Centre Number

Student Number

CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NEW SOUTH WALES

2008
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Morning Session
Monday, 11 August 2008

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1-8
- All questions are of equal value

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \quad \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Total marks – 120
Attempt Questions 1–8
All questions are of equal value

Answer each question in a SEPARATE writing booklet.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\int_0^{\frac{\pi}{4}} \cos x \sin^3 x \, dx$. **2**

(b) Find $\int_2^5 \frac{2\,dx}{x^2 - 4x + 13}$. **3**

(c) (i) Find the real numbers a, b and c such that $\frac{5}{x^2(2-x)} \equiv \frac{ax+b}{x^2} + \frac{c}{2-x}$. **2**

(ii) Hence, or otherwise, find $\int \frac{20}{x^2(2-x)}\,dx$. **3**

(d) (i) Use the substitution $x = \sin\theta$ to find $\int \frac{x^2}{\sqrt{1-x^2}}\,dx$. **3**

(ii) Use integration by parts to find $\int x \sin^{-1} x \, dx$. **2**

Marks

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) Simplify $(3-4i)^3$. **1**

(b) Solve $z^2 = 5 - 12i$, giving your answer in the form $x + iy$, where x and y are real. **3**

(c) (i) Express $1+i$ in modulus-argument form. **1**

(ii) Hence evaluate $(1+i)^{12}$. **2**

(d) Sketch the locus of all points z such that:

(i) $\arg z = \frac{\pi}{3}$. **1**

(ii) $\arg \overline{z} = \frac{\pi}{3}$. **1**

(iii) $\arg(-z) = \frac{\pi}{3}$. **1**

(e) The points O, A, Z and C on the Argand diagram represent the complex numbers 0, 1, z and $z+1$ respectively, where $z = \cos\theta + i\sin\theta$ is any complex number of modulus 1, with $0 < \theta < \pi$.

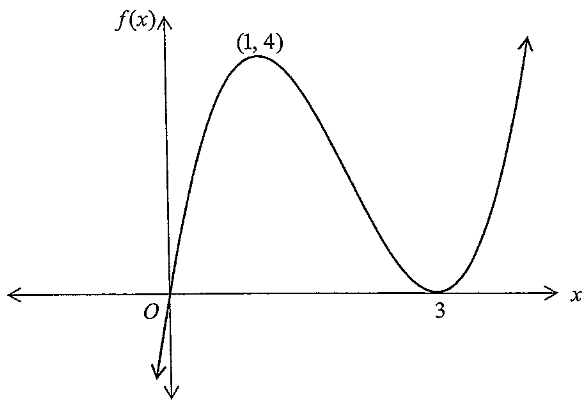
(i) Explain why $OACZ$ is a rhombus. **1**

(ii) Show that $\frac{z-1}{z+1}$ is purely imaginary. **2**

(iii) Find the modulus and argument of $z+1$. **2**

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The function defined by $f(x) = x(x-3)^2$ is drawn below.



(i) Draw separate, one-third page sketches, of the following:

(α) $y = f(|x|)$. **1**

(β) $y = \dfrac{1}{f(x)}$. **2**

(γ) $y^2 = f(x)$. **2**

(δ) $y = \tan^{-1} f(x)$. **2**

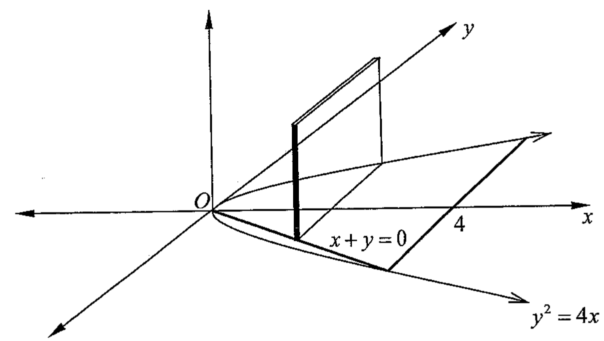
(ii) Find the values of k for which $f(x) = kx$ has exactly two distinct solutions. **2**

(b) When the polynomial $P(x)$ is divided by $(x+2)(x-3)$ the remainder is $4x+1$. **2**
What is the remainder when $P(x)$ is divided by $(x+2)$?

Question 3 continues on page 5

Question 3 (continued) **Marks**

(c) The base of a solid is the region bounded by the curve $y^2 = 4x$ and the lines $x + y = 0$ and $x = 4$. Every cross-sectional slice perpendicular to the x axis is a square having a side with one end-point on the line $x + y = 0$ and the other on the curve $y^2 = 4x$.



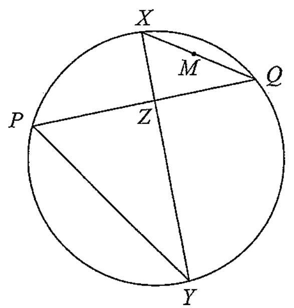
(i) Show that the area of the cross-section is given by $A(x) = 4x + x^2 - 4x^{\frac{3}{2}}$. **2**

(ii) Hence find the volume of the solid formed. **2**

End of Question 3

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Two perpendicular chords PQ and XY of a circle intersect at Z. **3**



NOT TO SCALE

Copy or trace the diagram into your writing booklet.

If M is the midpoint of the chord QX, prove that MZ produced is perpendicular to the chord PY.

(b) (i) $I_n = \int x^n e^{ax}\, dx$, where a is a constant. **2**

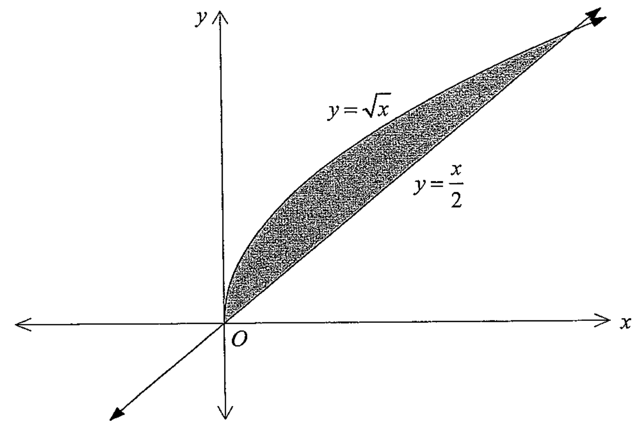
Prove that $I_n = \dfrac{x^n e^{ax}}{a} - \dfrac{n}{a} I_{n-1}$.

(ii) Hence find the value of $\int_0^1 x^3 e^{2x}\, dx$. **3**

Question 4 continues on page 7

Question 4 (continued) **Marks**

(c) By the method of cylindrical shells, find the volume of the solid generated by rotating the region bounded by $y = \dfrac{x}{2}$ and $y = \sqrt{x}$ about the x axis. **4**



(d) The polynomial equation $x^5 - ax^2 + b = 0$ has a multiple root. **3**

Show that $108a^5 = 3125b^3$.

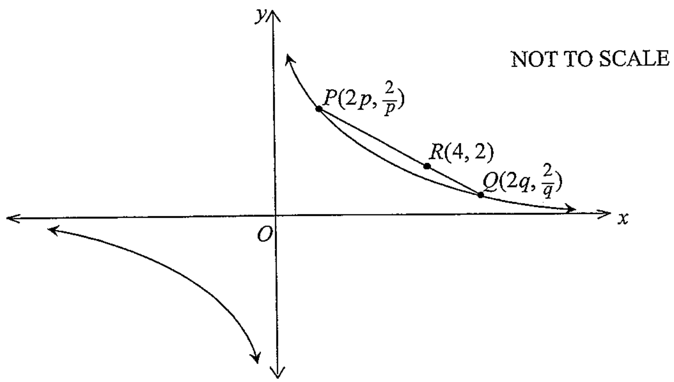
End of Question 4

Marks

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) If a, b and c are positive real numbers, prove that $(a+b)(b+c)(c+a) \geq 8abc$. **3**

(b) $P(2p, \frac{2}{p})$ and $Q(2q, \frac{2}{q})$ are points on the rectangular hyperbola $xy = 4$. M is the midpoint of the chord PQ. P and Q move on the hyperbola so that the chord PQ always passes through the point $R(4, 2)$.



