Centre Number

Student Number

CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NEW SOUTH WALES

2009
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics
Extension 2

Morning Session
Monday, 17 August 2009

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided as a separate page
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1-8
- All questions are of equal value

Total marks – 120
Attempt Questions 1–8
All questions are of equal value

Answer each question in a SEPARATE writing booklet.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\int_3^4 x\sqrt{x-3}\, dx$. **3**

(b) Use the substitution $t=\tan\frac{\theta}{2}$ to find $\int \frac{\tan\theta}{1+\cos\theta}\, d\theta$. **3**

(c) Use integration by parts to evaluate $\int_1^e \frac{\ln x}{\sqrt{x}}\, dx$. **3**

(d) (i) Find real numbers a and b such that $\frac{1}{x(x+1)} \equiv \frac{a}{x}+\frac{b}{x+1}$. **2**

(ii) Hence find $\int_1^N \frac{dx}{x(x+1)}$. **2**

(iii) Hence find the area bounded by the curve $y=\frac{1}{x(x+1)}$ and the x-axis to the right of $x=1$. **2**

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Given that $z=2+i$ and $w=-2$, express $\dfrac{z}{z+w}$ in the form $x+iy$ where x and y are real numbers. **1**

(b) (i) Express $w=4+4i$ in modulus-argument form. **1**

(ii) Hence, or otherwise, find all numbers z such that $z^5=4+4i$, giving your answer in modulus-argument form. **3**

(c) (i) On a single Argand diagram, sketch the locus of points for which $|z-3i|\le 2$ and $\arg(z+1)\le\dfrac{\pi}{4}$. **2**

(ii) Find the value of $\arg z$ at the point where $\arg z$ is a minimum. **2**

(d) (i) Given that ω is one of the complex roots of $z^3=1$, show that $1+\omega+\omega^2=0$. **2**

(ii) Hence, or otherwise, evaluate $1+\omega^4+\omega^8$. **1**

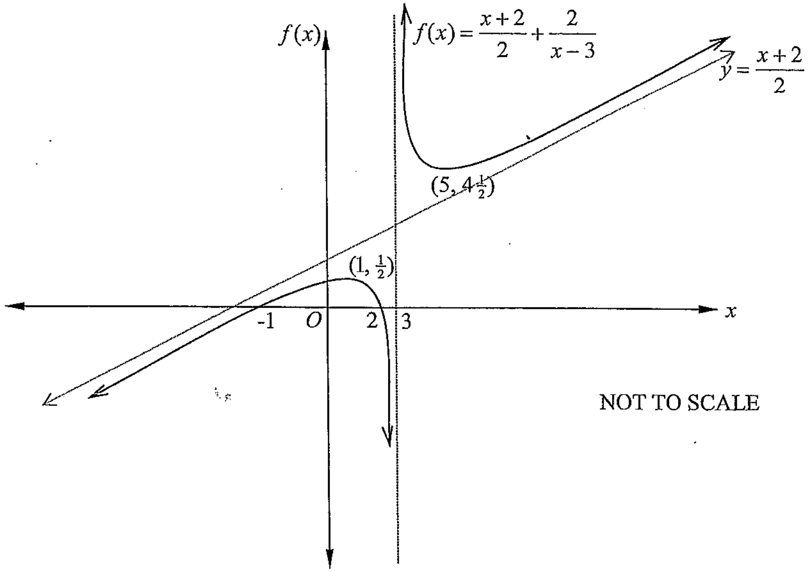
(e) The complex numbers z and w each have a modulus of 4. The arguments of z and w are $\dfrac{2\pi}{9}$ and $\dfrac{5\pi}{9}$, respectively. Find

(i) $\arg(z+w)$. **1**

(ii) $|z+w|$. **2**

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The function defined by $f(x)=\dfrac{x+2}{2}+\dfrac{2}{x-3}$ is drawn below.



Draw separate, one-third page sketches, of the following:

(i) $y=|f(x)|$. **1**

(ii) $y=[f(x)]^2$. **2**

(iii) $y=f(x^2)$. **2**

(iv) $y=\ln[f(x)]$. **2**

Question 3 continues on page 5

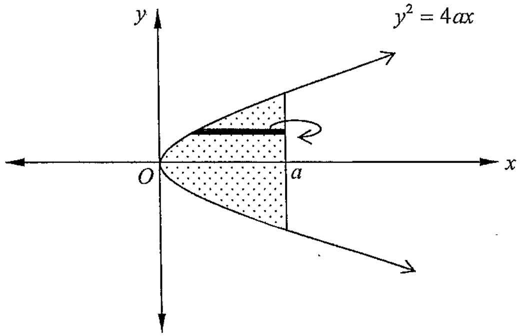
Question 3 (continued) **Marks**

(b) The cubic equation $3x^3-9x^2+6x+2=0$ has roots α, β and γ.

