equation $x + t^2y = 2ct$.

(ii)

$x = a\sec\theta \Rightarrow \frac{dx}{d\theta} = a\sec\theta\tan\theta$

$y = b\tan\theta \Rightarrow \frac{dy}{d\theta} = b\sec^2\theta$

$\therefore\ \frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{b\sec\theta}{a\tan\theta}$

Hence tangent l has gradient $\frac{b\sec\theta}{a\tan\theta}$ and equation $x\,b\sec\theta - y\,a\tan\theta = k$, k constant, where $Q(a\sec\theta, b\tan\theta)$ lies on l $\Rightarrow k = ab\sec^2\theta - ab\tan^2\theta = ab$. Hence l has equation $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$.

(iii) Comparing the two forms of the equation of line l, the coefficients must be in proportion. Hence

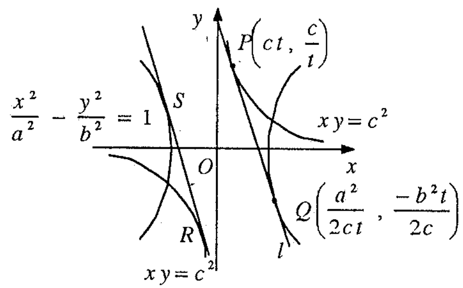
$$\frac{\left(\frac{\sec\theta}{a}\right)}{1} = \frac{\left(\frac{-\tan\theta}{b}\right)}{t^2} = \frac{1}{2ct} \qquad \therefore\ \frac{\sec\theta}{a} = \frac{-\tan\theta}{bt^2} = \frac{1}{2ct}.$$

(iv)

$Q(a\sec\theta, b\tan\theta) \equiv Q\left(\frac{a^2}{2ct}, \frac{-b^2t}{2c}\right)$, $\quad \sec^2\theta - \tan^2\theta = 1$, $\left(\frac{a}{2ct}\right)^2 - \left(\frac{-bt}{2c}\right)^2 = 1$ $\Rightarrow$ $a^2 - b^2t^4 = 4c^2t^2$, $\quad b^2t^4 + 4c^2t^2 - a^2 = 0$

This quadratic in t^2 has discriminant $\Delta = 16c^4 + 4a^2b^2 > 0$, and hence has two real roots which are opposite in sign (since their product is negative). But $t^2 > 0$, hence there is exactly one solution for t^2, and two solutions for t which are opposites of each other. Each such value of t gives a common tangent l to the two hyperbolas.

(v)



$R\left(-ct, \frac{-c}{t}\right)$, $\quad S\left(\frac{-a^2}{2ct}, \frac{b^2t}{2c}\right)$

(vi)

O is the common midpoint of diagonals PR and QS. Hence $PQRS$ is a parallelogram.

gradient PR $= \frac{2c}{t} \div 2ct = \frac{1}{t^2}$

gradient QS $= \frac{b^2t}{c} \div \frac{-a^2}{ct} = \frac{b^2}{a^2}(-t^2)$

$\therefore$ *gradient PR . gradient QS* $= -\frac{b^2}{a^2}$

Hence if $PQRS$ is a rhombus, $PR \perp QS$ and *gradient PR . gradient QS* $= -1 \Rightarrow b^2 = a^2$.

Then t satisfies $a^2t^4 + 4c^2t^2 - a^2 = 0$

$$t^4 + \frac{4c^2}{a^2}t^2 = 1$$

$$\left(t^2 + \frac{2c^2}{a^2}\right)^2 = 1 + \frac{4c^4}{a^4} < \left(1 + \frac{2c^2}{a^2}\right)^2$$

Hence $t^2 < 1$

(vii) If $PQRS$ is a square, then $PQRS$ is a rhombus with $R\hat{P}Q = 45°$. Then

$$\left.\begin{array}{l} \text{gradient } PR = \dfrac{1}{t^2} \\ \text{gradient } PQ = \dfrac{-1}{t^2} \end{array}\right\} \Rightarrow 1 = \left| \dfrac{\left(\frac{2}{t^2}\right)}{1+\left(\frac{1}{t^2}\right)\left(\frac{-1}{t^2}\right)} \right| = \frac{-2t^2}{t^4-1} \quad (\text{since } t^2<1 \text{ for } PQRS \text{ a rhombus})$$

Hence $t^4+2t^2-1=0$. But for $PQRS$ a rhombus, t satisfies $t^4+\dfrac{4c^2}{a^2}t^2-1=0$.

