Centre Number

Student Number

CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NEW SOUTH WALES

2002
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Morning Session
Monday 12 August 2002

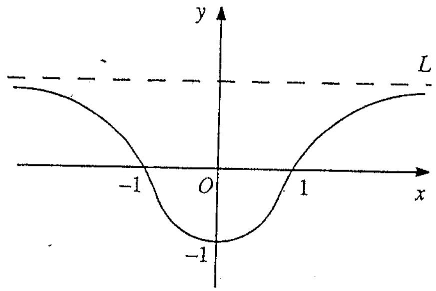
General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided separately
- All necessary working should be shown in every question

Total marks (120)
- Attempt Questions 1 – 8
- All questions are of equal value

Question 1 **Begin a new page** **Marks**

(a)



The diagram shows the graph of $y = f(x)$ where $f(x) = \dfrac{x^2-1}{x^2+1}$.

(i) Find the equation of the asymptote L. **1**

(ii) On separate diagrams sketch the following graphs, showing any intercepts on the coordinate axes and the equations of any asymptotes: **6**

$y = \{f(x)\}^2$, $y = \sqrt{f(x)}$, $y = \dfrac{1}{f(x)}$ and $y = e^{f(x)}$.

(iii) The function $f(x)$ with its domain restricted to $x \geq 0$ has an inverse $f^{-1}(x)$. Find $f^{-1}(x)$ as a function of x. **2**

(iv) If $g(x) = e^{f(x)}$, $x \geq 0$, write the inverse function g^{-1} in terms of f^{-1} and hence find $g^{-1}(x)$ as a function of x. **2**

(b) Consider the curve defined by the parametric equations $\left.\begin{aligned} x &= t^2+t-1 \\ y &= te^{2t} \end{aligned}\right\}$.

(i) Show that $\dfrac{dy}{dx} = e^{2t}$. **2**

(ii) Hence show that the tangent to the curve at the point on the curve where $t = -1$ passes through the origin. **2**

Question 2 **Begin a new page** **Marks**

(a) (i) Find $\int \frac{\cos^2 x}{1-\sin x}\,dx$. **2**

(ii) Find $\int \frac{1}{x(x^2+1)}\,dx$. **2**

(b) (i) Use the substitution $u=e^x$ to find $\int \frac{e^x}{\sqrt{e^{2x}+1}}\,dx$. **2**

(ii) Use the substitution $t=\tan\frac{x}{2}$ to evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{1-\cos x}\,dx$. **3**

(c) (i) If $I_n=\int_0^t \frac{1}{(1+x^2)^n}\,dx$, $n=1, 2, 3, \ldots$, show that **4**

$$2nI_{n+1}=(2n-1)I_n+\frac{t}{(1+t^2)^n} \quad \text{for} \quad n=1, 2, 3, \ldots.$$

(ii) Hence find the value of I_3 in terms of t. **2**

Question 3 **Begin a new page**

(a) Let $z=\sqrt{3}+i$

(i) Express z in modulus / argument form. **1**

(ii) Show that $z^7+64z=0$. **1**

(b) Find the complex number $z=a+ib$, where a and b are real, such that **2**

$$\text{Im}(z)+\bar{z}=\frac{1}{1-i}.$$

(c) The complex number z satisfies the condition $|z-8| = 2\,\text{Re}(z-2)$.

(i) Sketch the locus defined by this equation on an Argand diagram, showing any important features of the curve. State the type of curve and write down its equation. **2**

(ii) Write down the value of $|z+8|-|z-8|$. **1**

(iii) Find the possible values of $\arg z$. **2**

Marks

(d) P, Q represent complex numbers α, β respectively in an Argand diagram, where O is the origin and O, P, Q are not collinear. In ΔOPQ, the median from O to the midpoint M of PQ meets the median from Q to the midpoint N of OP in the point R, where R represents the complex number z.

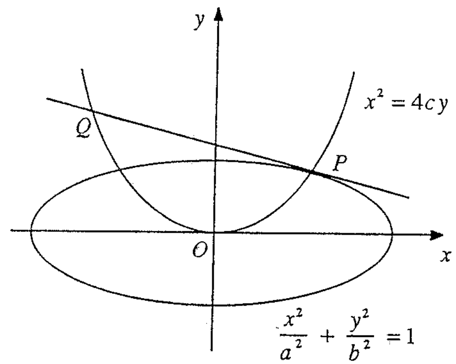
(i) Show this information on a sketch. **1**

(ii) Explain why there are positive real numbers k, l so that $kz=\frac{1}{2}(\alpha+\beta)$ and $l(z-\beta)=\frac{1}{2}\alpha-\beta$. **2**

(iii) Show that $z=\frac{1}{3}(\alpha+\beta)$. **2**

(iv) Deduce that R is the point of concurrence of the three medians of ΔOPQ. **1**

Question 4 **Begin a new page**



The parabola $x^2=4cy$ intersects the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(2cp,\ cp^2)$ in the first quadrant, where $a>b>0$ and $c>0$. The tangent to the ellipse at P meets the parabola again at $Q(2cq,\ cq^2)$.

(i) Show that the tangent to the ellipse at P has equation $\frac{2cpx}{a^2}+\frac{cp^2y}{b^2}=1$. **3**

(ii) If this tangent meets the parabola at $(2ct,\ ct^2)$ show that $\frac{p^2t^2}{b^2}+\frac{4pt}{a^2}-\frac{1}{c^2}=0$ and deduce that, considered as a quadratic in t, this equation has roots p and q. **2**

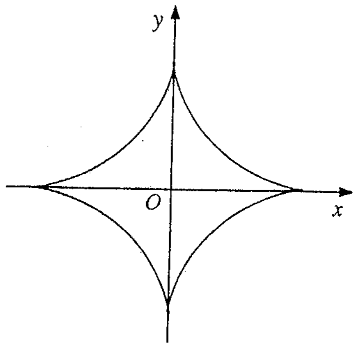
iii) If PQ subtends a right angle at the origin, show that $pq=-4$ and deduce that $\frac{1}{b^2}=\frac{1}{a^2}+\frac{1}{(4c)^2}$. **3**

iv) Hence show that if PQ subtends a right angle at the origin, then $p=2e$, where e is the eccentricity of the ellipse. **4**

v) Find a set of positive rational numbers a, b, c with $a>b$ so that PQ subtends a right angle at the origin, and sketch the corresponding parabola and ellipse showing the equations of the curves, the equation of the tangent at P, and the coordinates of P and Q **3**

Question 5   Begin a new page   Marks

(a)



The diagram shows the graph of the relation $|x|^{\frac{1}{2}}+|y|^{\frac{1}{2}}=L^{\frac{1}{2}}$ for some $L>0$.