(i) Find the cubic equation with roots α^2, β^2 and γ^2. **2**

(ii) Hence evaluate $\alpha^3\beta\gamma+\alpha\beta^3\gamma+\alpha\beta\gamma^3$. **2**

(c) The figure drawn below shows the shaded area enclosed by the parabola $y^2=4ax$ and the line $x=a$. **4**



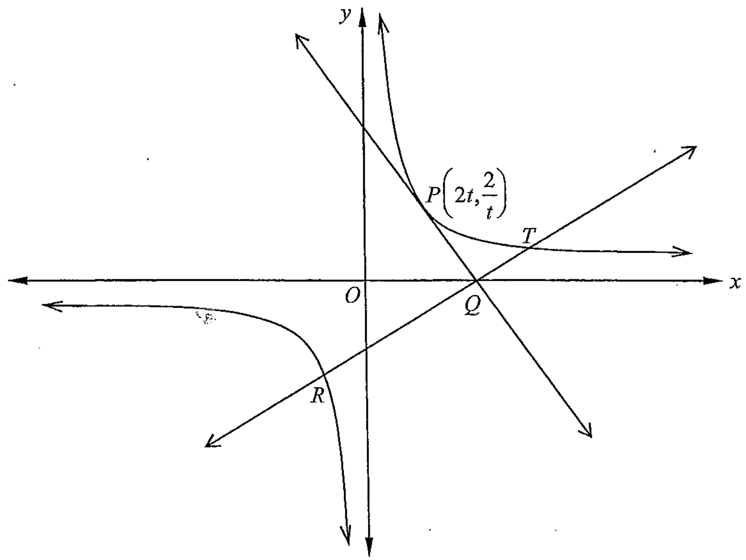
This area is rotated about the line $x=a$. By considering slices perpendicular to the axis of rotation, as shown above, find the volume of the solid of revolution.

End of Question 3

Marks

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) $P\left(2t,\frac{2}{t}\right)$ is a point on the rectangular hyperbola $xy=4$. The tangent at P cuts the x-axis at Q and the line through Q, perpendicular to the tangent at P, cuts the hyperbola at the points R and T as shown.



(i) Show that the equation of the tangent at P is $x+t^2y=4t$. **2**

(ii) Show that the line through Q, perpendicular to the tangent at P, has equation $t^2x-y=4t^3$. **3**

(iii) If M is the midpoint of RT, show that M has coordinates $(2t, -2t^3)$. **3**

(iv) Find the equation of the locus of M, as P moves on the curve $xy=4$. **1**

Question 4 continues on page 7

Question 4 (continued) **Marks**

(b) (i) State why the sum of n terms of $1+x+x^2+x^3+\ldots+x^{n-1}$ is equal to $\frac{1-x^n}{1-x}$. **1**

(ii) Show that, for all $n>1$, **2**

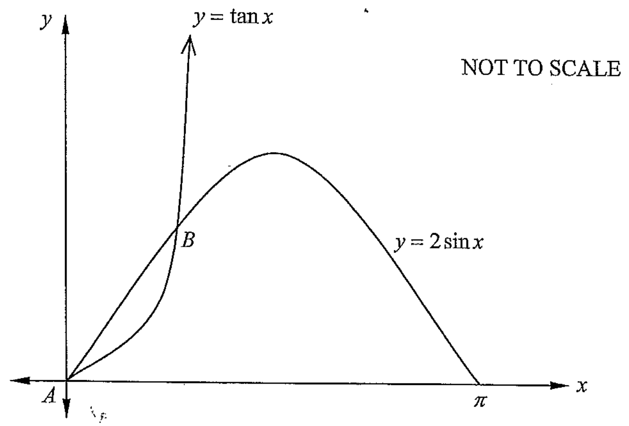
$$1+2x+3x^2+\ldots+(n-1)x^{n-2}=\frac{(n-1)x^n-nx^{n-1}+1}{(1-x)^2}.$$

(iii) Hence find an expression for $1+1+\frac{3}{4}+\frac{4}{8}+\frac{5}{16}+\ldots+\frac{n-1}{2^{n-2}}$ and show that this sum is always less than 4. **3**

End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The graphs of $y=2\sin x$ and $y=\tan x$ are shown below.



The base of a solid is the region bounded by the curves $y=2\sin x$ and $y=\tan x$ in the first quadrant between $A(0, 0)$ and $B\left(\frac{\pi}{3}, \sqrt{3}\right)$.

The solid is formed such that every cross-sectional slice perpendicular to the x-axis is a semicircle having a diameter with one end-point on the curve $y=2\sin x$ and the other on $y=\tan x$.

(i) Show that the area of the cross-section of the solid is given by **2**

$$A(x)=\frac{\pi}{8}\left(4\sin^2 x-4\sin x\tan x+\tan^2 x\right).$$

(ii) Hence find the volume of the solid formed. **4**

Question 5 continues on page 9

Question 5 (continued) **Marks**

(b) (i) Use De Moivre's theorem to show that $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ and $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$. **2**

(ii) Deduce that $\tan 3\theta = \dfrac{\tan^3\theta - 3\tan\theta}{3\tan^2\theta - 1}$. **2**

(iii) Hence, or otherwise, show that $\tan\dfrac{\pi}{12}$ is a root of the equation $x^3 - 3x^2 - 3x + 1 = 0$. **3**