By subtraction, $\left(\dfrac{4c^2}{a^2}-2\right)t^2=0$. But $t^2 \neq 0$. Hence $2c^2=a^2$.

Hence if $PQRS$ is a square (and hence a rhombus), then $b^2=a^2$, and the two hyperbolas have equations $x^2-y^2=a^2$ and $xy=c^2$, where $2c^2=a^2$.

This relationship between c^2 and a^2 means that the rectangular hyperbola $x^2-y^2=a^2$ rotated anticlockwise through $45°$ becomes the rectangular hyperbola $xy=c^2$.

Question 5

(a) **Outcomes Assessed:** (i) **E8** (ii) **H5** (iii) **E8**

Marking Guidelines

Criteria	Marks
(i) • one mark for integration by parts of $I-J$ • one mark for obtaining result	2
(ii) • one mark for finding $\int (x+1)e^x\,dx$ from the derivative of xe^x • one mark for finding the required expression for $I+J$	2
(iii) • one mark for value of I	1

Answer

(i) $I=\displaystyle\int_0^\pi x\,e^x\cos x\,dx$, $J=\displaystyle\int_0^\pi e^x\cos x\,dx$

$$I-J=\int_0^\pi (x-1)\,e^x\cos x\,dx$$

$$=\Big[(x-1)\,e^x\sin x\Big]_0^\pi - \int_0^\pi x\,e^x\sin x\,dx$$

$$=-\int_0^\pi x\,e^x\sin x\,dx$$

(ii) $\dfrac{d}{dx}xe^x = e^x+xe^x=(x+1)e^x$

$\therefore \displaystyle\int (x+1)e^x\,dx = xe^x+c$

$$I+J=\int_0^\pi (x+1)\,e^x\cos x\,dx$$

$$=\Big[x\,e^x\cos x\Big]_0^\pi - \int_0^\pi x\,e^x(-\sin x)\,dx$$

$$=-\pi\,e^\pi+\int_0^\pi x\,e^x\sin x\,dx$$

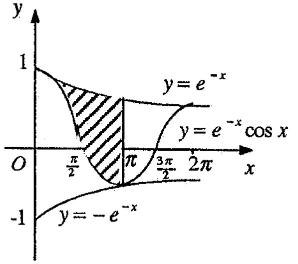
(iii) $I=\frac{1}{2}\{(I+J)+(I-J)\} = -\frac{1}{2}\pi\,e^\pi$

(b) **Outcomes Assessed:** (i) E6 (ii) E7 (iii) E8 (iv) E8

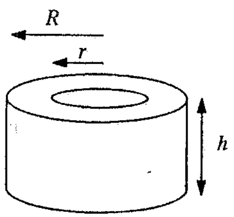
Marking Guidelines

Criteria	Marks
(i) • one mark for graphs of $y=e^{-x}$, $y=-e^{-x}$ • one mark for graph of $y=e^{-x}\cos x$ • one mark for shading region	3
(ii) • one mark for expression for volume of cylindrical shell δV in terms of x • one mark for using concept of limiting sum to form integral for V	2
(iii) • one mark for expressing integral for V in terms of $u=\pi-x$ • one mark for rearrangement to express V in terms of I	2
(iv) • one mark for integration by parts for $\int ue^{-u}\ du$ • one mark for evaluation of $\int ue^{-u}\ du$ • one mark for evaluating V	3

(i)



(ii)



$R=\pi-x+\delta x$, $r=\pi-x$

$h=e^{-x}-e^{-x}\cos x$

Cylindrical shell has volume

$\delta V=\pi\left(R^2-r^2\right)e^{-x}(1-\cos x)$

where

$$R^2-r^2=(R+r)(R-r)$$
$$=\{2(\pi-x)+\delta x\}\delta x$$
$$=2(\pi-x)\delta x$$

ignoring terms in $(\delta x)^2$.