(i) Show that the area of the region enclosed by the curve is $\frac{2}{3}L^2$. **3**

(ii) A stone building has height H metres. Its base is the region enclosed by the curve $|x|^{\frac{1}{2}}+|y|^{\frac{1}{2}}=L^{\frac{1}{2}}$, and the cross section taken parallel to the base at height h metres is a similar region enclosed by the curve $|x|^{\frac{1}{2}}+|y|^{\frac{1}{2}}=l^{\frac{1}{2}}$ where $l=L\left(1-\frac{h}{H}\right)$. **4**

Find the volume of the building.

(b) (i) Use De Moivre's Theorem to show that $(\cot\theta+i)^n+(\cot\theta-i)^n = \dfrac{2\cos n\theta}{\sin^n\theta}$. **2**

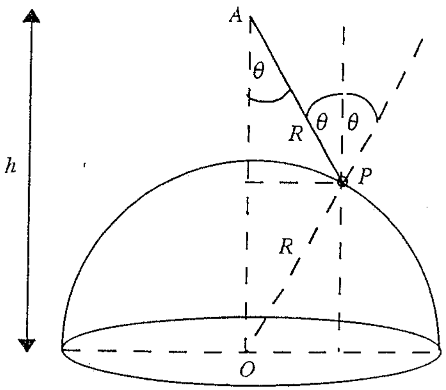
(ii) Show that the equation $(x+i)^5+(x-i)^5=0$ has roots $0,\ \pm\cot\frac{\pi}{10},\ \pm\cot\frac{3\pi}{10}$. **2**

(iii) Hence show that the equation $x^4-10x^2+5=0$ has roots $\pm\cot\frac{\pi}{10},\ \pm\cot\frac{3\pi}{10}$. **2**

(iv) Hence show that $\cot\frac{\pi}{10}=\sqrt{5+2\sqrt{5}}$. **2**

Question 6   Begin a new page   Marks

(a)



A particle P of mass m kg is connected to a fixed point A by a light, inextensible string. Point A is at a height h metres above the centre O of a smooth, hemispherical shell of radius R metres. P travels in a horizontal circle around the surface of the hemisphere with constant angular velocity ω radians per second. The length of the string is R metres, and the string makes an angle θ with OA.

(i) Draw a diagram showing the forces on P, and explain why the magnitude T of the tension in the string and the magnitude N of the normal reaction force between P and the surface of the sphere (measured in Newtons) satisfy the simultaneous equations **2**

$$T+N = \frac{mg}{\cos\theta}$$
$$T-N = mR\omega^2$$

(ii) If ω_0 is the maximum angular velocity for which P stays in contact with the surface, and $\omega=\lambda\omega_0$ for some $0<\lambda<1$, show that $\dfrac{N}{T}=\dfrac{1-\lambda^2}{1+\lambda^2}$. **4**

(iii) Describe qualitatively what would happen to the motion of the particle P if ω were to increase to ω_0 and then exceed ω_0. **1**

(b) An object of mass m kg is dropped from rest from the top of a cliff 30 metres high. The resistance to its motion has magnitude $\frac{1}{20}mv^2$ when the velocity of the object is v ms^{-1}. The object has fallen x metres after t seconds.

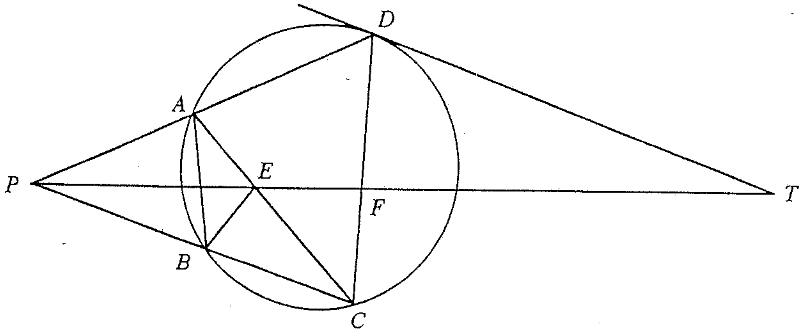
(i) Draw a diagram showing the forces on the object, and explain why $\ddot{x}=g-\frac{1}{20}v^2$. **1**

(ii) Find v as a function of x. **5**

(iii) What percentage of its terminal velocity V ms^{-1} will the object attain just before it hits the ground ? **2**

Question 7 **Begin a new page** Marks

(a)



$ABCD$ is a cyclic quadrilateral. DA produced and CB produced meet at P. T is a point on the tangent at D to the circle through A, B, C and D. PT cuts CA and CD at E and F respectively. $TF = TD$.

(i) Copy the diagram.
(ii) Show that $AEFD$ is a cyclic quadrilateral. 3
(iii) Show that $PBEA$ is a cyclic quadrilateral. 3

(b) (i) Use the method of Mathematical Induction to show that 5

$$a^2+(a+d)^2+(a+2d)^2+\ldots+\{a+(n-1)d\}^2=\tfrac{1}{6}n\{6a^2+6ad(n-1)+d^2(n-1)(2n-1)\}$$

for all positive integers $n \geq 1$.

(ii) Hence show that $1^2+3^2+5^2+\ldots+l^2=\frac{1}{6}l(l+1)(l+2)$ if l is odd, and 4
$2^2+4^2+6^2+\ldots+l^2=\frac{1}{6}l(l+1)(l+2)$ if l is even.

Question 8 **Begin a new page**

(a) If $P(x)=x^m(b^n-c^n)+b^m(c^n-x^n)+c^m(x^n-b^n)$ where m and n are positive integers, show that $x^2-(b+c)x+bc$ is a factor of $P(x)$. 3

(b) (i) If $f(x)=x-\ln\left(1+x+\frac{1}{2}x^2\right)$ show that $f(x)$ is an increasing function of x for $x<0$. 2
(ii) Hence show that $e^x < 1+x+\frac{1}{2}x^2$ for $x<0$. 3

(c) A goat grazes a rectangular paddock 10 metres by 20 metres. It is tethered to the fence at one corner of the paddock by an inextensible rope of length x metres, where $10<x<20$.