(i) Show that the equation of the chord PQ is $x + pqy = 2(p+q)$. **2**

(ii) Show that $pq = p + q - 2$. **1**

(iii) Hence sketch the locus of M, as P and Q move on the curve $xy = 4$. **3**

(c) (i) By considering $z^9 - 1$ as the difference of two cubes, or otherwise, write $1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + z^7 + z^8$ as a product of two polynomials with real coefficients, one of which is a quadratic. **2**

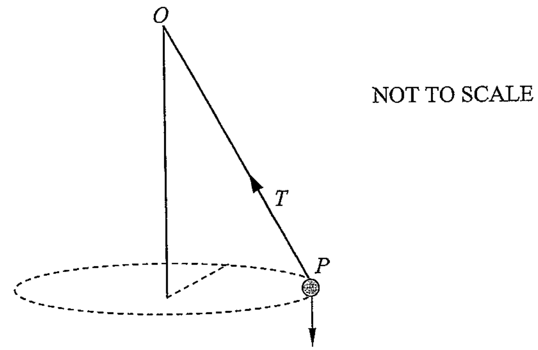
(ii) Solve $z^9 - 1 = 0$ and determine the six solutions of $z^6 + z^3 + 1 = 0$. **2**

(iii) Hence show that $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$. **2**

Marks

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) A body P of mass 0.5 kg is suspended from a fixed point O by means of a light rod of length 1 m. The mass is rotated in a horizontal circle at a constant speed v m s^{-1} and the rod makes an angle of θ with the downward direction of the vertical. Assume $g = 9.8$ ms^{-2} and $\theta = 30°$.



(i) Resolve the horizontal and vertical forces at P and show that $\tan\theta = \frac{v^2}{rg}$ where r is the radius of the circle. **3**

(ii) Find the tension T in the rod. **1**

(iii) Find the speed v of P. **1**

(iv) Find the period of the motion. **1**

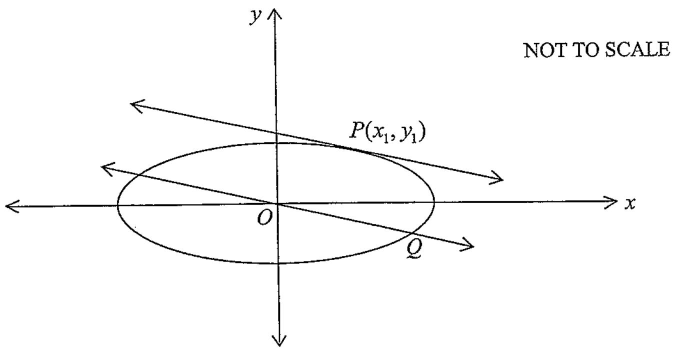
Question 6 continues on page 10

Marks

Question 6 (Continued)

(b) $P(x_1, y_1)$ is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with centre O. A line drawn from O, parallel to the tangent to the ellipse at P, meets the ellipse at Q.

The equation of the tangent to the ellipse at P is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.



(i) Show that the equation of the line OQ is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=0$. **2**

(ii) Show that the coordinates of Q, in terms of x_1 and y_1, are $\left(\frac{ay_1}{b}, \frac{-bx_1}{a}\right)$. **3**

(iii) Show that the distance between the tangent at P and the line OQ is $\frac{ab}{OQ}$. **3**

(iv) Hence prove that the area of the triangle OPQ is independent of the position P. **1**

End of Question 6

Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) A rock of mass 5 kg is propelled vertically upward into the air from the ground with initial speed 12 ms^{-1}. The rock is subject to air resistance of $\frac{v^2}{2}$ Newtons in the opposite direction to its velocity, v ms^{-1}. The rock is also subject to a downward gravitational force of 50 Newtons.

The equation of motion of the rock until it reaches its highest point is $\ddot{x}=-\frac{v^2}{10}-10$, where x metres is the height of the rock above the ground when its velocity is v ms^{-1}.

(i) Find the time taken by the rock to reach its maximum height. **3**

(ii) Show that $v^2=244e^{-\frac{x}{5}}-100$ while the rock is ascending. **3**

(iii) Find the maximum height reached by the rock. **2**

(b) The Fibonacci sequence 1, 1, 2, 3, 5, 8, . . . can be defined as $u_1=1$, $u_2=1$ and $u_{n+2}=u_n+u_{n+1}$ for integers $n>0$.

(i) Use induction to prove that $u_n<a^n$ for any $a>\frac{1+\sqrt{5}}{2}$. **3**

(ii) Assuming that $\frac{u_{n+1}}{u_n}$ approaches a limit as $n\to\infty$, **4**

show that $\frac{u_{n+1}}{u_n}\to\frac{1+\sqrt{5}}{2}$.

Marks

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the function $f(x)=(ax-b)^2+(cx-d)^2$ where a, b, c and d are real.

(i) Explain why $f(x)>0$ for all x. **1**

(ii) Hence, or otherwise, prove that $|ab+cd|\le\sqrt{a^2+c^2}\,\sqrt{b^2+d^2}$. **3**

(b) (i) Write out the binomial expansion of $\left(1+\frac{1}{n}\right)^n$, where n is a positive integer. **1**

(ii) Show that the $(k+1)^{th}$ term, T_{k+1} in your expansion, is given by **2**

$$T_{k+1}=\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right).$$

(iii) Let the $(k+1)^{th}$ term in the expansion of $\left(1+\frac{1}{n}\right)^{n+1}$ be U_{k+1}. **3**

Show that $U_{k+1}>T_{k+1}$.

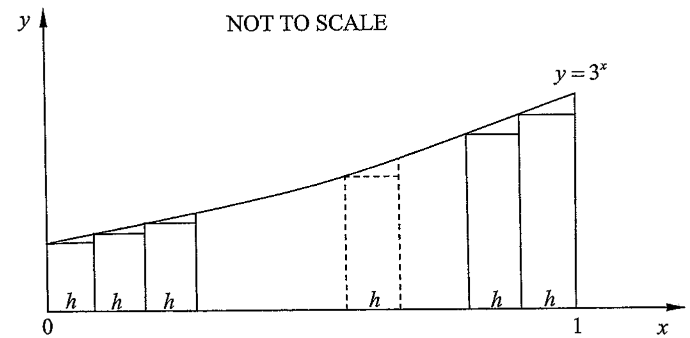
Question 8 continues on page 13

Marks

Question 8 (continued)

(c) The diagram shows the curve with equation $y=3^x$ for $0\le x\le 1$.

The area A under the curve between these limits is divided into n strips, each of width h where $nh=1$.