Hence volume of solid of revolution is given by

$$V=\lim_{\delta x\to 0}\sum_{x=0}^{x=\pi}\delta V=2\pi\int_0^{\pi}(\pi-x)e^{-x}(1-\cos x)\ dx.$$

(iii)

$u=\pi-x$ $\quad du=-dx$

$x=0\Rightarrow u=\pi$

$x=\pi\Rightarrow u=0$

$$1-\cos x=1-\cos(\pi-u)$$
$$=1+\cos u$$

$$V=2\pi\int_{\pi}^{0}u\,e^{u-\pi}\{1+\cos u\}\ (-du)$$
$$=2\pi\ e^{-\pi}\int_0^{\pi}u\,e^{u}\{1+\cos u\}\ du$$
$$=2\pi\ e^{-\pi}\left\{\int_0^{\pi}u\,e^{u}\ du+\int_0^{\pi}u\,e^{u}\cos u\ du\right\}$$
$$=2\pi\ e^{-\pi}\left\{\int_0^{\pi}u\,e^{u}\ du+I\right\}$$

(iv)

$$\int_0^{\pi}u\,e^{u}\ du=\left[u\,e^{u}\right]_0^{\pi}-\int_0^{\pi}e^{u}\ du$$
$$=\pi\,e^{\pi}-\left[e^{u}\right]_0^{\pi}$$
$$=\pi\,e^{\pi}-\left(e^{\pi}-1\right)$$

$$V=2\pi e^{-\pi}\left\{\pi e^{\pi}-e^{\pi}+1+I\right\}$$
$$=2\pi e^{-\pi}\left\{\pi e^{\pi}-e^{\pi}+1-\tfrac{1}{2}\pi e^{\pi}\right\}$$
$$=\pi(\pi-2)+2\pi e^{-\pi}$$

Hence volume is $\pi(\pi-2)+2\pi e^{-\pi}$ cu. units.

Question 6

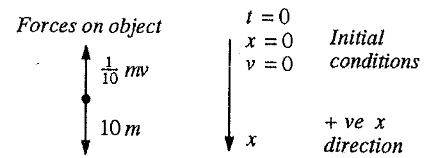
(a) **Outcomes Assessed:** (i) E2, E5 (ii) E2, E5 (iii) PE3

Marking Guidelines

Criteria	Marks
(i) • one mark for expression for $\ddot{x}$ in terms of v	1
(ii) • one mark for obtaining expression for $\frac{dv}{dx}$ • one mark for integration using initial conditions to find expression for x in terms of v • one mark for obtaining required equation for speed V on entry to water	3
(iii) • one mark for showing there is a solution for V lying between 20 and 30 • one mark for applying Newton's method to find expression for next approximation • one mark for obtaining value of V	3

Answer

(i)



$m\ddot{x}=10m-\frac{1}{10}mv$ $\quad\therefore\ \ddot{x}=10-\frac{1}{10}v$

(ii) $\ddot{x}=v\dfrac{dv}{dx}=10-\frac{1}{10}v\ \Rightarrow\ 10\dfrac{dv}{dx}=\dfrac{100-v}{v}$

$$\frac{-1}{10}\frac{dx}{dv}=\frac{-v}{100-v}=1+\frac{-100}{100-v}$$

$-\frac{1}{10}x=v+100\ln(100-v)+c$, c constant

$t=0,\ x=0,\ v=0\ \Rightarrow\ c=-100\ \ln 100$

$\therefore\ -\frac{1}{10}x=v+100\ln\left(1-\frac{v}{100}\right)$

$\left.\begin{array}{l}t=40\\ v=V\end{array}\right\}\Rightarrow\ \begin{array}{l}-4=V+100\ln\left(1-\frac{V}{100}\right)\\ -0\cdot 04=\frac{V}{100}+\ln\left(1-\frac{V}{100}\right)\end{array}$