(i) Show that the goat can graze an area of A m^2 where $A=\frac{1}{2}x^2\sin^{-1}\left(\frac{10}{x}\right)+5\sqrt{x^2-100}$. 3
(ii) If the goat can graze an area equal to half the area of the paddock, find the length of the rope using one application of Newton's method with an initial value of $x=10$. Give your answer correct to one decimal place. 4

Question 1

1(a) **Outcomes Assessed:** (i) **P5** (ii) **E6** (iii) **HE4** (iv) **HE4**

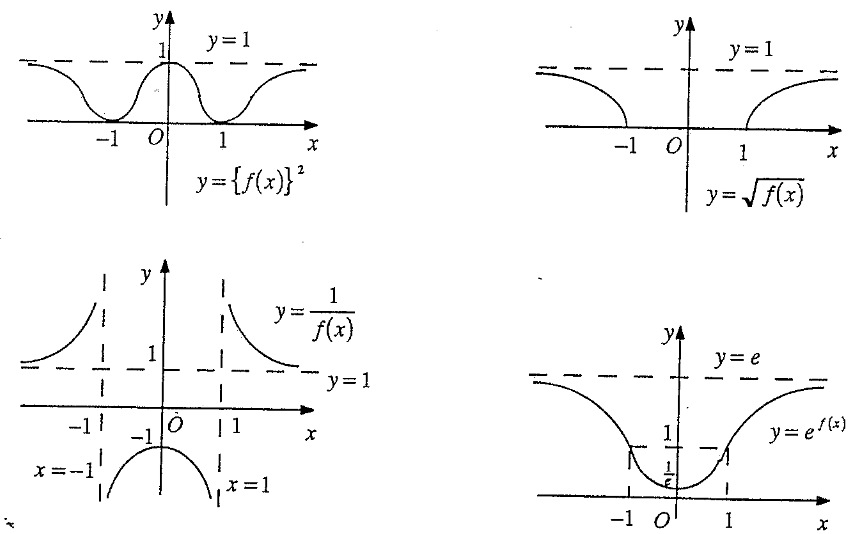
Marking Guidelines

Criteria	Marks
(i) • equation of asymptote	1
(ii) • graph of $y=\{f(x)\}^2$	1
• graph of $y=\sqrt{f(x)}$	1
• asymptotes of $y=\dfrac{1}{f(x)}$	1
• shape and intercept of $y=\dfrac{1}{f(x)}$	1
• shape of $y=e^{f(x)}$	1
• intercept and asymptote of $y=e^{f(x)}$	1
(iii) • interchanging variables with attempt to rearrange	1
• successful completion of rearranement to find inverse function	1
(iv) • writing g^{-1} in terms of f^{-1}	1
• obtaining expression for $g^{-1}(x)$	1

Answer

(i) $\lim\limits_{x\to\infty}\dfrac{x^2-1}{x^2+1}=\lim\limits_{x\to\infty}\dfrac{1-\left(\frac{1}{x}\right)^2}{1+\left(\frac{1}{x}\right)^2}=\dfrac{1-0}{1+0}=1$ Hence asymptote L has equation $y=1$.

(ii)



(iii)

$$y=\frac{x^2-1}{x^2+1},\quad x\ge 0 \Rightarrow \begin{cases} yx^2+y=x^2-1\\ x^2(y-1)=-1-y \\ x^2=\dfrac{1+y}{1-y}\\ x=\sqrt{\dfrac{1+y}{1-y}}\end{cases}$$

$$\therefore\ f^{-1}(x)=\sqrt{\frac{1+x}{1-x}}$$

(iv)

$$y=e^{f(x)},\quad x\ge 0 \Rightarrow \begin{cases}\ln y = f(x)\\ f^{-1}(\ln y)=f^{-1}(f(x))\\ f^{-1}(\ln y)=x\end{cases}$$

$$\therefore\ g^{-1}(x)=f^{-1}(\ln x)$$

$$\therefore\ g^{-1}(x)=\sqrt{\frac{1+\ln x}{1-\ln x}}$$

1(b) **Outcomes Assessed:** (i) **PE3, PE5** (ii) **E3**

Marking Guidelines

Criteria	Marks
(i) • obtaining both $\frac{dx}{dt}$, $\frac{dy}{dt}$	1
• obtaining $\frac{dy}{dx}$	1
(ii) • finding the equation of the tangent in any form	1
• showing tangent passes through origin	1

Answer

(i)

$$x=t^2+t-1 \Rightarrow \frac{dx}{dt}=2t+1$$

$$y=te^{2t} \Rightarrow \frac{dy}{dt}=e^{2t}+2te^{2t}$$

$$\therefore\ \frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}=\frac{e^{2t}(1+2t)}{2t+1}=e^{2t}$$

(ii)

$$\left.\begin{array}{l} t=-1\\ x=-1\\ y=-e^{-2}\\ \dfrac{dy}{dx}=e^{-2}\end{array}\right\}\Rightarrow$$

required tangent has equation

$$y+e^{-2}=e^{-2}(x+1)$$

$$y=e^{-2}x$$

which passes through $(0,0)$.

Question 2

a) **Outcomes Assessed:** (i) **H5** (ii) **E8**

Marking Guidelines

Criteria	Marks
(i) • simplification of integrand	1
• finding indefinite integral	1
(ii) • decomposition into partial fractions	1
• finding indefinite integral	1

Answer

(i)

$$\int\frac{\cos^2 x}{1-\sin x}dx=\int\frac{1-\sin^2 x}{1-\sin x}dx$$

$$=\int(1+\sin x)\,dx$$

$$=x-\cos x+c$$

(ii)

$$\int\frac{1}{x(x^2+1)}dx=\int\left(\frac{1}{x}-\frac{x}{x^2+1}\right)dx$$

$$=\ln|x|-\tfrac{1}{2}\ln(x^2+1)+c$$

2(b) **Outcomes Assessed:** (i) **HE6** (ii) **HE6**

Marking Guidelines

Criteria	Marks
(i) • change of variable to obtain integral with respect to u	1
• indefinite integral expressed as function of x	1
(ii) • converting x limits into t limits and replacing dx by $\frac{2}{1+t^2}dt$	1
• converting the integrand to function of t in simplest form	1
• calculation of definite integral	1

Answer

(i)

$$u=e^x \Rightarrow du=e^x\,dx$$

$$\int\frac{e^x}{\sqrt{e^{2x}+1}}dx=\int\frac{1}{\sqrt{u^2+1}}du$$

$$=\ln\left(u+\sqrt{u^2+1}\right)+c$$

$$=\ln\left(e^x+\sqrt{e^{2x}+1}\right)+c$$

(using the table of standard integrals)

(ii)

$$t=\tan\tfrac{x}{2}$$

$$dt=\tfrac{1}{2}\sec^2\tfrac{x}{2}\,dx$$

$$dx=\frac{2}{1+t^2}\,dt$$

$$x=\tfrac{\pi}{3}\Rightarrow t=\tfrac{1}{\sqrt3}$$

$$x=\tfrac{\pi}{2}\Rightarrow t=1$$

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{1}{1-\cos x}dx$$

$$=\int_{\frac{1}{\sqrt3}}^{1}\frac{1}{1-\frac{1-t^2}{1+t^2}}\,\frac{2}{1+t^2}\,dt$$

$$=\int_{\frac{1}{\sqrt3}}^{1}\frac{1}{t^2}\,dt$$

$$=\left[\frac{-1}{t}\right]_{\frac{1}{\sqrt3}}^{1}$$

$$=-1+\sqrt3$$

2(c) **Outcomes Assessed:** (i) **E8** (ii) **E8**

Marking Guidelines

Criteria	Marks
(i) • correct use of integration by parts	1
• substitution of limits	1
• rearrangement of remaining integral to express it in terms of I_n, I_{n+1}	1
• rearrangement of resultant recurrence formula	1
(ii) • evaluation of I_1	1
• using recurrence formula to find I_3	1