(i) Show that $A>\frac{2h}{3^h-1}$. **2**

(ii) Hence show that $\frac{h}{3^h-1}<\frac{1}{\ln 3}<\frac{h\,3^h}{3^h-1}$. **3**

End of paper

CATHOLIC SECONDARY SCHOOLS ASSOCIATION
2008 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION
MATHEMATICS EXTENSION 2

Question 1 (15 marks)
(a) (2 marks)
Outcomes assessed: E8
Targeted Performance Bands: E2-E3

Criteria	Marks
• finds the correct primitive	1
• evaluates the integral correctly	1

Sample Answer:

$$\int_0^{\frac{\pi}{4}} \cos x \ \sin^3 x \, dx = \tfrac{1}{4}\left[\sin^4 x\right]_0^{\frac{\pi}{4}}$$
$$= \tfrac{1}{4}\left[\left(\sin\tfrac{\pi}{4}\right)^4 - \left(\sin 0\right)^4\right]$$
$$= \tfrac{1}{4}\left[\left(\tfrac{1}{\sqrt{2}}\right)^4 - 0\right]$$
$$= \tfrac{1}{16}$$

(b) (3 marks)
Outcomes assessed: E8
Targeted Performance Bands: E2-E3

Criteria	Marks
• completes the square	1
• finds the correct primitive	1
• evaluates the integral correctly	1

Sample Answer:

$$\int_2^5 \frac{2}{x^2-4x+13}\,dx = \int_2^5 \frac{2}{(x-2)^2+3^2}\,dx$$
$$= 2\left[\frac{1}{3}\tan^{-1}\left(\frac{x-2}{3}\right)\right]_2^5$$
$$= \frac{2}{3}(\tan^{-1}1 - \tan^{-1}0)$$
$$= \frac{2}{3}\left(\frac{\pi}{4} - 0\right)$$
$$= \frac{\pi}{6}$$

(c) (i) (2 marks)
Outcomes assessed: E8
Targeted Performance Bands: E2-E3

Criteria	Marks
• significant progress towards answer	1
• evaluates the pronumerals	1

Sample Answer:

$$\frac{5}{x^2(2-x)} \equiv \frac{ax+b}{x^2} + \frac{c}{2-x}$$
$$5 \equiv (2-x)(ax+b) + cx^2$$
$$x=0 \ \Rightarrow \ 2b=5 \ \text{ i.e. } b=\frac{5}{2}$$
$$x=2 \ \Rightarrow \ 4c=5 \ \text{ i.e. } c=\frac{5}{4}$$
$$x=1 \ \Rightarrow \ 5=a+b+c \ \text{ i.e. } a=\frac{5}{4}$$

OR

$$5 \equiv 2ax + 2b - ax^2 - bx + cx^2$$
$$= 2b + x(2a-b) + x^2(c-a)$$

equating coefficients

$$b = \frac{5}{2}$$
$$c = a$$
$$a = \frac{b}{2} = \frac{5}{4}$$
$$\therefore c = \frac{5}{4}$$

(c) (ii) (3 marks)
Outcomes assessed: E8
Targeted Performance Bands: E2-E3

Criteria	Marks
• correctly uses previous result	1
• establishes correct integrand	1
• finds the primitive	1

Sample Answer:

$$\int \frac{20}{x^2(2-x)}\,dx = \int 4\left(\frac{\frac{5}{4}x+\frac{5}{2}}{x^2} + \frac{\frac{5}{4}}{2-x}\right)dx$$
$$= 5\int\left(\frac{x+2}{x^2} + \frac{1}{2-x}\right)dx$$
$$= 5\int\left(\frac{1}{x} + 2x^{-2} + \frac{1}{2-x}\right)dx$$
$$= 5\left[\ln|x| - \frac{2}{x} - \ln|2-x|\right] + C$$
$$= 5\ln\left|\frac{x}{2-x}\right| - \frac{10}{x} + C$$

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(d) (i) (3 marks)
Outcomes assessed: E8
Targeted Performance Bands: E2-E3

Criteria	Marks
• reduces integral to $\int \sin^2\theta\,d\theta$	1
• finds primitive of $\sin^2\theta$	1
• finds correct primitive	1

Sample Answer:

$$x = \sin\theta \qquad dx = \cos\theta\,d\theta \qquad \theta = \sin^{-1}x$$

$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta$$
$$= \int \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta$$
$$= \int \sin^2\theta\,d\theta$$
$$= \frac{1}{2}\int(1-\cos 2\theta)\,d\theta$$
$$= \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right)$$
$$= \frac{1}{2}\sin^{-1}x - \frac{1}{4}\sin(2\sin^{-1}x) + C$$

(d) (ii) (2 marks)
Outcomes assessed: E8
Targeted Performance Bands: E2-E3

Criteria	Marks
• correctly identifies u and v, u' and v' or significant progress towards solution	1
• finds correct primitive	1

Sample Answer:

$$u = \sin^{-1}x \qquad \frac{dv}{dx} = x$$
$$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad v = \frac{x^2}{2}$$

$$\int x\sin^{-1}x\,dx = \int \sin^{-1}x\,\tfrac{d}{dx}\left(\tfrac{x^2}{2}\right)dx$$
$$= \frac{x^2}{2}\sin^{-1}x - \int \frac{x^2}{2}\times\frac{1}{\sqrt{1-x^2}}\,dx$$
$$= \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\int\frac{x^2}{\sqrt{1-x^2}}\,dx$$
$$= \frac{x^2}{2}\sin^{-1}x - \frac{1}{4}\sin^{-1}x + \frac{1}{8}\sin(2\sin^{-1}x) + C$$

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Question 2 (15 marks)
(a) (1 mark)
Outcomes assessed: E3
Targeted Performance Bands: E2-E3

Criteria	Mark
• gives the correct solution	1

Sample Answer:

$$(3-4i)^3$$
$$= 3^3 - 3\times 3^2\times 4i + 3\times 3\times(4i)^2 - (4i)^3$$
$$= 27 - 108i - 144 + 64i$$
$$= -117 - 44i$$

(b) (3 marks)
Outcomes assessed: E3
Targeted Performance Bands: E2-E3

Criteria	Marks
• forms equations from the decomposition of real and imaginary parts	1
• solves equation to find values of x^2 or y^2	1
• gives correct solutions	1

Sample Answer:

$z^2 = 5 - 12i$

let $z = x + iy$

$\therefore x^2 - y^2 + 2xyi = 5 - 12i$

equating coefficients gives

$x^2 - y^2 = 5$ and $2xy = -12$

i.e. $y = -\dfrac{6}{x}$

$\therefore x^2 - \dfrac{36}{x^2} = 5$

$x^4 - 5x^2 - 36 = 0$

$(x^2+4)(x^2-9) = 0$

$x^2 = 9, -4$ but x is real $\therefore$ solutions from $x^2 = 9$ only

i.e. $x = 3$ or $x = -3$

when $x = 3$, $y = -2$

when $x = -3$, $y = 2$

$\therefore z = 3 - 2i$ or $z = -3 + 2i$

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(c) (i) (1 mark)
Outcomes assessed: E3
Targeted Performance Bands: E2-E3

Criteria	Mark
• gives correct modulus and argument	1

Sample Answer:

$$z = 1 + i$$

$$|z| = \sqrt{2} \text{ and } \arg z = \frac{\pi}{4}$$

$$\therefore z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

(c) (ii) (2 marks)
Outcomes assessed: E3
Targeted Performance Bands: E2-E3

Criteria	Marks
• uses De Moivre's Theorem to determine correct modulus and argument	1
• gives correct solution	1

Sample Answer:

$$\text{let } (1+i)^{12} = r$$

$$|r| = \left(\sqrt{2}\right)^{12} = 64$$

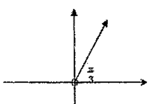
$$\arg r = 12 \times \frac{\pi}{4} = 3\pi$$

$$\therefore (1+i)^{12} = 64(\cos 3\pi + i\sin 3\pi) = -64$$

(d) (i) (1 mark)
Outcomes assessed: E3
Targeted Performance Bands: E2-E3

Criteria	Mark
• gives correct solution	1

Sample Answer:

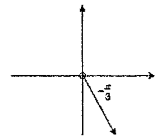


$$\arg z = \frac{\pi}{3}$$

(d) (ii) (1 mark)
Outcomes assessed: E3
Targeted Performance Bands: E2-E3

Criteria	Mark
• gives correct solution	1

Sample Answer:

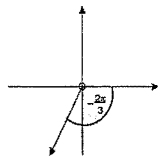


$$\arg \bar{z} = \frac{\pi}{3}$$

(d) (iii) (1 mark)
Outcomes assessed: E3
Targeted Performance Bands: E3-E4

Criteria	Mark
• gives correct solution	1

Sample Answer:



$$\arg(-z) = \frac{\pi}{3}$$

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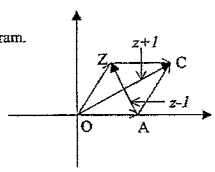
(e) (i) (1 mark)
Outcomes assessed: E3
Targeted Performance Bands: E2-E3

Criteria	Mark
• gives correct explanation	1

Sample Answer:
On addition of vectors, *OACZ* forms a parallelogram.
Since $|OA| = |z| = 1$, i.e. adjacent sides equal
OACZ must be a rhombus.