$\therefore$ Speed V ms^{-1} just before entering water satisfies

$$\frac{V}{100}+\ln\left(1-\frac{V}{100}\right)+0\cdot 04=0\quad **$$

(iii)

Let $\lambda=\frac{V}{100}$, $f(\lambda)=\lambda+\ln(1-\lambda)+0\cdot 04$

$f(0\cdot 2)\approx 0\cdot 02>0$ $\quad f(0\cdot 3)\approx -0\cdot 02<0$

and $f(\lambda)$ is a continuous function. Hence $f(\lambda)=0$ has a solution for λ between $0\cdot 2$ and $0\cdot 3$, and ** has a solution for V between 20 and 30. Using Newton's Method with a first approximation $\lambda=0\cdot 25$ $(V=25)$

$$f(\lambda)=\lambda+\ln(1-\lambda)+0\cdot 04$$
$$f'(\lambda)=1-\frac{1}{1-\lambda}=\frac{-\lambda}{1-\lambda}$$
$$\frac{f(\lambda)}{f'(\lambda)}=\{\lambda+\ln(1-\lambda)+0\cdot 04\}\left(\frac{1-\lambda}{-\lambda}\right)$$
$$=\lambda-1-\frac{(1-\lambda)\{\ln(1-\lambda)+0\cdot 04\}}{\lambda}$$
$$\lambda-\frac{f(\lambda)}{f'(\lambda)}=1+\frac{(1-\lambda)\{\ln(1-\lambda)+0\cdot 04\}}{\lambda}$$

λ	$1+\frac{1-\lambda}{\lambda}\{\ln(1-\lambda)+0\cdot 04\}$
$0\cdot 25$	$1+3(\ln 0\cdot 75+0\cdot 04)=0\cdot 257$
$0\cdot 257$	$1+\frac{0\cdot 743}{0\cdot 257}(\ln 0\cdot 743+0\cdot 04)=0\cdot 257$

Hence $\lambda=0\cdot 257\Rightarrow V=25\cdot 7$ to one decimal place.

(b) **Outcomes Assessed:** (i) E2, E5 (ii) E2, E5 (iii) E5

Marking Guidelines

Criteria	Marks
(i) • one mark for expression for $\ddot{x}$ in terms of v • one mark for deducing object slows on entry to water • one mark for finding terminal velocity	3
(ii) • one mark for obtaining expression for $\frac{dv}{dt}$ • one mark for expressing $\frac{dv}{dt}$ in terms of partial fractions • one mark for integration using initial conditions to find expression for t in terms of v	3
(iii) • one mark for selecting correct value of v to substitute in expression for t • one mark for value of t	2

Answer

(i) After entering the water,

Forces on object: $\frac{1}{10}mv^2$ (upward), $10m$ (downward)

$t=0$, $x=0$, $v=V$ *Initial conditions*; x: *+ve x direction* (downward)

$$m\ddot{x}=10m-\tfrac{1}{10}mv^2 \quad \therefore\ \ddot{x}=10-\tfrac{1}{10}v^2$$

$\ddot{x} = 10-\frac{1}{10}V^2<0$ and $\dot{x}=V>0$

Hence object slows on entry to the water.

$\ddot{x}\to 0$ as $v\to 10$

Hence terminal velocity in the water is $10\ \text{ms}^{-1}$.