Answer

(i)

$$I_n=\int_0^t\frac{1}{(1+x^2)^n}dx$$

$$=\left[x\cdot\frac{1}{(1+x^2)^n}\right]_0^t-\int_0^t x\cdot\frac{-2nx}{(1+x^2)^{n+1}}dx$$

$$=\frac{t}{(1+t^2)^n}+2n\int_0^t\frac{1+x^2-1}{(1+x^2)^{n+1}}dx$$

$$I_n=\frac{t}{(1+t^2)^n}+2n\left(I_n-I_{n+1}\right)$$

$$2nI_{n+1}=(2n-1)I_n+\frac{t}{(1+t^2)^n}$$

(ii)

$$I_1=\int_0^t\frac{1}{1+x^2}dx=\left[\tan^{-1}x\right]_0^t=\tan^{-1}t$$

$$n=1\Rightarrow 2I_2=I_1+\frac{t}{1+t^2}$$

$$I_2=\tfrac{1}{2}\left(\tan^{-1}t+\frac{t}{1+t^2}\right)$$

$$n=2\Rightarrow 4I_3=3I_2+\frac{t}{(1+t^2)^2}$$

$$\therefore\ I_3=\tfrac{1}{8}\left(3\tan^{-1}t+\frac{3t}{1+t^2}+\frac{2t}{(1+t^2)^2}\right)$$

3(a) **Outcomes Assessed:** (i) **E3** (ii) **E3**

Marking Guidelines

Criteria	Marks
(i) • expressing z in modulus / argument form	1
(ii) • showing required result	1

Answer

(i)

$z=\sqrt{3}+i$

$z=2\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

(ii)

$z^7=2^7\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)=-64\left(\sqrt{3}+i\right)$

$\therefore\ z^7+64z=0$

3(b) **Outcomes Assessed:** **E3**

Marking Guidelines

Criteria	Marks
• equating real and imaginary parts of the two expressions	1
• finding z	1

Answer

$z=a+ib$

$\text{Im}(z)+\bar{z}=b+(a-ib)=(a+b)+i(-b)$

$\frac{1}{1-i}=\frac{1+i}{2}=\frac{1}{2}+\frac{1}{2}i$

Equating real and imaginary parts,

$\left.\begin{array}{l}a+b=\frac{1}{2}\\-b=\frac{1}{2}\end{array}\right\}\Rightarrow\ a=1,\ b=-\frac{1}{2}$

$\therefore\ z=1-\frac{1}{2}i$

3(c) **Outcomes Assessed:** (i) **E3** (ii) **E3, E4** (iii) **E3, E4**

Marking Guidelines

Criteria	Marks
(i) • name and equation of locus	1
• sketch of locus	1
(ii) • value of expression	1
(iii) • finding the asymptotes	1
• using the asymptotes to deduce the values of $\arg z$	1

Answer

(i)

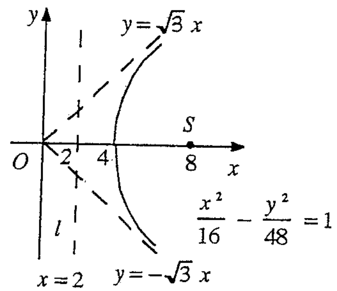
$|z-8|=2\,\text{Re}(z-2)$

Distance from z to $S(8,0)$ is twice distance from z to line l with equation $x=2$, and $\text{Re}\,z\geq 2$.

Locus is the right hand branch of the hyperbola with focus S, directrix l, and eccentricity $e=2$.

$l:\ x=2=\frac{4}{e}$ and $S:\ (8,0)=(4e,0)$

Hence the hyperbola is centred on the origin with $a=4,\ b^2=4^2(2^2-1)=48$, and equation $\frac{x^2}{16}-\frac{y^2}{48}=1$.



(ii) If P represents z, then $|z+8|-|z-8|=PS'-PS=2(PM'-PM)=8$ where $S'(-8,0)$ is the second focus of the hyperbola, $l':\ x=-2$ is the second directrix, and M', M are the feet of the perpendiculars from P to l', l respectively.

(iii) $\frac{b}{a}=\frac{4\sqrt{3}}{4}=\sqrt{3}$. Hence the asymptotes of the hyperbola have equations $y=\pm\sqrt{3}x$

The asymptotes each make angle $\frac{\pi}{3}$ with the x axis. Hence $-\frac{\pi}{3}<\arg z<\frac{\pi}{3}$.

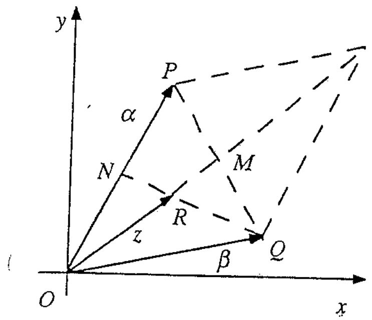
3(d) **Outcomes Assessed:** (i) **E3** (ii) **E3** (iii) **E3** (iv) **E2, E3, E9**

Marking Guidelines

Criteria	Marks
(i) • sketch showing vectors	1
(ii) • obtaining relationship for kz	1
• obtaining relationship for $l(z-\beta)$	1
(iii) • finding values for k and l	1
• deducing required result	1
(iv) • explaining required deduction	1

Answer

(i)



(ii)

Completing the parallelogram to add the vectors representing α and β, $\overrightarrow{OM}=\frac{1}{2}(\alpha+\beta)$ since the diagonals bisect each other.

Vectors $\overrightarrow{OR}$, $\overrightarrow{OM}$ are parallel. Hence $kz=\frac{1}{2}(\alpha+\beta)$ for some real $k>0$.

Also $\overrightarrow{ON}=\frac{1}{2}\alpha\ \Rightarrow\ \overrightarrow{QN}=\frac{1}{2}\alpha-\beta$,

$\overrightarrow{QR}=z-\beta$ and $\overrightarrow{QR}$, $\overrightarrow{QN}$ are parallel.

Hence $l(z-\beta)=\frac{1}{2}\alpha-\beta$ for some real $l>0$.