(e) (ii) (2 marks)
Outcomes assessed: E3
Targeted Performance Bands: E3-E4

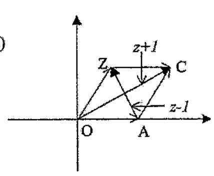
Criteria	Marks
• states perpendicular property of diagonals in rhombus	1
• gives correct explanation	1

Sample Answer:

$z + 1 \perp z - 1$ (diagonals perpendicular in rhombus)

$$\therefore \quad \arg\frac{z-1}{z+1} = \arg(z-1) - \arg(z+1) = \frac{\pi}{2}$$

$$\therefore \quad \frac{z-1}{z+1} \text{ is purely imaginary}$$



(e) (iii) (2 marks)
Outcomes assessed: E3
Targeted Performance Bands: E3-E4

Criteria	Marks
• gives correct modulus	1
• gives correct argument	1

Sample Answer:

$\arg z = \theta \therefore \arg(z+1) = \dfrac{\theta}{2}$ (diagonals bisect angles through which they pass in a rhombus)

$\angle OAC = 180 - \theta$

$\therefore |z+1| = \sqrt{1^2 + 1^2 - 2\cos(180 - \theta)}$ (using the cosine rule)

$= \sqrt{2 + 2\cos\theta}$

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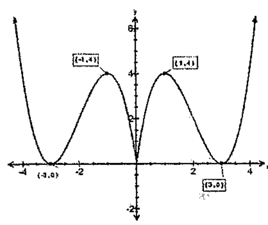
Question 3 (15 marks)
(a) (i) (α) (1 mark)
Outcomes assessed: E6
Targeted Performance Bands: E2-E3

Criteria	Mark
• correctly sketches the graph including all important points	1

Sample Answer:
$y = f(|x|)$

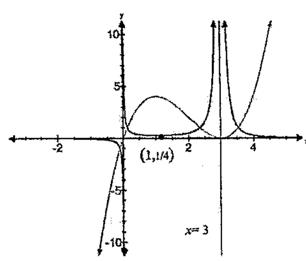


(a) (i) (β) (2 marks)
Outcomes assessed: E6
Targeted Performance Bands: E2-E3

Criteria	Marks
• significant progress towards the graph	1
• correctly sketches the graph	1

Sample Answer:

$$y = \frac{1}{f(x)}$$



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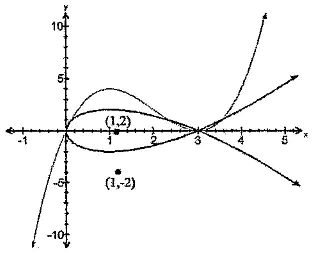
(a) (i) (γ) (2 marks)
Outcomes assessed: E6
Targeted Performance Bands: E2-E3

Criteria	Marks
• significant progress towards the graph	1
• correctly sketches the graph	1

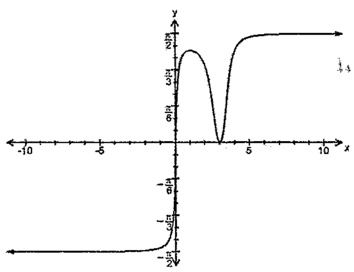
Sample Answer:
$y^2 = f(x)$



(a) (i) (δ) (2 marks)
Outcomes assessed: E6
Targeted Performance Bands: E3-E4

Criteria	Marks
• significant progress towards the graph	1
• correctly sketches the graph	1

Sample Answer:
$y = \tan^{-1} f(x)$



(a) (ii) (2 marks)
Outcomes assessed: E6
Targeted Performance Bands: E3-E4

Criteria	Marks
• significant progress towards answer or finds a value for k	1
• establishes correct values for k	1

Sample Answer:

$f(x) = kx$ has 2 distinct solutions if $k = 0$ (horizontal line through the origin) and another 2 distinct solutions when $y = kx$ is a tangent.

$f(x) = x(x-3)^2$
$= x(x^2 - 6x + 9)$
$= x^3 - 6x^2 + 9x$
$f'(x) = 3x^2 - 6x + 9$
at $x = 0$, $m = 9$ $\therefore$ the equation of the tangent is $y = 9x$
$\therefore k = 0$ or 9

OR

Solve simultaneously $f(x) = kx$ and $f(x) = x(x-3)^2$, i.e. $kx = x(x-3)^2$
$\therefore x\left((x-3)^2 - k\right) = 0$
So $x = 0$ or $x = 3 \pm \sqrt{k}$
For two distinct solutions: $k = 0$, solutions $x = 0$ or $x = 3$
OR
$k = 9$, solutions $x = 0$ or $x = 6$
$\therefore k = 0$ or 9

(b) (2 marks)
Outcomes assessed: E4
Targeted Performance Bands: E2-E3

Criteria	Marks
• uses the remainder theorem	1
• evaluates the remainder	1

Sample Answer:
$P(x) = Q(x)(x+2)(x-3) + (4x+1)$
by the Remainder Theorem, the remainder is $P(-2)$
$P(-2) = 0 + (4 \times -2 + 1)$
$= -7$
$\therefore R(x) = -7$

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(c) (i) (2 marks)
Outcomes assessed: E7
Targeted Performance Bands: E2-E3

Criteria	Marks
• significant progress towards solution	1
• gives correct expression for $A(x)$	1

Sample Answer:
The cross-section is a square with side length $y_1 + y_2$ where y_1 is on the curve $y^2 = 4x$ and y_2 is on the line $x + y = 0$.

$A(x) = (y_1 + y_2)^2$
$= (2\sqrt{x} - x)^2$
$= 4x + x^2 - 4x^{\frac{3}{2}}$

(c) (ii) (2 marks)
Outcomes assessed: E7
Targeted Performance Bands: E2-E3

Criteria	Marks
• establishes correct expression for V	1
• finds the volume (correct numerical equivalence)	1

Sample Answer:

Volume of slice $\Delta V = \left(4x + x^2 - 4x^{\frac{3}{2}}\right)\Delta x$

$$V = \lim_{\Delta x \to 0} \sum_{x=0}^{4} \Delta V$$
$$= \lim_{\Delta x \to 0} \sum_{x=0}^{4} \left(x^2 + 4x - 4x^{\frac{3}{2}}\right)\Delta x$$
$$= \int_0^4 \left(x^2 + 4x - 4x^{\frac{3}{2}}\right) dx$$
$$= \left[\frac{x^3}{3} + 2x^2 - \frac{8x^{\frac{5}{2}}}{5}\right]_0^4$$
$$= \left(\frac{64}{3} + 32 - \frac{8 \times 32}{5}\right) - (0)$$
$$= \frac{32}{15} \text{ cubic units}$$

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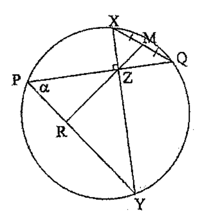
Question 4 (15 marks)
(a) (3 marks)
Outcomes assessed: PE3, E2
Targeted Performance Bands: E3-E4

Criteria	Marks
• gives complementary connection between $\angle P$ and $\angle Q$ with reasons	1
• gives connection between $\angle MZQ$ and $\angle Q$ with reasons	1
• gives correct conclusion with reasons	1

Sample Answer:



Construction: Join XQ, PY and MR
To prove: $MR \perp PY$
Proof: Let $\angle YPQ = \alpha$
$\therefore$ $\angle YXQ = \alpha$ (angles on the same arc QY)
$\therefore$ $\angle XQZ = 90 - \alpha$ (given $XY \perp PQ$)
Since $\angle XZQ$ is a right angle, XQ is the diameter and M is the centre of a circle passing through X, Z and Q
$\therefore$ $ZM = MQ$ (equal radii)
$\therefore \angle MZQ = \angle XQZ = 90 - \alpha$ (isosceles triangle)
$\therefore \angle PZR = 90 - \alpha$ (vertically opposite angles)
$\therefore \angle PRZ = 90°$ (angle sum of triangle)
ie $MR \perp PY$ as required

(b) (i) (2 marks)
Outcomes assessed: E8
Targeted Performance Bands: E3-E4

Criteria	Marks
• gives correct expressions for u, u', v, v' or significant progress towards solution	1
• gives correct steps to conclusion	1

Sample Answer:

$$I_n = \int x^n e^{ax}\, dx$$
Let $u = x^n$, $v' = e^{ax}$, $v = \frac{1}{a}e^{ax}$, $u' = nx^{n-1}$
$$\therefore \quad I_n = \frac{x^n}{a}e^{ax} - \int \frac{n}{a} x^{n-1} e^{ax}\, dx$$
$$= \frac{x^n}{a}e^{ax} - \frac{n}{a}\int x^{n-1} e^{ax}\, dx$$
$$= \frac{x^n}{a}e^{ax} - \frac{n}{a}I_{n-1} \quad \text{as required}$$

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(b) (ii) (3 marks)
Outcomes assessed: E8
Targeted Performance Bands: E2-E3

Criteria	Marks
• some progress towards solution e.g. one correct step say from I_3 to I_2	1
• gives correct expression for I_0	1
• gives correct solution	1

Sample Answer:

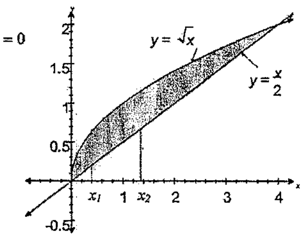
$$I_3 = \left[\frac{x^3}{2}e^{2x}\right]_0^1 - \frac{3}{2}I_2 = \frac{e^2}{2} - \frac{3}{2}I_2$$
$$I_2 = \left[\frac{x^2}{2}e^{2x}\right]_0^1 - \frac{2}{2}I_1 = \frac{e^2}{2} - I_1$$
$$I_1 = \left[\frac{x}{2}e^{2x}\right]_0^1 - \frac{1}{2}I_0 = \frac{e^2}{2} - \frac{1}{2}I_0$$
$$I_0 = \int_0^1 e^{2x}\,dx = \left[\frac{1}{2}e^{2x}\right]_0^1 = \frac{e^2}{2} - \frac{1}{2}$$
$$\therefore I_1 = \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4}$$
$$I_2 = \frac{e^2}{2} - \frac{e^2}{2} + \frac{e^2}{4} - \frac{1}{4} = \frac{e^2}{4} - \frac{1}{4}$$
$$I_3 = \frac{e^2}{2} - \frac{3}{2}\left[\frac{e^2}{4} - \frac{1}{4}\right] = \frac{e^2}{8} + \frac{3}{8}$$

(c) (4 marks)
Outcomes assessed: E7
Targeted Performance Bands: E2-E3

Criteria	Marks
• gives correct points of intersection	1
• gives correct expression for δV	1
• gives correct expression for V	1
• gives correct solution	1

Sample Answer:

Find points of intersection: solve $\sqrt{x} = \frac{x}{2}$ i.e. $x^2 - 4x = 0$
$x = 0,\ 4$ and hence $y = 0,\ 2$ respectively
i.e. points of intersection are $(0, 0)$ and $(4, 2)$.
Consider a slice, thickness δy, parallel to the x-axis, from $x_2 = 2y$ to $x_1 = y^2$, rotated about the x-axis to form a cylindrical shell.



Volume of the slice, $\delta V \approx 2\pi y\,(x_2 - x_1)\,\delta y$

$$V = \lim_{\delta y \to 0} \sum_{y=0}^{2} 2\pi y (x_2 - x_1)\,\delta y$$
$$= \int_0^2 2\pi y\,(2y - y^2)\,dy$$
$$= 2\pi \int_0^2 (2y^2 - y^3)\,dy$$
$$= 2\pi\left[\frac{2y^3}{3} - \frac{y^4}{4}\right]_0^2$$
$$= 2\pi\left(\frac{16}{3} - 4\right) - (0)$$
$$= \frac{8\pi}{3} \text{ cubic units}$$

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(d) (3 marks)
Outcomes assessed: E4
Targeted Performance Bands: E3-E4

Criteria	Marks
• determines the solutions of $P'(x) = 0$	1
• substitutes the correct value of x into $P(x)$	1
• gives correct steps to conclusion	1

Sample Answer:

Let $P(x) = x^5 - ax^2 + b$
$P'(x) = 5x^4 - 2ax$
multiple root $\therefore P'(x) = 0 \Rightarrow x(5x^3 - 2a) = 0$
i.e. $x = 0$ or $5x^3 = 2a$
but $P(0) \neq 0 \quad \therefore x = \left(\frac{2a}{5}\right)^{\frac{1}{3}}$ must be a solution to $P(x) = 0$
substituting into $P(x)$ gives:

$$\left(\frac{2a}{5}\right)^{\frac{5}{3}} - a \times \left(\frac{2a}{5}\right)^{\frac{2}{3}} + b = 0$$
$$\left(\frac{2a}{5}\right)^{\frac{5}{3}} - a \times \left(\frac{2a}{5}\right)^{\frac{2}{3}} = -b$$
$$a^{\frac{5}{3}}\left(\frac{2}{5}\right)^{\frac{5}{3}} - a^{\frac{5}{3}}\left(\frac{2}{5}\right)^{\frac{2}{3}} = -b$$
$$a^{\frac{5}{3}}\left(\frac{2}{5}\right)^{\frac{2}{3}}\left(\frac{2}{5} - 1\right) = -b$$

cubing this results gives:

$$a^5\left(\frac{2}{5}\right)^2\left(-\frac{3}{5}\right)^3 = (-b)^3$$
$$\frac{-108a^5}{3125} = -b^3$$

i.e. $108a^5 = 3125b^3$ as required

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Question 5 (15 marks)
(a) (3 marks)
Outcomes assessed: E9
Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes result for $a+b$ i.e. $a + b \geq 2\sqrt{ab}$ or some progress towards result	1
• generalises result to b and c	1
• establishes correct result	1

Sample Answer:

$(a-b)^2 \geq 0$ for all a, b
i.e. $a^2 - 2ab + b^2 \geq 0$
hence $a^2 + 2ab + b^2 \geq 4ab$ i.e. $(a+b)^2 \geq 4ab$
$\therefore a + b \geq 2\sqrt{ab}$ since a and b are both positive
Similarly $b + c \geq 2\sqrt{bc}$ and $c + a \geq 2\sqrt{ca}$
$\therefore (a+b)(b+c)(c+a) \geq 8\sqrt{ab}\sqrt{bc}\sqrt{ca}$
$\geq 8abc$

(b) (i) (2 marks)
Outcomes assessed: E4
Targeted Performance Bands: E2-E3

Criteria	Marks
• finds correct gradient of PQ	1
• derives the given equation	1

Sample Answer:

Given $P(2p, \frac{2}{p})$ and $Q(2q, \frac{2}{q})$

$$m_{PQ} = \frac{\frac{2}{p} - \frac{2}{q}}{2p - 2q}$$
$$= \frac{\frac{2(q-p)}{pq}}{2(p-q)}$$
$$= \frac{-1}{pq}$$

Equation of chord PQ

$$y - \frac{2}{p} = \frac{-1}{pq}(x - 2p)$$
$$pqy - 2q = -x + 2p$$
$$x + pqy = 2(p+q)$$