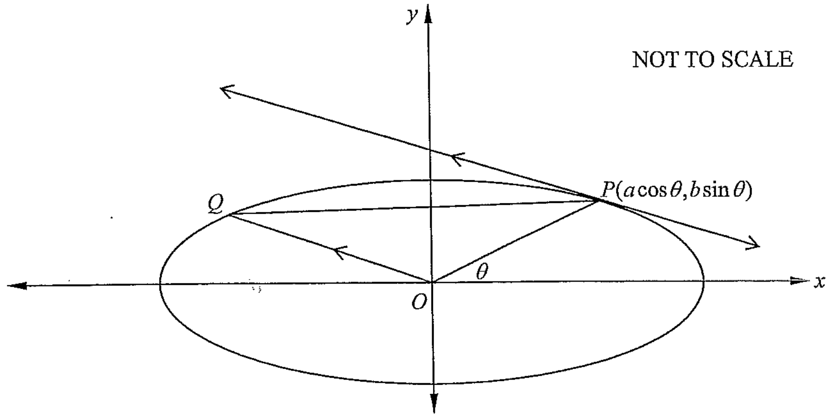
(iv) Show that $\tan\dfrac{\pi}{12} + \tan\dfrac{5\pi}{12} = 4$. **2**

End of Question 5

Marks

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) $P(a\cos\theta,\ b\sin\theta)$ is a point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $a \geq b$ where a and b are positive. A line through the origin and parallel to the tangent at P meets the ellipse at Q.



(i) Show that the gradient of the tangent at P is $\dfrac{-b}{a}\cot\theta$. **1**

(ii) Show that the coordinates of Q are $(-a\sin\theta,\ b\cos\theta)$. **2**

(iii) Find the length of PQ and state the maximum and minimum values of this length for $-\pi \leq \theta \leq \pi$. **3**

Question 6 continues on page 11

Question 6 (continued)

Marks

(b) (i) Given that $x > 0$ and $y > 0$ show that $x + y \ge 2\sqrt{xy}$. **1**

(ii) Hence show that for $x > 0,\ y > 0,\ z > 0$ and $w > 0$ **2**

$x + y + z + w \ge 4\sqrt[4]{xyzw}$.

(iii) Consider x, y, z and $w = \dfrac{x+y+z}{3}$. Apply the result in (ii) to show **2**

that $\dfrac{x+y+z}{3} \ge \sqrt[3]{xyz}$.

(c) Consider $I = \displaystyle\int_{-1}^{1} \frac{x^2 e^x}{e^x + 1}\,dx$ and $J = \displaystyle\int_{-1}^{1} \frac{x^2}{e^x + 1}\,dx$.

(i) Use the substitution $u = -x$ in I to show that $I = J$. **3**

(ii) Hence evaluate I. **1**

End of Question 6

Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that $2\cos A \sin B = \sin(A+B) - \sin(A-B)$. **1**

(ii) Show that $\dfrac{\sin\left(\frac{3x}{2}\right) - \sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)} = \cos x$. **2**

(iii) Prove by induction that for all positive integers n, **4**

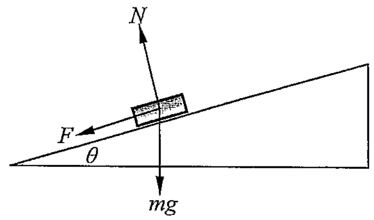
$$\cos x + \cos 2x + \cos 3x + \ldots + \cos nx = \frac{\sin\left(\frac{2n+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)} - \frac{1}{2}.$$

Question 7 continues on page 13

Question 7 (continued)

Marks

(b) A car of mass m is travelling around a circular banked track and is subject to a reaction force, N, perpendicular to the track and a frictional force, F, parallel to the track as indicated in the diagram.



The track has a radius of 300 metres and is banked at an angle of $\theta = 5^\circ$ to the horizontal.

(i) Use a diagram to show that **2**

$$mg = N\cos\theta - F\sin\theta$$

and $$\frac{mv^2}{r} = F\cos\theta + N\sin\theta.$$

(ii) By squaring the equations above find an expression for $N^2 + F^2$. **2**

(iii) If the car is travelling at a speed of 72 km/h and the ratio of N to F is 6:1, show that in this situation $N = \frac{6m}{\sqrt{37}}\sqrt{g^2 + \frac{160\,000}{r^2}}$. **2**

(iv) A driver wants to reduce the frictional force acting on the car to zero. Find the speed at which the car must travel around the track in this situation. (You may assume that $g = 10\,\text{m}\,\text{s}^{-2}$.) **2**

End of Question 7

Marks

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) The only force acting on a particle moving horizontally in a straight line is a resistance of $mk(c+v)$ acting in that line, where m is the mass of the particle, v is the velocity and k and c are positive constants. Initially the particle moves with positive velocity U and comes to rest at time T.

At time $\frac{1}{2}T$ its velocity is $\frac{1}{4}U$.

(i) Show that the acceleration is given by $a = -k(c+v)$. **1**

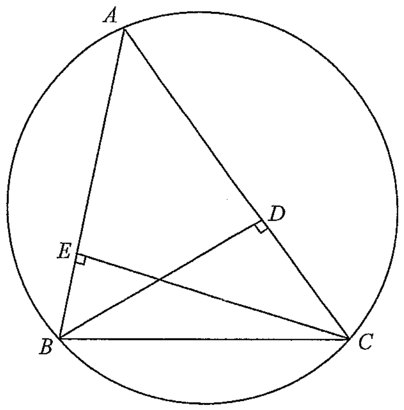
(ii) Show that $c = \frac{1}{8}U$. **4**

(iii) Show that at time t, $\frac{8v}{U} = 9e^{-kt} - 1$. **2**

Question 8 continues on page 15

Marks

(b) In the diagram, BC is a fixed chord of a circle and A is a variable point on the major arc of the chord BC. The point D lies on AC, between A and C, such that BD is perpendicular to AC. The point E lies on AB, between A and B, such that CE is perpendicular to AB.