(iii)

$kz=\frac{1}{2}(\alpha+\beta)$

$l(z-\beta)=\frac{1}{2}\alpha-\beta\ \Rightarrow\ lz=\frac{1}{2}\alpha+(l-1)\beta$

By subtraction, $(k-l)\,z=\left(\frac{3}{2}-l\right)\beta$

But $\overrightarrow{OR}$, $\overrightarrow{OQ}$ are not parallel. Hence

$k-l=\frac{3}{2}-l=0.\quad\therefore\ k=l=\frac{3}{2}$

Now $kz=\frac{1}{2}(\alpha+\beta)\Rightarrow z=\frac{1}{3}(\alpha+\beta)$

(iv)

Similarly if the median from P to OQ meets the median OM in T, then T represents the complex number $\frac{1}{3}(\alpha+\beta)$. Hence T and R are the same point, and the three medians are concurrent in R.

Question 4

Outcomes Assessed: (i) **E3, E4** (ii) **PE3, E2** (iii) **PE4, E2** (iv) **E2, E3, E4** (v) **E3, E4**

Marking Guidelines

Criteria	Marks
(i) • finding the gradient of the tangent	1
• finding the equation of the tangent	1
• substitution of the coordinates of P	1
(ii) • obtaining required result	1
• explaining required deduction	1
(iii) • showing $pq=-4$	1
• substitution of $t=q$ and $pq=-4$ in expresion in (ii)	1
• rearrangement to obtain required result	1
(iv) • obtaining an expression for pq as the product of the roots of the quadratic in (ii)	1
• expressing p in terms of b and c.	1
• expressing the eccentricity e in terms of b and c.	1
• deducing that $p=2e$	1
(v) • finding an appropriate set of values of a, b, c	1
• finding the corresponding values of parameters p, q	1
• sketch with required information	1

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{x}{a^2}\Big/\frac{y}{b^2}$$

Hence tangent at (x_1, y_1) has equation $\frac{x_1 x}{a^2}+\frac{y_1 y}{b^2}=k$, k constant

(x_1, y_1) lies on the tangent $\Rightarrow k=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$

Hence tangent at $P(2cp,\ cp^2)$ has equation $\frac{2cpx}{a^2}+\frac{cp^2 y}{b^2}=1$

(ii)

$(2ct,\ ct^2)$ lies on tangent at P $\Rightarrow \frac{2cp(2ct)}{a^2}+\frac{cp^2(ct^2)}{b^2}=1 \quad \therefore \frac{p^2t^2}{b^2}+\frac{4pt}{a^2}-\frac{1}{c^2}=0$ **

Since $(2ct,\ ct^2)$ lies on the tangent for both $t=p$ (at P) and $t=q$ (at Q), equation ** , considered as a quadratic in t, has roots p and q.

(iii)

$P(2cp,\ cp^2)$, $Q(2cq,\ cq^2)$ $\Rightarrow$ *gradient* OP. *gradient* $OQ=\frac{p}{2}.\frac{q}{2}$ $\quad \therefore OP\perp OQ \Rightarrow pq=-4.$

Since q is a root of **, substituting $t=q$ gives $\frac{(pq)^2}{b^2}+\frac{4pq}{a^2}-\frac{1}{c^2}=0$.

Now $pq=-4 \Rightarrow \frac{(-4)^2}{b^2}+\frac{4(-4)}{a^2}-\frac{1}{c^2}=0.$ Hence $P\hat{O}Q=90° \Rightarrow \frac{1}{b^2}=\frac{1}{a^2}+\frac{1}{(4c)^2}$

(iv) Quadratic equation in t $\quad \frac{p^2t^2}{b^2}+\frac{4pt}{a^2}-\frac{1}{c^2}=0$ has roots $p,\ q$. (from (ii))

$\therefore pq=\frac{-1}{c^2}\Big/\frac{p^2}{b^2}=-\frac{b^2}{c^2p^2} \qquad \therefore pq=-4 \Rightarrow p^2=\frac{b^2}{4c^2}$

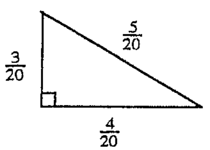
Since P is in the first quadrant, $p>0$. Hence $P\hat{O}Q=90° \Rightarrow p=\frac{b}{2c}$.

The eccentricity e of the ellipse is given by $e^2=1-\frac{b^2}{a^2}$.

From (iii), $P\hat{O}Q=90° \Rightarrow \frac{1}{b^2}=\frac{1}{a^2}+\frac{1}{(4c)^2} \Rightarrow 1-\frac{b^2}{a^2}=\frac{b^2}{(4c)^2} \Rightarrow e=\frac{b}{4c}$

Hence if $P\hat{O}Q=90°$, $\quad p=2e$.

(v) A suitable set of $a,\ b,\ c$ can be formed from any Pythagorean triad using $\frac{1}{b^2}=\frac{1}{a^2}+\frac{1}{(4c)^2}$.

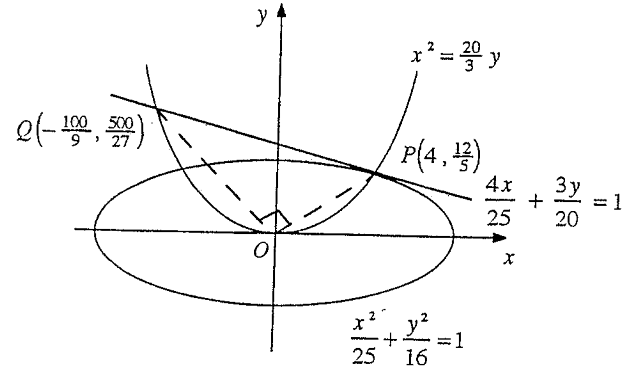


Using $(3,\ 4,\ 5)$ and dividing by the product of 4 and 5

$\frac{1}{b}=\frac{5}{20}=\frac{1}{4}$, $\quad \frac{1}{a}=\frac{4}{20}=\frac{1}{5}$, $\quad \frac{1}{4c}=\frac{3}{20}$ $\quad$ satisfy

$$\frac{1}{b^2}=\frac{1}{a^2}+\frac{1}{(4c)^2}$$

Now $a=5,\ b=4,\ c=\frac{5}{3},\ p=\frac{b}{2c}=\frac{6}{5}$. Also $pq=-4 \Rightarrow q=-\frac{10}{3}$



Question 5

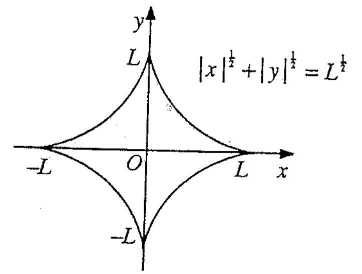
5(a) **Outcomes Assessed:** (i) **H8** (ii) **E7**