(ii)
$$\ddot{x}=\frac{dv}{dt}=10-\tfrac{1}{10}v^2 \Rightarrow 10\frac{dv}{dt}=100-v^2$$
$$\frac{1}{10}\frac{dt}{dv}=\frac{1}{(10+v)(10-v)}$$
$$=\frac{1}{20}\left\{\frac{1}{(10+v)}+\frac{1}{(10-v)}\right\}$$
$$2\frac{dt}{dv}=\frac{1}{(v+10)}-\frac{1}{(v-10)}$$
$$2t=\ln\left\{\frac{(v+10)}{(v-10)}A\right\},\quad A \text{ constant}$$
$$\left.\begin{matrix}t=0\\ v=V\end{matrix}\right\} \Rightarrow \frac{(V+10)}{(V-10)}A=1 \Rightarrow A=\frac{(V-10)}{(V+10)}$$
$$\therefore\ 2t=\ln\left\{\frac{(v+10)(V-10)}{(v-10)(V+10)}\right\}$$

(iii) $v=105\%$ of $10 \Rightarrow v=10{\cdot}5$ and $2t\approx\ln\left\{\frac{(20{\cdot}5)(15{\cdot}7)}{(0{\cdot}5)(35{\cdot}7)}\right\} \Rightarrow t\approx 1{\cdot}4$.

Hence particle slows to 105% of its terminal velocity $1{\cdot}4$ seconds after entering the water.

Question 7

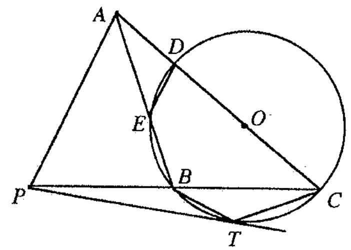
(a) **Outcomes Assessed:** (ii) PE2, PE3, E2, E9 (iii) PE2, PE3, E2, E9 (iv) PE2, PE3, E2, E9

Marking Guidelines

Criteria	Marks
(i) • no marks fo copying diagram	
(ii) • one mark for $\angle BTP=\angle TCP$ with reason • one mark for completing deduction of similarity with reasons	2
(iii) • one mark for $\frac{PB}{PT}=\frac{PT}{PC}$ with reason • one mark for $\frac{PB}{PA}=\frac{PA}{PC}$ with reason • one mark for completing deduction of similarity with reasons	3
(iv) • one mark for $\angle PAE=\angle BCD$ with reason • one mark for $\angle BCD=\angle DEA$ with reason • one mark for reason for $DE \parallel AP$	3

Answer

(i)



(ii) In ΔPBT, ΔPTC

$T\hat{P}B=C\hat{P}T$ (common angle)

$B\hat{T}P=T\hat{C}P$ (angle between chord BT and tangent PT is equal to angle in alternate segment)

$\therefore\ \Delta PBT\ |||\ \Delta PTC$ (two pairs of corresponding angles are equal)

(iii) In ΔAPB, ΔCPA

$\frac{PB}{PT}=\frac{PT}{PC}$ (corresponding sides of similar triangles ΔPBT, ΔPTC are in proportion)

$\therefore\ \frac{PB}{PA}=\frac{PA}{PC}$ (given $PT=PA$)

$A\hat{P}B=C\hat{P}A$ (common angle)

$\therefore\ \Delta APB\ |||\ \Delta CPA$ (two pairs of corresponding sides in proportion and included angles are equal)

(iv) $P\hat{A}E=B\hat{C}D$ (corresponding angles of similar triangles ΔAPB, ΔCPA are equal)

$B\hat{C}D=D\hat{E}A$ (exterior angle of cyclic quadrilateral $BCDE$ is equal to interior opposite angle)

$\therefore\ P\hat{A}E=D\hat{E}A$

$\therefore\ DE\parallel AP$ (equal alternate angles on transversal AE)

(b) **Outcomes Assessed:** (i) HE2, E2, E9 (ii) H5, E2, E9

Marking Guidelines

Criteria	Marks
(i) • one mark for showing statement $A(n): u_n=4^n-2^n$ is true for $n=1,\ n=2$ • one mark for using reduction formula to express u_{k+1} in terms of expressions for u_k, u_{k-1} when $A(n)$ is true for $n\le k$ • one mark for concluding that if $A(n)$ is true for $n\le k$, then $A(k+1)$ is true • one mark for deducing that $A(n)$ is true for $n\ge 1$	4
(ii) • one mark for recognising S_n as partial sum of the difference of two geometric series • one mark for finding expression for S_n in terms of the individual partial sums • one mark for values of $a,\ b,\ c$	3

Answer

Let $A(n)$ be the statement: $u_n = 4^n - 2^n$, $n = 1, 2, 3, \ldots$

(i) *Consider* $A(1)$, $A(2)$: $4^1 - 2^1 = 2 = u_1$, $4^2 - 2^2 = 12 = u_2$ $\quad \therefore A(1), A(2)$ are both true.