NOT TO SCALE

Copy or trace this diagram into your writing booklet.

(i) Show that $BCDE$ is a cyclic quadrilateral on a circle with BC as diameter. **2**

(ii) Show that as A varies ED has a constant length. **3**

(iii) Show that the locus of the midpoint of ED is part of a circle whose centre is the midpoint of BC. **3**

End of paper

CATHOLIC SECONDARY SCHOOLS ASSOCIATION

2009 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

MATHEMATICS EXTENSION 2

Question 1 (15 marks)
(a) (3 marks)
Outcomes assessed: E8
Targeted Performance Bands: E2-E3

Criteria	Marks
• uses an appropriate substitution	1
• establishes correct integral	1
• evaluates correctly	1

Sample Answer:

$$\int_3^4 x\sqrt{x-3}\,dx = \int_0^1 (u+3)\sqrt{u}\,du \qquad \text{let } u = x-3 \qquad \text{when } x=3,\ u=0$$

$$= \int_0^1 u^{\frac{3}{2}} + 3u^{\frac{1}{2}}\,du \qquad \frac{du}{dx}=1 \qquad \text{when } x=4,\ u=1$$

$$= \left[\frac{2}{5}u^{\frac{5}{2}} + 2u^{\frac{3}{2}}\right]_0^1$$

$$= \left(\frac{2}{5}+2\right) - (0-0)$$

$$= 2\frac{2}{5}$$

(b) (3 marks)
Outcomes assessed: E8
Targeted Performance Bands: E2-E3

Criteria	Marks
• substitutes using t formula	1
• integrates correctly	1
• finds correct primitive (+C not essential)	1

Sample Answer:

If $t=\tan\frac{\theta}{2}$ then $\tan\theta = \frac{2t}{1-t^2}$, $\cos\theta = \frac{1-t^2}{1+t^2}$ and $\frac{d\theta}{dt} = \frac{2}{1+t^2}$.

$$\int \frac{\tan\theta}{1+\cos\theta}\,d\theta = \int \frac{2t}{1-t^2} \div \left(1+\frac{1-t^2}{1+t^2}\right) \times \frac{2}{1+t^2}\,dt$$

$$= \int \frac{2t}{1-t^2} \div \left(\frac{1+t^2+1-t^2}{1+t^2}\right) \times \frac{2}{1+t^2}\,dt$$

$$= \int \frac{2t}{1-t^2} \times \left(\frac{1+t^2}{2}\right) \times \frac{2}{1+t^2}\,dt$$

$$= \int \frac{2t}{1-t^2}\,dt$$

$$= -\ln\left|1-t^2\right| + C$$

$$= -\ln\left|1-\tan^2\frac{\theta}{2}\right| + C$$

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(c) (3 marks)
Outcomes assessed: E8
Targeted Performance Bands: E2-E3

Criteria	Marks
• correctly applies integration by parts or significant progress towards solution	1
• finds correct primitive	1
• evaluates the integral	1

Sample Answer:

$$\int_1^e \frac{\ln x}{\sqrt{x}}\,dx = \int_1^e \ln x \frac{d}{dx}\left(2\sqrt{x}\right)dx \qquad u = \ln x \qquad \frac{dv}{dx} = \frac{1}{\sqrt{x}}$$

$$= \left[2\sqrt{x}\ln x\right]_1^e - \int_1^e \frac{1}{x}\times 2\sqrt{x}\,dx \qquad \frac{du}{dx} = \frac{1}{x} \qquad v = 2\sqrt{x}$$

$$= \left(2\sqrt{e}\ln e - 2\ln 1\right) - 2\int_1^e \frac{1}{\sqrt{x}}\,dx$$

$$= 2\sqrt{e} - 2\left[2\sqrt{x}\right]_1^e$$

$$= 2\sqrt{e} - 2\left(2\sqrt{e} - 2\right)$$

$$= 4 - 2\sqrt{e}$$

(d) (i) (2 marks)
Outcomes assessed: E8
Targeted Performance Bands: E2-E3

Criteria	Marks
• significant progress towards answer	1
• evaluates the pronumerals	1

Sample Answer:

$$\frac{1}{x(x+1)} \equiv \frac{a}{x} + \frac{b}{x+1}$$

i.e. $1 \equiv a(x+1) + bx$

$a = 1$ and $a + b = 0$

$\therefore b = -1$

$$\therefore \frac{1}{x(x+1)} \equiv \frac{1}{x} - \frac{1}{x+1}$$

(d) (ii) (2 marks)
Outcomes assessed: E8
Targeted Performance Bands: E2-E3

Criteria	Marks
• finds correct integral	1
• finds correct expression	1

Sample Answer:

$$\int_1^N \frac{dx}{x(x+1)} = \int_1^N\left(\frac{1}{x} - \frac{1}{x+1}\right)dx$$

$$= \left[\ln|x| - \ln|x+1|\right]_1^N$$

$$= \ln N - \ln(N+1) - (\ln 1 - \ln 2)$$

$$= \ln N - \ln(N+1) + \ln 2$$

$$= \ln\left(\frac{2N}{N+1}\right)$$

(d) (iii) (2 marks)
Outcomes assessed: E8
Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes correct limit	1
• finds correct area	1