Marking Guidelines

Criteria	Marks
(i) • expressing area as a definite integral	1
• carrying out the integration	1
• substitution of the limits to evaluate the area	1
(ii) • expressing the area of cross section in terms of L, h and H	1
• expressing volume as limiting sum of volumes of slices	1
• carrying out integration	1
• substitution of limits	1
	1

Answer

(i)



For $x>0,\ y>0$

$|x|^{\frac12}+|y|^{\frac12}=L^{\frac12} \Rightarrow y=\left(L^{\frac12}-x^{\frac12}\right)^2$

Hence area enclosed by the curve is given by

$$A=4\int_0^L\left(L-2L^{\frac12}x^{\frac12}+x\right)\,dx$$

$$=4\left[Lx-\tfrac{4}{3}L^{\frac12}x^{\frac32}+\tfrac12 x^2\right]_0^L$$

$$A=4\left(L^2-\tfrac43 L^{\frac12}L^{\frac32}+\tfrac12 L^2\right)=\tfrac23 L^2$$

(ii)

At height h area of cross section A is given by

$$A=\tfrac23 l^2=\tfrac23 L^2\left(1-\tfrac{h}{H}\right)^2$$

Hence volume of slice at height h is

$$\delta V=\tfrac23 L^2\left(1-\tfrac{h}{H}\right)^2\delta h$$

and volume of solid is

$$V=\lim_{\delta h\to 0}\sum_{h=0}^{h=H}\tfrac23 L^2\left(1-\frac{h}{H}\right)^2\delta h$$

Hence

$$V=\int_0^H \tfrac23 L^2\left(1-\frac{h}{H}\right)^2\,dh$$

$$=\frac{-2L^2H}{9}\left[\left(1-\frac{h}{H}\right)^3\right]_0^H$$

$$V=-\tfrac29 L^2H(0-1)=\tfrac29 L^2H$$

5(b) **Outcomes Assessed:** (i) E3 (ii) E3, E4 (iii) E3, E4 (iv) E2, E4

Marking Guidelines

Criteria	Marks
(i) • rearrangement in preparation for use of De Moivre's Theorem	1
• using De Moivre's Theorem and simplifying to obtain required result	1
(ii) • substitution of $x=\cot\theta$ to obtain equation $\cos 5\theta=0$	1
• solution of this trig. equation to obtain the values of x	1
(iii) • binomial expansion and simplification	1
• deducing required result	1
(iv) • solving equation as quadratic in x^2	1
• deducing value of $\cot\frac{\pi}{10}$ in surd form	1

Answer

(i)

$$(\cot\theta+i)^n=\left(\frac{\cos\theta+i\sin\theta}{\sin\theta}\right)^n=\frac{\cos n\theta+i\sin n\theta}{\sin^n\theta}$$

$$(\cot\theta-i)^n=\left(\frac{\cos\theta-i\sin\theta}{\sin\theta}\right)^n=\frac{\cos n\theta-i\sin n\theta}{\sin^n\theta}$$

$$\therefore\quad(\cot\theta+i)^n+(\cot\theta-i)^n=\frac{2\cos n\theta}{\sin^n\theta}$$

(ii)

$$x=\cot\theta\ \Rightarrow\ (x+i)^5+(x-i)^5=\frac{2\cos 5\theta}{\sin^5\theta}$$

Then $(x+i)^5+(x-i)^5=0\ \Leftrightarrow\ \cos 5\theta=0$

$\therefore\ x=\cot\theta$ where $5\theta=n\frac{\pi}{2},\ n=\pm1,\pm3,\pm5,\ldots$

$\therefore\ x=\cot n\frac{\pi}{10}\ \Rightarrow\ x=\pm\cot\frac{\pi}{10},\ \pm\cot\frac{3\pi}{10},\ \pm\cot\frac{5\pi}{10},\ldots$

$\therefore\ x=0,\ \pm\cot\frac{\pi}{10},\ \pm\cot\frac{3\pi}{10}$

(iii)

$$(x+i)^5+(x-i)^5$$
$$=x^5(1+1)+5x^4\{i+(-i)\}+10x^3\{i^2+(-i)^2\}+10x^2\{i^3+(-i)^3\}+5x\{i^4+(-i)^4\}+\{i^5+(-i)^5\}$$

Hence $(x+i)^5+(x-i)^5=2x^5-20x^3+10x=2x(x^4-10x^2+5)$

$\therefore\ x^4-10x^2+5=0$ has roots $\pm\cot\frac{\pi}{10},\ \pm\cot\frac{3\pi}{10}$

(iv) Considered as a quadratic in x^2, $x^4-10x^2+5=0$ has solution $x^2=\frac{10\pm\sqrt{80}}{2}=\frac{10\pm4\sqrt5}{2}$.

Hence the roots of $x^4-10x^2+5=0$ are $\pm\sqrt{5+2\sqrt5}$, $\pm\sqrt{5-2\sqrt5}$. Since the largest positive root of the equation is $\cot\frac{\pi}{10}$, $\cot\frac{\pi}{10}=\sqrt{5+2\sqrt5}$.

Question 6

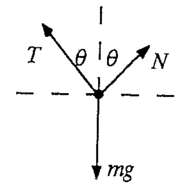
6(a) **Outcomes Assessed:** (i) E5 (ii) E2, E5 (iii) E5, E9

Marking Guidelines

Criteria	Marks
(i) • using the diagram and vertical components of forces to deduce first equation	1
• using the diagram and horizontal components of forces to deduce second equation	1
(ii) • finding expression for N in terms of ω	1
• finding expression for T in terms of ω	1
• finding expression for ω_0^2	1
• substitution in expression for $\frac{N}{T}$ to obtain required result	1
(iii) • correct description	1

Answer

(i) The resultant force on P is directed toward the centre of its circle of motion, and has magnitude $mr\omega^2=m(R\sin\theta)\,\omega^2$, since the radius r of the circle of motion is $R\sin\theta$.



Hence the resultant force on P has

vertical component zero $\Rightarrow\quad T\cos\theta+N\cos\theta=mg$

horizontal component $mr\omega^2\Rightarrow\quad T\sin\theta-N\sin\theta=mR\sin\theta\,\omega^2$

Hence $\quad T+N=\frac{mg}{\cos\theta}$

$\quad T-N=mR\omega^2$

(ii)

$$2N=\frac{mg}{\cos\theta}-mR\omega^2\ \Rightarrow\ N=\frac{mR}{2}\left(\frac{g}{R\cos\theta}-\omega^2\right)$$

$$2T=\frac{mg}{\cos\theta}+mR\omega^2\ \Rightarrow\ T=\frac{mR}{2}\left(\frac{g}{R\cos\theta}+\omega^2\right)$$

The particle P stays in contact with the surface if $N\ge0$, that is if $\omega^2\le\frac{g}{R\cos\theta}$. Hence $\omega_0^2=\frac{g}{R\cos\theta}$.

$$\omega=\lambda\omega_0\ \Rightarrow\ \begin{cases}N=\dfrac{mR}{2}\cdot\dfrac{g}{R\cos\theta}\left(1-\lambda^2\right)\\[2mm] T=\dfrac{mR}{2}\cdot\dfrac{g}{R\cos\theta}\left(1+\lambda^2\right)\end{cases}\ \Rightarrow\ \frac{N}{T}=\frac{1-\lambda^2}{1+\lambda^2}$$

(iii) As ω increases, N decreases while T increases, until ω reaches ω_0 when the particle is on the point of losing contact with the surface. For $\omega>\omega_0$, the particle is no longer on the surface of the sphere, and it moves in a horizontal circle above the hemisphere, the string making an angle θ with the vertical which increases as ω increases.