If $A(n)$ *is true for positive integers* $n \le k$ (k some positive integer, $k \ge 2$), *then*

$$u_n = 4^n - 2^n, \quad n = 1, 2, 3, \ldots, k \quad **$$

Consider $A(k+1)$, $k \ge 2$: $u_{k+1} = 6u_k - 8u_{k-1}$

$$\therefore u_{k+1} = 6(4^k - 2^k) - 8(4^{k-1} - 2^{k-1}) \quad \text{if } A(n) \text{ is true for } n \le k, \text{ using } **$$
$$= 6.4^k - 6.2^k - 2.4.4^{k-1} + 4.2.2^{k-1}$$
$$= (6-2)4^k - (6-4)2^k$$
$$= 4^{k+1} - 2^{k+1}$$

Hence if $A(n)$ is true for $n \le k$ (k some integer, $k \ge 2$), then $A(k+1)$ is true. But $A(1)$ and $A(2)$ are true, and hence $A(3)$ is true; then $A(n)$ is true for $n = 1, 2, 3$ and hence $A(4)$ is true and so on. Hence by mathematical induction, $A(n)$ is true for all positive integers $n \ge 1$.

(ii) $S_n = \sum_{k=1}^{n} u_k = \sum_{k=1}^{n} (4^k - 2^k) = \sum_{k=1}^{n} 4^k - \sum_{k=1}^{n} 2^k$

$\sum_{k=1}^{n} 4^k = \frac{4(4^n - 1)}{4-1} = \frac{4}{3}(4^n - 1)$ (sum of n terms of geometric progression, $a = 4$, $r = 4$)

$\sum_{k=1}^{n} 2^k = \frac{2(2^n - 1)}{2-1} = 2(2^n - 1)$ (sum of n terms of geometric progression, $a = 2$, $r = 2$)

$\therefore S_n = \frac{4}{3}(4^n - 1) - 2(2^n - 1) = \frac{1}{3}2^{2n+2} - \frac{4}{3} - 2^{n+1} + 2 = \frac{1}{3}2^{2n+2} - 2^{n+1} + \frac{2}{3}$

Question 8

(a) **Outcomes Assessed:** (i) **H5** (ii) **PE3, E2, E9**

Marking Guidelines

Criteria	Marks
(i) • one mark for differentiation • one mark for simplification to obtain required result	2
(ii) • one mark for using $\frac{dy}{dx} < 0$ to deduce function is decreasing for $0 < x < \frac{\pi}{2}$ • one mark for establishing $y = 0$ when $x = 0$ • one mark for deducing the required inequality	3

Answer

(i) $y = x - \ln(\sec x + \tan x)$, $0 \le x < \frac{\pi}{2}$

$$\frac{dy}{dx} = 1 - \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$$
$$= 1 - \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}$$
$$= 1 - \sec x$$

(ii) $x = 0 \Rightarrow y = 0 - \ln(1+0) = 0$

$\frac{dy}{dx} = 0$ for $x = 0$, and $\frac{dy}{dx} < 0$ for $0 < x < \frac{\pi}{2}$

Hence $y = x - \ln(\sec x + \tan x)$ is a decreasing function, and hence $y < 0$, for $0 < x < \frac{\pi}{2}$.

$x < \ln(\sec x + \tan x)$ for $0 < x < \frac{\pi}{2}$.