Sample Answer:

$$A = \int_1^N \frac{dx}{x(x+1)} = \ln\left(\frac{2N}{N+1}\right)$$

Evaluate integral as $N \to \infty$ i.e. find the limit of $\ln\left(\frac{2N}{N+1}\right)$ as $N \to \infty$.

$\ln\left(\frac{2N}{N+1}\right) = \ln 2 + \ln\left(\frac{N}{N+1}\right)$ and $\frac{N}{N+1} = \frac{N+1-1}{N+1} = 1 - \frac{1}{N+1}$

as $N \to \infty$, $1 - \frac{1}{N+1} \to 1$; hence $\ln\left(\frac{N}{N+1}\right) \to 0$ and

$\therefore A = \ln 2$

Area bounded by $y = \frac{1}{x(x+1)}$ to the right of $x = 1$ is $\ln 2$ square units.

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Question 2 (15 marks)
(a) (1 mark)
Outcomes assessed: E3
Targeted Performance Bands: E2-E3

Criteria	Mark
• substitutes and correctly realises the denominator (any correct expression)	1

Sample Answer:

$$\frac{z}{z+w}=\frac{2+i}{2+i-2}$$
$$=\frac{2+i}{i}\times\frac{i}{i}$$
$$=\frac{-1+2i}{-1}$$
$$=1-2i$$

(b) (i) (1 mark)
Outcomes assessed: E3
Targeted Performance Bands: E2-E3

Criteria	Marks
• finds correct modulus and argument	1

Sample Answer:

$$w=4+4i$$
$$|w|=\sqrt{4^2+4^2}=4\sqrt{2}$$
$$\arg w=\tan^{-1}1=\frac{\pi}{4}$$
$$\therefore w=4\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$

(b) (ii) (3 marks)
Outcomes assessed: E3
Targeted Performance Bands: E2-E3

Criteria	Marks
• uses De Moivre's Theorem to establish correct relationships	1
• progress towards solutions	1
• gives correct solutions	1

Sample Answer:

$$z^5=4+4i$$
Let $z=r(\cos\theta+i\sin\theta)$

Then by De Moivre's Theorem $z^5=r^5(\cos 5\theta+i\sin 5\theta)$

$$\therefore r^5(\cos 5\theta+i\sin 5\theta)=4\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$

i.e. $r^5=4\sqrt{2}$ and $5\theta=\frac{\pi}{4}+2k\pi,\quad k=0,1,2,3,4$

$r=\sqrt{2}$ $\qquad \theta=\frac{\pi+8k\pi}{20}$

i.e. $z_1=\sqrt{2}\left(\cos\frac{\pi}{20}+i\sin\frac{\pi}{20}\right)$

$$z_2=\sqrt{2}\left(\cos\frac{9\pi}{20}+i\sin\frac{9\pi}{20}\right)$$
$$z_3=\sqrt{2}\left(\cos\frac{17\pi}{20}+i\sin\frac{17\pi}{20}\right)$$
$$z_4=\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)\right)$$
$$z_5=\sqrt{2}\left(\cos\left(-\frac{7\pi}{20}\right)+i\sin\left(-\frac{7\pi}{20}\right)\right)$$

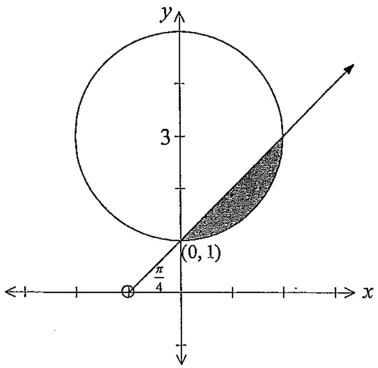
(c) (i) (2 mark)

Outcomes assessed: E3

Targeted Performance Bands: E2-E3

Criteria	Mark
• correctly displays one locus	1
• correctly displays both loci with shading	1

Sample Answer:



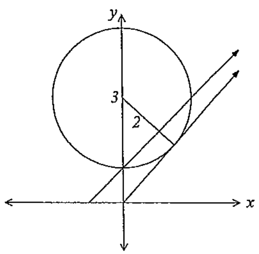
(c) (ii) (2 marks)

Outcomes assessed: E3

Targeted Performance Bands: E3-E4

Criteria	Marks
• progress towards solution	1
• gives correct answer	1

Sample Answer:



For minimum argument $\Rightarrow$ tangent to the circle i.e. perpendicular to radius

Forms right-angled triangle $\Rightarrow$ angle between tangent and imaginary axis is $\sin^{-1}\frac{2}{3}$

$$\therefore \arg z = \frac{\pi}{2} - \sin^{-1}\frac{2}{3}$$

(d) (i) (2 marks)

Outcomes assessed: E3

Targeted Performance Bands: E2-E3

Criteria	Mark
• factorises correctly or other significant progress towards result	1
• identifies complex root and shows the correct result	1

Sample Answer:

$z^3 = 1 \quad \Rightarrow \quad z^3 - 1 = 0$

i.e. $(z-1)(z^2+z+1) = 0$

$\therefore$ as ω is a complex root, ω is a solution of $(z^2+z+1) = 0$

i.e. $\omega^2 + \omega + 1 = 0$

OR

If ω is a complex root of $z^3 = 1$ then $\omega^3 = 1$

Thus ω^2 is also a root of $z^3 = 1$ since $(\omega^2)^3 = (\omega^3)^2 = 1$.