6(b) **Outcomes Assessed:** (i) E5 (ii) E5 (iii) E5

Marking Guidelines

Criteria	Marks
(i) • diagram and explanation	1
(ii) • choice of appropriate expression for $\ddot{x}$	1
• carry out integration to find x in terms of v^2	1
• identify initial conditions to find constant of integration	1
• obtain relation between $e^{-\frac{1}{10}x}$, v^2	1
• rearrangement to find expression for v in terms of x.	1
(iii) • expression for terminal velocity	1
• calculation of percentage of terminal velocity attained just before impact	1

Answer

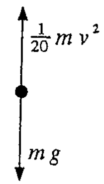
(i)

Initial conditions

$t=0$

$x=0,\ v=0$

Forces



By Newton's second law, the resultant downward force on the object has magnitude $m\ddot{x}$.

Hence

$$m\ddot{x} = mg - \tfrac{1}{20}mv^2$$

$$\ddot{x} = g - \tfrac{1}{20}v^2$$

Terminal velocity

$\ddot{x} \to 0$ as $v \to \sqrt{20g}$

Hence the object has terminal velocity $V = \sqrt{20g}$

(ii)

$$\tfrac{1}{2}\frac{dv^2}{dx} = g - \tfrac{1}{20}v^2$$

$$10\frac{dv^2}{dx} = 20g - v^2$$

$$-\tfrac{1}{10}\frac{dx}{d(v^2)} = \frac{-1}{20g-(v^2)}$$

$-\tfrac{1}{10}x = \ln\{A(20g-v^2)\}$, A constant

$t=0,\ x=0,\ v=0 \ \Rightarrow\ A.20g=1$

$$\therefore\ -\tfrac{1}{10}x = \ln\left(1-\frac{v^2}{20g}\right)$$

$$e^{-\frac{1}{10}x} = 1 - \frac{v^2}{20g}$$

$$\frac{v^2}{20g} = 1 - e^{-\frac{1}{10}x}$$

$$\therefore\ v>0 \ \Rightarrow\ v = \sqrt{20g\left(1-e^{-\frac{1}{10}x}\right)}$$

iii) Terminal velocity is $V=\sqrt{20g}$. $\quad x=30 \Rightarrow \frac{v}{V} = \sqrt{1-e^{-3}} \approx 0{\cdot}975$.

Hence object attains $97{\cdot}5\%$ of its terminal velocity just before impact with the ground.

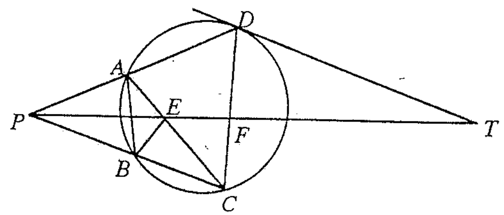
Question 7

7(a) **Outcomes Assessed:** (ii) PE2, PE3 (iii) PE2, PE3

Marking Guidelines

Criteria	Marks
(ii) • one mark for each deduction * with reason	3
(iii) • one mark for each deduction * with reason	3

Answer



(ii)

$T\hat{F}D = T\hat{D}F$ (*equal* $\angle$*s lie opp. given equal sides TF, TD in isosceles* ΔTFD) *

$T\hat{D}C = D\hat{A}C$ ($\angle$ *between tangent TD and chord DC is equal to* $\angle$ *subtended by DC at circumference in alternate segment.*) *

$\therefore\ T\hat{F}D = D\hat{A}E$ (*D, F, C collinear* $\Rightarrow T\hat{D}F = T\hat{D}C$; *A, E, C collinear* $\Rightarrow D\hat{A}E = D\hat{A}C$)

$\therefore$ *AEFD is a cyclic quadrilateral* (*exterior angle,* $T\hat{F}D$, *equal to opp. interior angle,* $D\hat{A}E$) *

(iii)

$A\hat{E}P = A\hat{D}F$ (*exterior* $\angle$ *equal to opp. interior* $\angle$ *in cyclic quad. AEFD*) *

$A\hat{B}P = A\hat{D}C$ (*exterior* $\angle$ *equal to opp. interior* $\angle$ *in cyclic quad. ABCD*) *

$\therefore\ A\hat{E}P = A\hat{B}P$ (*D, F, C collinear* $\Rightarrow A\hat{D}F = A\hat{D}C$)

$\therefore$ *PBEA is a cyclic quadrilateral* (*equal* $\angle$*s subtended by AP at B, E on same side of AP*) *

7(b) Outcomes Assessed: (i) HE2, E2 (ii) E2

Marking Guidelines

Criteria	Marks
(i) • showing $S(1)$ is true	1
• incorporating result if $S(k)$ is true in expression for $S(k+1)$	1
• algebraic rearrangement to show $S(k)$ true implies $S(k+1)$ true	2
• correct concluding statement of process of mathematical induction	1
(ii) • correct choice of $a,\ d,\ l$ when l is odd	1
• substitution and rearrangement to obtain result for l odd	1
• correct choice of $a,\ d,\ l$ when l is even	1
• substitution and rearrangement to obtain result for l even	1

Answer

(i) Let the sequence of statements $S(n)$, $n=1,\ 2,\ 3,\ \dots$ be defined by

$$S(n):\quad a^2+(a+d)^2+(a+2d)^2+\dots+\{a+(n-1)d\}^2=\tfrac{1}{6}n\{6a^2+6ad(n-1)+d^2(n-1)(2n-1)\}$$

Consider $S(1)$. $\quad LHS = a^2=\tfrac{1}{6}.1.\{6a^2\}= RHS \quad \therefore S(1)$ is true.