(b) **Outcomes Assessed:** (i) **H5** (ii) **H5, E2** (iii) **H5**

Marking Guidelines

Criteria	Marks
(i) • one mark for establishing required identity	1
(ii) • one mark for repeated use of this identity • one mark for simplification to obtain stated result	2
(iii) • one mark for using this result to rearrange integrand • one mark for evaluation of integral	2

Answer

(i) $\left.\begin{array}{l}\sin(A+B) = \sin A\cos B + \cos A\sin B\\ \sin(A-B) = \sin A\cos B - \cos A\sin B\end{array}\right\} \Rightarrow \begin{array}{l}\sin(A+B) - \sin(A-B) = 2\cos A\sin B\\ \dfrac{\sin(A+B) - \sin(A-B)}{2\sin B} = \cos A\end{array}$

(ii) Let $A = (2n-1)x$, $B = x$. Then

$$\left.\begin{array}{l}A = (2n-1)x\\ B = x\end{array}\right\} \Rightarrow \cos(2n-1)x = \frac{\sin 2nx - \sin 2(n-1)x}{2\sin x} = \frac{\sin 2nx}{2\sin x} - \frac{\sin 2(n-1)x}{2\sin x}$$

Hence

$$\cos x + \cos 3x + \cos 5x + \ldots + \cos(2n-3)x + \cos(2n-1)x$$
$$= \left(\frac{\sin 2x}{2\sin x} - \frac{\sin 0}{2\sin x}\right) + \left(\frac{\sin 4x}{2\sin x} - \frac{\sin 2x}{2\sin x}\right) + \left(\frac{\sin 6x}{2\sin x} - \frac{\sin 4x}{2\sin x}\right) + \ldots$$
$$\ldots + \left(\frac{\sin 2(n-1)x}{2\sin x} - \frac{\sin 2(n-2)x}{2\sin x}\right) + \left(\frac{\sin 2nx}{2\sin x} - \frac{\sin 2(n-1)x}{2\sin x}\right)$$
$$\therefore \cos x + \cos 3x + \ldots + \cos(2n-1)x = \frac{\sin 2nx}{2\sin x}$$

(iii)

$$\int_0^{\frac{\pi}{2}} \frac{\sin 8x}{\sin x}\,dx = 2\int_0^{\frac{\pi}{2}} (\cos x + \cos 3x + \cos 5x + \cos 7x)\,dx = 2\left[\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \tfrac{1}{7}\sin 7x\right]_0^{\frac{\pi}{2}}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin 8x}{\sin x}\,dx = 2\left(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7}\right) = \frac{152}{105}$$

(c) **Outcomes Assessed:** (i) **PE3, E2** (ii) **E2, E9**

Marking Guidelines

Criteria	Marks
(i) • one mark for obtaining equations for A and B • one mark for values of A and B	2
(ii) • one mark for expressing $2^{14}+1$ in form $4\times 8^4 + 1$ • one mark for using the polynomial factorisation to obtain factors 145×113 • one mark for prime factors 5, 29, 113	3

Answer

(i) $4x^4 + 1 \equiv (2x^2 + Ax + 1)(2x^2 + Bx + 1) \equiv 4x^4 + 2(A+B)x^3 + (AB+4)x^2 + (A+B)x + 1$

Equating coefficients: $\left.\begin{array}{l}A + B = 0\\ AB + 4 = 0\end{array}\right\} \Rightarrow \begin{array}{l}B = -A\\ -A^2 + 4 = 0\end{array} \quad \therefore \begin{array}{l}A = 2,\ B = -2\\ \text{or}\\ A = -2,\ B = 2\end{array}$

Hence $4x^4 + 1 \equiv (2x^2 + 2x + 1)(2x^2 - 2x + 1)$ **

(ii) $2^{14} + 1 = 4(2^3)^4 + 1 = \{2(2^3)^2 + 2(2^3) + 1\}\{2(2^3)^2 - 2(2^3) + 1\}$, putting $x = (2^3)$ in **.

$$\therefore 2^{14} + 1 = (2\times 64 + 16 + 1)(2\times 64 - 16 + 1) = 145\times 113 = 5\times 29\times 113$$