$\therefore$ the roots of $z^3 - 1 = 0$ are ω, ω^2 and 1.

Sum of roots: $\frac{-b}{a} = 0$

$\therefore \omega^2 + \omega + 1 = 0$

(d) (ii) (1 mark)

Outcomes assessed: E3

Targeted Performance Bands: E3-E4

Criteria	Mark
• gives correct result	1

Sample Answer:

$\omega^4 = \omega \times \omega^3$

$= \omega \quad$ as $\omega^3 = 1$

$\omega^8 = \omega^2(\omega^3)^2$

$= \omega^2 \quad$ as $\omega^3 = 1$

Thus

$$1 + \omega^4 + \omega^8 = 1 + \omega + \omega^2$$
$$= 0$$

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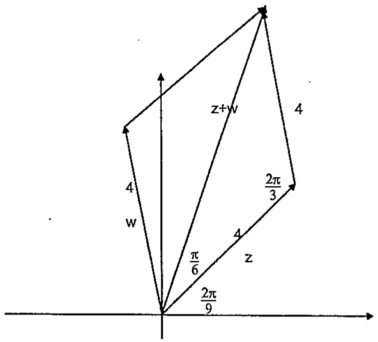
(e) (i) (1 mark)

Outcomes assessed: E3

Targeted Performance Bands: E3-E4

Criteria	Mark
• gives correct result	1

Sample Answer:



$$\arg(z+w)=\frac{2\pi}{9}+\frac{\pi}{6}$$
$$=\frac{7\pi}{18}$$

(e) (ii) (2 marks)

Outcomes assessed: E3

Targeted Performance Bands: E3-E4

Criteria	Mark
• applies cosine rule correctly	1
• gives correct solution	1

Sample Answer:

$$|z+w|^2=4^2+4^2-2\times 4\times 4\times\cos\frac{2\pi}{3}$$
$$=32+32\times\frac{1}{2}$$
$$=48$$
$$\therefore\ |z+w|=4\sqrt{3}$$

Question 3 (15 marks)

(a) (i) (1 mark)

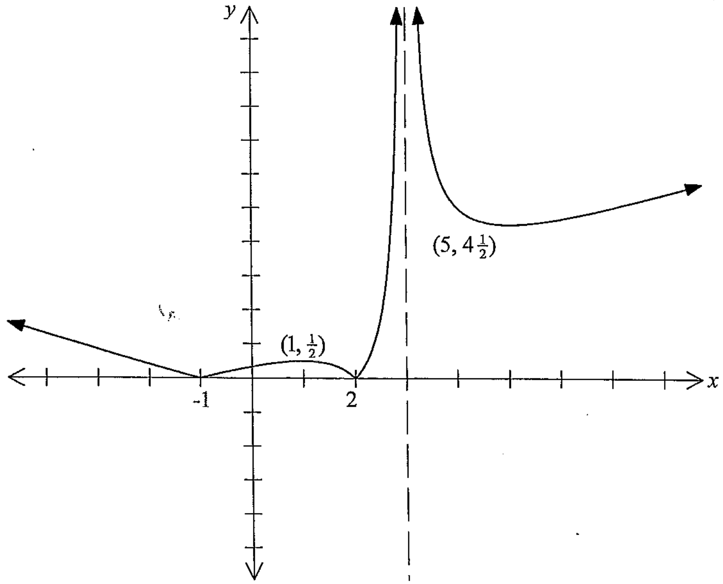
Outcomes assessed: E6

Targeted Performance Bands: E2-E3

Criteria	Mark
• correctly sketches the graph including all important points	1

Sample Answer:

$y=|f(x)|$

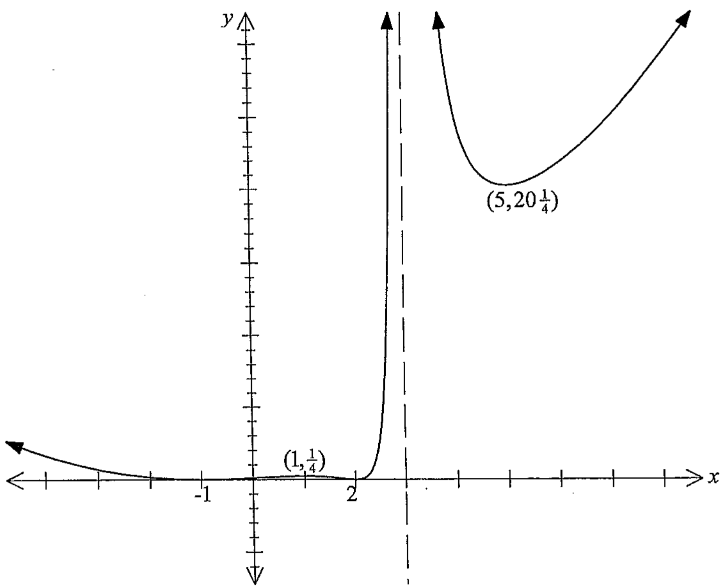


(a) (ii) (2 marks)
Outcomes assessed: E6
Targeted Performance Bands: E2-E3

Criteria	Marks
• significant progress towards the graph	1
• correctly sketches the graph	1