If $S(k)$ is true, then $a^2+(a+d)^2+\dots+\{a+(k-1)d\}^2=\tfrac{1}{6}k\{6a^2+6ad(k-1)+d^2(k-1)(2k-1)\}$ **

Consider $S(k+1)$.

$$\begin{aligned}LHS &= a^2+(a+d)^2+(a+2d)^2+\dots+\{a+(k-1)d\}^2+(a+kd)^2\\
&=\tfrac{1}{6}k\{6a^2+6ad(k-1)+d^2(k-1)(2k-1)\}+(a+kd)^2 \qquad \textit{if } S(k) \textit{ is true, using } **\\
&=\tfrac{1}{6}\{6a^2k+6ad\,k\,(k-1)+d^2k(k-1)(2k-1)+6(a^2+2akd+k^2d^2)\}\\
&=\tfrac{1}{6}\{6a^2(k+1)+6ad[k\,(k-1)+2k]+d^2k[(k-1)(2k-1)+6k]\}\\
&=\tfrac{1}{6}\{6a^2(k+1)+6ad[k\,(k+1)]+d^2k[(k+1)(2k+1)]\}\\
&=\tfrac{1}{6}(k+1)\{6a^2+6ad\,k+d^2k\,(2k+1)\}\\
&=\tfrac{1}{6}(k+1)\{6a^2+6ad[(k+1)-1]+d^2[(k+1)-1][2(k+1)-1]\}\\
&= RHS\end{aligned}$$

Hence if $S(k)$ is true, then $S(k+1)$ is true.
But $S(1)$ is true, hence $S(2)$ is true, and then $S(3)$ is true and so on.
Hence by Mathematical Induction, $S(n)$ is true for all positive integers n.

(ii) If l is odd, putting $a=1,\ d=2,\ l=2n-1$ in $S(n)$ gives

$$1^2+3^2+5^2+\dots+l^2=\tfrac{1}{6}n\{6+12(n-1)+4(n-1)(2n-1)\}=\tfrac{1}{6}n\{8n^2-2\}$$

$$\therefore\ 1^2+3^2+5^2+\dots+l^2=\tfrac{1}{6}(2n-1).2n.(2n+1)=\tfrac{1}{6}l(l+1)(l+2)$$

If l is even, putting $a=2,\ d=2,\ l=2n$ in $S(n)$ gives

$$2^2+4^2+6^2+\dots+l^2=\tfrac{1}{6}n\{24+24(n-1)+4(n-1)(2n-1)\}=\tfrac{1}{6}n\{8n^2+12n+4\}$$

$$\therefore\ 2^2+4^2+6^2+\dots+l^2=\tfrac{1}{6}2n\,(2n+1)(2n+2)=\tfrac{1}{6}l(l+1)(l+2)$$

Question 8

8(a) **Outcomes Assessed: E2, E4**

Marking Guidelines

Criteria	Marks
• showing $x-b$ is a factor of $P(x)$	1
• showing $x-c$ is a factor of $P(x)$	1
• deducing required result	1

Answer

$$P(x)=x^m(b^n-c^n)+b^m(c^n-x^n)+c^m(x^n-b^n)$$

$$P(b)=b^m(b^n-c^n)+b^m(c^n-b^n)+c^m(b^n-b^n)=0 \Rightarrow (x-b) \text{ is a factor of } P(x)$$

$$P(c)=c^m(b^n-c^n)+b^m(c^n-c^n)+c^m(c^n-b^n)=0 \Rightarrow (x-c) \text{ is a factor of } P(x)$$

$$\therefore (x-b)(x-c)=x^2-(b+c)x+bc \text{ is a factor of } P(x).$$

(b) **Outcomes Assessed: (i) H6, E2 (ii) PE3, E2**

Marking Guidelines

Criteria	Marks
(i) • differentiation	1
• rearrangement to show $f'(x)>0$	1
(ii) • use $f(0)=0$ as an upper bound for $f(x),\ x<0$	1
• obtain inequality for x in terms of $\ln(1+x+\frac{1}{2}x^2)$	1
• use the fact that e^x is an increasing function of x to obtain required result	1

Answer

(i)

$$f(x)=x-\ln(1+x+\tfrac{1}{2}x^2)$$

$$f'(x)=1-\frac{1+x}{1+x+\frac{1}{2}x^2}=\frac{\frac{1}{2}x^2}{1+x+\frac{1}{2}x^2}$$

$$\therefore f'(x)=\frac{x^2}{2+2x+x^2}=\frac{x^2}{1+(1+x)^2}$$

$$\therefore x<0 \Rightarrow f'(x)>0 \Rightarrow f \text{ is increasing.}$$

(ii)

$$\left.\begin{array}{c}f(0)=0\\ f\text{ is increasing}\\ x<0\end{array}\right\}\Rightarrow \begin{array}{r}f(x)<0\\ x-\ln(1+x+\frac{1}{2}x^2)<0\\ x<\ln(1+x+\frac{1}{2}x^2)\end{array}$$

$\therefore$ for $x<0$, $\quad e^x<1+x+\frac{1}{2}x^2$

(since e^x is also an increasing function of x)

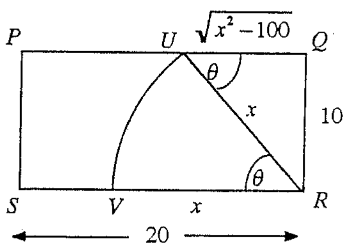
8(c) **Outcomes Assessed:** (i) **E2** (ii) **PE3, E2**

Marking Guidelines

Criteria	Marks
(i) • deduction from diagram that region comprises a sector and a right triangle	1
• obtaining expression for area of triangle	1
• obtaining expression for area of sector	1
(ii) • forming function on which Newton's method can be applied	1
• differentiation	1
• substitution into approximating formula for x	1
• calculation of length of rope	1

Answer

(i)



(ii)

The goat tethered at R can graze the sector RUV and the right-angled triangle UQR.
In triangle UQR, $UQ=\sqrt{x^2-100}$ using Pythagoras' Theorem, and $\theta=\sin^{-1}\left(\frac{10}{x}\right)$.

Hence $A=\frac{1}{2}x^2\sin^{-1}\left(\frac{10}{x}\right)+5\sqrt{x^2-100}$.

ii) Let $f(x)=A-100$. Then the goat can graze half the area of the paddock when $f(x)=0$.
Using Newton's method with initial value $x=10$, $f(x)=0$ has solution $x\approx 10-\frac{f(10)}{f'(10)}$.

$$f'(x)=x\sin^{-1}\left(\frac{10}{x}\right)+\frac{1}{2}x^2\frac{1}{\sqrt{1-\frac{100}{x^2}}}\left(\frac{-10}{x^2}\right)+\frac{5x}{\sqrt{x^2-100}} = x\sin^{-1}\left(\frac{10}{x}\right)$$

$$x\approx 10-\frac{f(10)}{f'(10)}=10-\frac{50.\frac{\pi}{2}-100}{10.\frac{\pi}{2}}=5+\frac{20}{\pi}\approx 11\cdot 4$$

Hence the length of the rope is $11\cdot 4$ m (correct to 1 decimal place).