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(b) (ii) (1 mark)
Outcomes assessed: E4
Targeted Performance Bands: E2-E3

Criteria	Marks
• substitutes R to determine the relationship	1

Sample Answer:

The point $R(4, 2)$ satisfies the equation of PQ

$$4+2pq=2(p+q)$$
$$2+pq=p+q$$
$$\text{ie } pq=p+q-2$$

(b) (iii) (3 marks)
Outcomes assessed: E4
Targeted Performance Bands: E2-E3

Criteria	Marks
• finds the midpoint	1
• determines the locus	1
• sketches the hyperbola	1

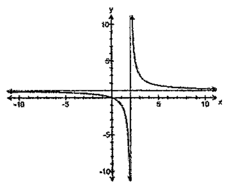
Sample Answer:

Find midpoint, M

$$M=\left(\frac{2p+2q}{2}, \frac{\frac{2}{p}+\frac{2}{q}}{2}\right)$$
$$=\left(p+q, \frac{p+q}{pq}\right)$$

i.e. $x=p+q$ and $y=\dfrac{p+q}{pq}$ so $y=\dfrac{p+q}{p+q-2}$ using result from (ii)

$$\therefore y=\frac{x}{x-2} \quad \text{i.e.} \quad y=1+\frac{2}{x-2}$$



(c) (i) (2 marks)
Outcomes assessed: E4
Targeted Performance Bands: E2-E3

Criteria	Marks
• factorises by difference of 2 cubes	1
• uses an alternative factorisation to show the result	1

Sample Answer:

$$z^9-1=(z^3-1)(z^6+z^3+1)$$
$$=(z-1)(z^2+z+1)(z^6+z^3+1)$$

$$\text{also } z^9-1=(z-1)(z^8+z^7+z^6+z^5+z^4+z^3+z^2+z+1)$$
$$\therefore z^8+z^7+z^6+z^5+z^4+z^3+z^2+z+1=(z^2+z+1)(z^6+z^3+1)$$

(c) (ii) (2 marks)
Outcomes assessed: E3, E4
Targeted Performance Bands: E3-E4

Criteria	Marks
• correctly solves $z^9-1=0$	1
• gives correct solution	1

Sample Answer:

Solve $z^9-1=0$ ie solve $z^9=1$

$1=\text{cis}\,0$

If $z=\text{cis}\,\theta$ then $z^9=\text{cis}\,9\theta$

$\therefore r=1$ and $9\theta=2n\pi$

$$\theta=0, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{6\pi}{9}, \frac{8\pi}{9}, \frac{-8\pi}{9}, \frac{-6\pi}{9}, \frac{-4\pi}{9}, \frac{-2\pi}{9}$$

For $z^9-1=0$ solutions are:

$$1, \text{cis}\frac{2\pi}{9}, \text{cis}\frac{4\pi}{9}, \text{cis}\frac{2\pi}{3}, \text{cis}\frac{8\pi}{9}, \text{cis}\frac{-8\pi}{9}, \text{cis}\frac{-2\pi}{3}, \text{cis}\frac{-4\pi}{9}, \text{cis}\frac{-2\pi}{9}$$

Solutions of $z^3-1=0$ are $1, \text{cis}\dfrac{2\pi}{3}, \text{cis}\dfrac{-2\pi}{3}$

$\therefore$solutions of $z^6+z^3+1=0$ are $\text{cis}\pm\dfrac{2\pi}{9}, \text{cis}\pm\dfrac{4\pi}{9}, \text{cis}\pm\dfrac{8\pi}{9}$

(c) (iii) (2 marks)
Outcomes assessed: E4
Targeted Performance Bands: E3-E4

Criteria	Marks
• uses sum of roots to establish initial result	1
• deduces correct identity	1

Sample Answer:

Sum of the roots of z^6+z^3+1 equals 0

$$\therefore \text{cis}\frac{2\pi}{9}+\text{cis}\frac{4\pi}{9}+\text{cis}\frac{8\pi}{9}+\text{cis}\frac{-8\pi}{9}+\text{cis}\frac{-4\pi}{9}+\text{cis}\frac{-2\pi}{9}=0$$
$$\text{ie } 2\left(\cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}+\cos\frac{8\pi}{9}\right)=0$$
$$\cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}=-\cos\frac{8\pi}{9}$$
$$\text{So } \cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}=\cos\frac{\pi}{9} \text{ since } \cos(\pi-\theta)=-\cos\theta$$

Question 6 (15marks)
(a) (i) (3 marks)
Outcomes assessed: E5
Targeted Performance Bands: E3-E4

Criteria	Marks
• gives correct expression for vertical resolution	1
• gives correct expression for horizontal resolution	1
• gives correct conclusion	1

Sample Answer:

Resolving horizontally:

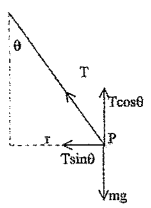
$$\frac{mv^2}{r}=T\sin\theta \qquad (1)$$

Resolving vertically:

$$mg=T\cos\theta \Rightarrow T=\frac{mg}{\cos\theta} \qquad (2)$$

Substitute (2) into (1)

$$\frac{mv^2}{r}=\frac{mg}{\cos\theta}\sin\theta$$
$$\frac{v^2}{rg}=\tan\theta \quad \text{as required}$$



(a) (ii) (1 mark)
Outcomes assessed: E5
Targeted Performance Bands: E3-E4

Criteria	Mark
• gives correct solution	1

Sample Answer:

$$T=\frac{mg}{\cos\theta}$$
$$=\frac{0.5\times9.8}{\sqrt{3}/2}$$
$$=\frac{9.8}{\sqrt{3}}=5.66N$$

(a) (iii) (1 mark)
Outcomes assessed: E5
Targeted Performance Bands: E3-E4

Criteria	Mark
• gives correct solution	1

Sample Answer:

$$v^2=rg\tan\theta$$
$$=\frac{1}{2}\times9.8\times\frac{1}{\sqrt{3}} \quad \text{since } \frac{r}{1}=\sin30^\circ=\frac{1}{2}$$
$$=2.82901\ldots$$
$$\therefore v=1.682\,\text{ms}^{-1}$$

(a) (iv) (1 mark)
Outcomes assessed: E5
Targeted Performance Bands: E3-E4

Criteria	Mark
• gives correct solution	1

Sample Answer:

$$\text{Period}=\frac{2\pi}{\omega}=\frac{2\pi r}{v}$$
$$=\frac{2\pi\times0.5}{1.682}$$
$$=1.9 \text{ seconds}$$

(b) (i) (2 marks)
Outcomes assessed: E3
Targeted Performance Bands: E2-E3

Criteria	Marks
• gives equation of family of lines parallel to tangent at P or equivalent merit	1
• gives correct solution	1

Sample Answer:

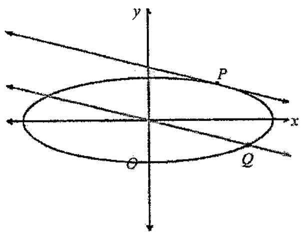
The tangent at P has equation $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

Lines parallel to P have equation

$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=c$

Substituting in (0,0) $\Rightarrow$ $c=0$

$\therefore \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=0$ is the equation of OQ.