Sample Answer:

$y=[f(x)]^2$

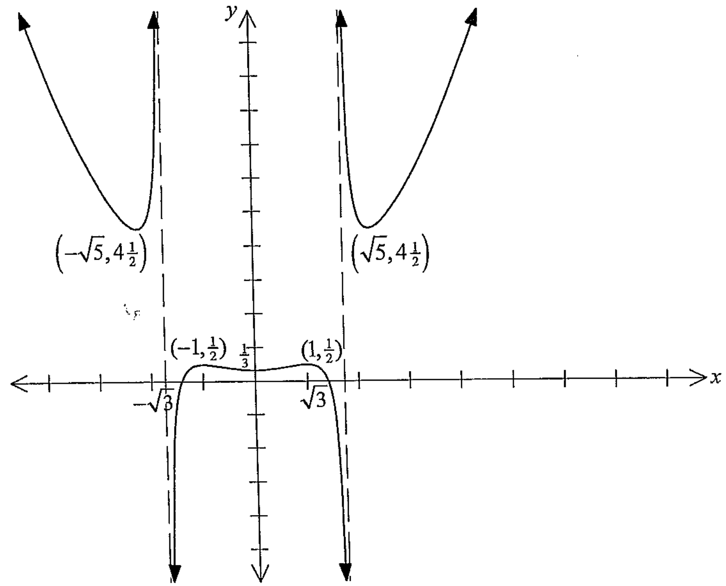


(a) (iii) (2 marks)
Outcomes assessed: E6
Targeted Performance Bands: E3-E4

Criteria	Marks
• significant progress towards the graph	1
• correctly sketches the graph	1

Sample Answer:

$y=f(x^2)$

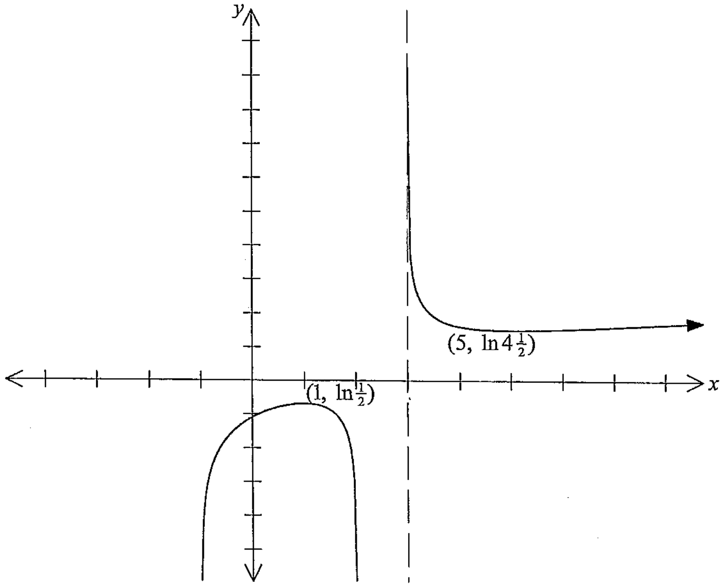


(a) (iv) (2 marks)
Outcomes assessed: E6
Targeted Performance Bands: E3-E4

Criteria	Marks
• significant progress towards the graph	1
• correctly sketches the graph	1

Sample Answer:

$y = \ln[f(x)]$



(b) (i) (2 marks)
Outcomes assessed: E4
Targeted Performance Bands: E2-E3

Criteria	Mark
• correct substitution or significant progress toward solution	1
• gives the correct result	1

Sample Answer:

$3x^3 - 9x^2 + 6x + 2 = 0$

The cubic equation with roots α^2, β^2 and γ^2 is given by replacing x with $\sqrt{x}$

i.e. $3\left(\sqrt{x}\right)^3 - 9\left(\sqrt{x}\right)^2 + 6\left(\sqrt{x}\right) + 2 = 0$

$3x\sqrt{x} - 9x + 6\sqrt{x} + 2 = 0$

$3\sqrt{x}\,(x+2) = 9x - 2$

$9x(x^2 + 4x + 4) = 81x^2 - 36x + 4$ on squaring both sides

$9x^3 - 45x^2 + 72x - 4 = 0$

(b) (ii) (2 marks)
Outcomes assessed: E4
Targeted Performance Bands: E2-E3

Criteria	Mark
• factorises correctly	1
• substitutes and gives the correct solution (correct numerical equivalence)	1