(b) (ii) (3 marks)
Outcomes assessed: E4
Targeted Performance Bands: E3-E4

Criteria	Marks
• gives correct substitution of one equation into the other	1
• gives correct x value	1
• gives correct y value	1

Sample Answer:

Solving simultaneously $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=0 \Rightarrow y=-\frac{xx_1b^2}{y_1a^2}$ ......(1)

and $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ......(2)

Substituting (1) into (2)

$$\frac{x^2}{a^2}+\frac{x^2x_1^2b^4}{b^2y_1^2a^4}=1$$

$$x^2\left(\frac{1}{a^2}+\frac{x_1^2b^2}{y_1^2a^4}\right)=1$$

$$x^2\left(\frac{y_1^2a^2+x_1^2b^2}{y_1^2a^4}\right)=1$$

$$\therefore x^2\left(\frac{a^2b^2}{y_1^2a^4}\right)=1 \quad \text{since } y_1^2a^2+x_1^2b^2=a^2b^2$$

i.e. $x^2=\frac{y_1^2a^2}{b^2}$

hence $x=\frac{ay_1}{b}$ (in 4[th] quadrant, $x>0$)

$$\therefore y=-\frac{ay_1}{b}\times\frac{x_1b^2}{y_1a^2}$$

$$=-\frac{x_1b}{a}$$

i.e. $Q=\left(\frac{ay_1}{b}, \frac{-bx_1}{a}\right)$ as required

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(b) (iii) (3 marks)
Outcomes assessed: E4
Targeted Performance Bands: E3-E4

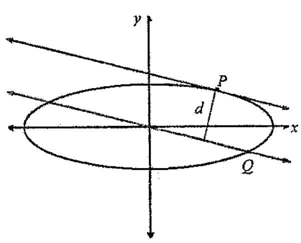
Criteria	Marks
• gives correct expression for perpendicular distance d between $P(x_1, y_1)$ and line OQ	1
• gives correct expression for OQ	1
• gives correct conclusion	1

Sample Answer:

$$d=\frac{\left|\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+0\right|}{\sqrt{\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}}}$$

$$=\frac{1}{\sqrt{\frac{x_1^2b^4+y_1^2a^4}{a^4b^4}}} \quad \text{since } \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$$

$$=\frac{a^2b^2}{\sqrt{x_1^2b^4+y_1^2a^4}}$$



$$OQ=\sqrt{\frac{a^2y_1^2}{b^2}+\frac{x_1^2b^2}{a^2}}$$

$$=\frac{\sqrt{a^4y_1^2+x_1^2b^4}}{ab}$$

Now $d=ab\times\frac{ab}{\sqrt{a^4y_1^2+x_1^2b^4}}$

$$=\frac{ab}{\frac{\sqrt{a^4y_1^2+x_1^2b^4}}{ab}}$$

$$=\frac{ab}{OQ} \quad \text{as required}$$

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(b) (iv) (1 mark)
Outcomes assessed: E4
Targeted Performance Bands: E2-E3

Criteria	Mark
• gives correct solution	1

Sample Answer:

$$\text{Area of } \Delta OPQ=\frac{1}{2}\times d\times OQ$$

$$=\frac{1}{2}\times\frac{ab}{OQ}\times OQ$$

$$=\frac{1}{2}ab$$

which is independent of P as required

Question 7 (15 marks)
(a) (i) (3 marks)
Outcomes assessed: E5
Targeted Performance Bands: E3-E4

Criteria	Marks
• sets up the integrand with the correct limits or some progress towards solution	1
• significant progress towards solution	1
• finds the time	1

Sample Answer:

$$\frac{dv}{dt}=-\frac{v^2+100}{10}$$

$$\text{So } \frac{dt}{dv}=-\frac{10}{v^2+100}$$

$$\int_{12}^{0}\frac{dt}{dv}\,dv=-\int_{12}^{0}\frac{10}{v^2+100}\,dv$$

$$t=\left[-\tan^{-1}\frac{v}{10}\right]_{12}^{0}$$

$$t=-\tan^{-1}0+\tan^{-1}\frac{12}{10}$$

$$t=\tan^{-1}\frac{12}{10}\approx 0.876\text{s}$$

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(a) (ii) (3 marks)
Outcomes assessed: E5
Targeted Performance Bands: E3-E4

Criteria	Marks
• some progress towards solution	1
• significant progress towards solution	1
• justifies the result	1

Sample Answer:

$$\ddot{x} = -\frac{v^2}{10} - 10 \text{ and } \ddot{x} = v\frac{dv}{dx}$$

$$\therefore v\frac{dv}{dx} = \frac{-v^2-100}{10}$$

$$\frac{dv}{dx} = \frac{-v^2-100}{10v}$$

$$\text{So } \frac{dx}{dv} = -\frac{10v}{v^2+100}$$

$$\int\frac{dx}{dv}\,dv = -\int\frac{10v}{v^2+100}\,dv$$

$$\therefore x = -5\ln(v^2+100)+C$$

when $x=0, v=12$ so $0=-5\ln(144+100)+C$ i.e. $C=5\ln 244$

$$\therefore x = -5\ln\left(\frac{v^2+100}{244}\right)$$

$$e^{-\frac{x}{5}} = \frac{v^2+100}{244}$$

$$\text{so } v^2 = 244e^{-\frac{x}{5}} - 100$$

(a) (iii) (2 marks)
Outcomes assessed: E5
Targeted Performance Bands: E3-E4

Criteria	Marks
• uses velocity is zero at maximum height	1
• finds the maximum height	1

Sample Answer:

Maximum height when $v=0$ i.e. solve $e^{-\frac{x}{5}} = \frac{100}{244}$

$$-\frac{x}{5} = \ln\left(\frac{100}{244}\right)$$

$$x = -5\ln\left(\frac{100}{244}\right) \approx 4.46\,\text{m}$$

(b) (i) (3 marks)
Outcomes assessed: E2
Targeted Performance Bands: E3-E4

Criteria	Marks
• shows true for two values of n	1
• uses the assumptions correctly to set up proof	1
• proves final result	1

Sample Answer:

1, 1, 2, 3, 5, 8, . . . defined as $u_1 = 1$, $u_2 = 1$ and $u_{n+2} = u_n + u_{n+1}$

Prove that $u_n < a^n$ for any $a > \frac{1+\sqrt{5}}{2}$

Show true for $n=1$ and $n=2$: $u_1 = 1 < \frac{1+\sqrt{5}}{2}$ and $u_2 = 1 < \left(\frac{1+\sqrt{5}}{2}\right)^2$

$\therefore$ true for $n=1$ and $n=2$

Assume true for $n=k$ and $n=k+1$ i.e. $u_k < a^k$ and $u_{k+1} < a^{k+1}$

Show true for $n=k+2$ i.e. $u_{k+2} < a^{k+2}$

$$\begin{aligned} u_{k+2} &= u_k + u_{k+1} \\ &< a^k + a^{k+1} \quad \text{using assumptions} \\ &= a^k + a \times a^k \\ &= a^k(1+a) \\ &< a^k \times a^2 \quad \text{since } (1+a) < a^2 \text{ when } a > \frac{1+\sqrt{5}}{2} \\ &= a^{k+2} \end{aligned}$$

$$\therefore u_{k+2} < a^{k+2}$$

Hence if result is true for $n=k$ and $n=k+1$ then it is true for $n=k+2$.
Thus since it is true for $n=1$ and $n=2$, it follows by induction that it is true for all positive integral n.

(b) (ii) (4 marks)
Outcomes assessed: E2
Targeted Performance Bands: E3-E4

Criteria	Marks
• some progress towards solution	1
• further progress towards solution	1
• significant progress towards solution	1
• establishes the result	1

Sample Answer:

If $\frac{u_{n+1}}{u_n}$ approaches a limit as $n \to \infty$ then $\frac{u_{n+2}}{u_{n+1}}$ approaches the same limit

So as $n \to \infty$ $\frac{u_{n+2}}{u_{n+1}} = \frac{u_{n+1}}{u_n}$ i.e. $\frac{u_{n+1}+u_n}{u_{n+1}} = \frac{u_{n+1}}{u_n}$ since $u_{n+2} = u_n + u_{n+1}$

$$\therefore 1 + \frac{u_n}{u_{n+1}} = \frac{u_{n+1}}{u_n}$$

Let $\frac{u_{n+1}}{u_n} = x$ as $n \to \infty$

$$\therefore 1 + \frac{1}{x} = x$$

i.e. $x^2 - x - 1 = 0$

$$x = \frac{1 \pm \sqrt{5}}{2}$$

but $x > 0$ $\therefore$ limit is $\frac{1+\sqrt{5}}{2}$

i.e. as $n \to \infty$, $\frac{u_{n+1}}{u_n} \to \frac{1+\sqrt{5}}{2}$

Question 8 (15 marks)
(a) (i) (1 marks)
Outcomes assessed: E6
Targeted Performance Bands: E2-E3

Criteria	Mark
• gives correct explanation	1

Sample Answer:

$f(x) = (ax-b)^2 + (cx-d)^2$

Since $(ax-b)^2 \geq 0$ and $(cx-d)^2 \geq 0$ for all x

$f(x) \geq 0$ for all x

(a) (ii) (3 marks)
Outcomes assessed: E2
Targeted Performance Bands: E3-E4

Criteria	Marks
• some progress towards solution	1
• significant progress towards solution	1
• establishes the result	1

Sample Answer:

$$\begin{aligned} f(x) &= a^2x^2 - 2abx + b^2 + c^2x^2 - 2cdx + d^2 \\ &= (a^2+c^2)x^2 - (2ab+2cd)x + b^2 + d^2 \end{aligned}$$

Since $f(x) \geq 0$ for all x, the function has no real roots, i.e. $\Delta \leq 0$

$$\Delta = (-(2ab+2cd))^2 - 4(a^2+c^2)(b^2+d^2) \leq 0$$

$$4(ab+cd)^2 - 4(a^2+c^2)(b^2+d^2) \leq 0$$

$$(ab+cd)^2 \leq (a^2+c^2)(b^2+d^2)$$

$$\sqrt{(ab+cd)^2} \leq \sqrt{a^2+c^2}\sqrt{b^2+d^2}$$

i.e. $|ab+cd| \leq \sqrt{a^2+c^2}\sqrt{b^2+d^2}$ as required

(b) (i) (1 mark)
Outcomes assessed: E2
Targeted Performance Bands: E2-E3

Criteria	Marks
• gives the correct expansion	1

Sample Answer:

$$\left(1+\frac{1}{n}\right)^n = 1 + {}^nC_1\frac{1}{n} + {}^nC_2\frac{1}{n^2} + \ldots + {}^nC_k\frac{1}{n^k} + \ldots + \frac{1}{n^n}$$

(b) (ii) (2 marks)
Outcomes assessed: E2, E9
Targeted Performance Bands: E3-E4

Criteria	Marks
• significant progress towards solution	1
• establishes the result	1

Sample Answer:

$$\begin{aligned}
T_{k+1} &= {}^nC_k\frac{1}{n^k}\\
&= \frac{n!}{k!(n-k)!}\times\frac{1}{n^k}\\
&= \frac{n(n-1)(n-2)\ldots(n-(k-2))(n-(k-1))(n-k)!}{k!(n-k)!}\times\frac{1}{n^k}\\
&= \frac{1}{k!}\times\frac{n}{n}\times\frac{(n-1)}{n}\times\frac{(n-2)}{n}\times\ldots\times\frac{(n-(k-2))}{n}\times\frac{(n-(k-1))}{n}\\
&= \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{k-2}{n}\right)\left(1-\frac{k-1}{n}\right) \quad \text{as required}
\end{aligned}$$

(b) (iii) (3 marks)
Outcomes assessed: E2, E9
Targeted Performance Bands: E3-E4

Criteria	Marks
• some progress towards solution	1
• significant progress towards solution	1
• establishes the result	1

Sample Answer:

$$\begin{aligned}
U_{k+1} &= {}^{n+1}C_k\frac{1}{n^k}\\
&= \frac{(n+1)!}{k!((n+1)-k)!}\times\frac{1}{n^k}\\
&= \frac{(n+1)n(n-1)(n-2)\ldots((n+1)-(k-2))((n+1)-(k-1))((n+1)-k)!}{k!((n+1)-k)!}\times\frac{1}{n^k}\\
&= \frac{1}{k!}\times\frac{(n+1)}{n}\times\frac{n}{n}\times\frac{(n-1)}{n}\times\frac{(n-2)}{n}\times\ldots\times\frac{(n-(k-3))}{n}\times\frac{(n-(k-2))}{n}\\
&= \frac{1}{k!}\left(1+\frac{1}{n}\right)\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{k-3}{n}\right)\left(1-\frac{k-2}{n}\right)\\
&= \frac{T_{k+1}}{\left(1-\frac{k-1}{n}\right)}\times\left(1+\frac{1}{n}\right)\\
&= T_{k+1}\times\frac{n}{n-k+1}\times\frac{n+1}{n}\\
&= T_{k+1}\times\frac{n+1}{n-k+1}
\end{aligned}$$

$$\therefore \frac{U_{k+1}}{T_{k+1}} = \frac{n+1}{n-k+1} > 1 \qquad \text{since } n+1 > n-k+1$$

$$\therefore U_{k+1} > T_{k+1} \qquad \text{as required}$$

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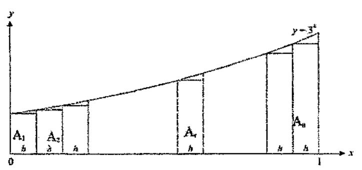
(c) (i) (2 marks)
Outcomes assessed: E2
Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes inequality using summation of area of rectangles inside the curve	1
• shows the required result using the sum of a geometric series	1

Sample Answer:

Let A_r be the area of the r^{th} rectangle as shown ($r = 1$ to n)

Then the area under the curve, $A > \sum_{r=1}^{n} A_r$



$$\begin{aligned}
\text{i.e. } A &> 3^0\times h + 3^h\times h + 3^{2h}\times h + \ldots + 3^{rh}\times h + \ldots + 3^{(n-1)h}\times h\\
&= h(1+3^h+3^{2h}+\ldots+3^{(n-1)h})\\
&= \frac{h\times 1(3^{nh}-1)}{3^h-1} \quad (\text{geometric series: } a=1, r=3^h)\\
&= \frac{h\times 1(3-1)}{3^h-1} \quad (\text{given } nh=1)\\
&= \frac{2h}{3^h-1}
\end{aligned}$$

$$\text{i.e. } A > \frac{2h}{3^h-1}$$

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(c) (ii) (3 marks)
Outcomes assessed: E2
Targeted Performance Bands: E3-E4

Criteria	Marks
• uses sum of rectangles above the curve to find the upper bound of A	1
• establishes that A lies between the sum of the area of the inner and outer rectangles and that $A = \int_0^1 3^x\,dx$	1
• completes the proof	1

Sample Answer:

Similarly, using rectangles above the curve:

$$\begin{aligned}
A &< 3^h\times h + 3^{2h}\times h + \ldots + 3^{(n-1)h}\times h + 3^{nh}\times h\\
&= h(3^h+3^{2h}+\ldots+3^{(n-1)h}+3^{nh})\\
&= \frac{h\times 3^h(3^{nh}-1)}{3^h-1} \quad (\text{geometric series: } a=3^h, r=3^h)\\
&= \frac{h\times 3^h(3-1)}{3^h-1} \quad (\text{given } nh=1)\\
&= \frac{2h\times 3^h}{3^h-1}
\end{aligned}$$

$$\text{i.e. } A < \frac{2h\times 3^h}{3^h-1}$$

$$\text{Thus } \frac{2h}{3^h-1} < A < \frac{2h\times 3^h}{3^h-1}$$

$$\begin{aligned}
\text{But } A &= \int_0^1 3^x\,dx\\
&= \frac{1}{\ln 3}\left[3^x\right]_0^1\\
&= \frac{1}{\ln 3}(3-1)\\
&= \frac{2}{\ln 3}
\end{aligned}$$

$$\text{Thus } \frac{2h}{3^h-1} < \frac{2}{\ln 3} < \frac{2h\times 3^h}{3^h-1}$$

$$\text{i.e. } \frac{h}{3^h-1} < \frac{1}{\ln 3} < \frac{h3^h}{3^h-1}$$

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