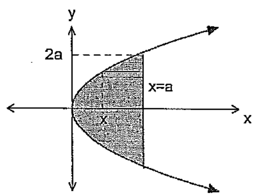
Sample Answer:

$$\alpha^3\beta\gamma + \alpha\beta^3\gamma + \alpha\beta\gamma^3 = \alpha\beta\gamma(\alpha^2 + \beta^2 + \gamma^2)$$
$$= -\frac{2}{3} \times \frac{45}{9}$$
$$= -\frac{10}{3}$$

(c) (4 marks)
Outcomes assessed: E7
Targeted Performance Bands: E2-E3

Criteria	Marks
• derives a correct expression for area or volume of slice	1
• gives a correct integral for volume	1
• gives the correct primitive	1
• evaluates correctly (correct numerical equivalence)	1

Sample Answer:



Consider a slice perpendicular to $x = a$
Slice rotated about $x = a$; radius of slice $a - x$, thickness of slice δy

Volume of slice $\delta V \approx \pi(a-x)^2\delta y$

$$\delta V = \pi(a^2 - 2ax + x^2)\delta y$$
$$= \pi\left(a^2 - \frac{y^2}{2} + \frac{y^4}{16a^2}\right)\delta y$$
$$\therefore V = \lim_{\delta y \to 0}\sum_{-2a}^{2a}\pi\left(a^2 - \frac{y^2}{2} + \frac{y^4}{16a^2}\right)\delta y$$
$$= 2\pi\int_0^{2a}\left(a^2 - \frac{y^2}{2} + \frac{y^4}{16a^2}\right)dy$$
$$= 2\pi\left[a^2y - \frac{y^3}{6} + \frac{y^5}{80a^2}\right]_0^{2a}$$
$$= 2\pi\left(2a^3 - \frac{8a^3}{6} + \frac{32a^3}{80} - (0 - 0 + 0)\right)$$
$$= \frac{32a^3}{15}\pi \text{ cubic units}$$

Question 4 (15 marks)
(a) (i) (2 marks)
Outcomes assessed: E4
Targeted Performance Bands: E2-E3

Criteria	Marks
• finds correct gradient or significant progress towards solution	1
• shows correct equation	1

Sample Answer:

$$y = 4x^{-1}$$
$$\frac{dy}{dx} = -4x^{-2}$$

At $x = 2t$, $m = \dfrac{-1}{t^2}$

$\therefore$ the equation of the tangent is

$$y - \frac{2}{t} = \frac{-1}{t^2}(x - 2t)$$
$$t^2y - 2t = -x + 2t$$
$$\therefore x + t^2y = 4t$$

(a) (ii) (3 marks)
Outcomes assessed: E4
Targeted Performance Bands: E2-E3

Criteria	Marks
• finds correct coordinates	1
• finds correct gradient or significant progress towards solution	1
• shows correct equation	1

Sample Answer:

At Q, $y = 0$ $\therefore x = 4t$, i.e. $Q(4t, 0)$

The line through Q, perpendicular to the tangent at P has gradient $m = t^2$.

$\therefore$ the equation of the line is

$$y - 0 = t^2(x - 4t)$$
$$y = t^2x - 4t^3$$
$$\therefore t^2x - y = 4t^3$$

(a) (iii) (3 marks)
Outcomes assessed: E4
Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes correct equation	1
• finds x or y coordinate	1
• finds the midpoint	1

Sample Answer:

R and T lie on both the hyperbola and the line through Q.

Solve simultaneously $t^2x - y = 4t^3$ and $xy = 4$

$y = t^2x - 4t^3$

$x\left(t^2x - 4t^3\right) = 4$

$t^2x^2 - 4t^3x - 4 = 0$

Let the roots of this equation be α and β.

The sum of the roots: $\alpha + \beta = \dfrac{4t^3}{t^2} = 4t$

$M(x,y)$ is the midpoint of RT i.e. halfway between the roots

$\therefore\ x = \dfrac{\alpha+\beta}{2} = 2t$

As y lies on RT: $y = t^2 \times 2t - 4t^3 = -2t^3$

i.e. $M(2t, -2t^3)$

(a) (iv) (1 mark)
Outcomes assessed: E4
Targeted Performance Bands: E3-E4

Criteria	Marks
• finds the locus	1

Sample Answer:

At M, $x = 2t$, $\therefore\ t = \dfrac{x}{2}$, and $y = -2t^3$

$y = -2\times\left(\dfrac{x}{2}\right)^3 = -\dfrac{x^3}{4}$

Thus the locus of M is the curve $y = -\dfrac{x^3}{4}$, $x \neq 0$, $y \neq 0$.

(b) (i) (1 mark)
Outcomes assessed: E2, E9
Targeted Performance Bands: E2-E3

Criteria	Marks
• correctly identifies sum of geometric series	1

Sample Answer:

$1 + x + x^2 + x^3 + \ldots + x^{n-1}$ is a geometric series with n terms, $a = 1$ and $r = x$

$\therefore$ sum to n terms, $S_n = \dfrac{1(1-x^n)}{1-x}$

i.e. $1 + x + x^2 + x^3 + \ldots + x^{n-1} = \dfrac{1-x^n}{1-x}$

(b) (ii) (2 marks)
Outcomes assessed: E2, E9
Targeted Performance Bands: E2-E3

Criteria	Marks
• identifies LHS is differentiated	1
• differentiates RHS correctly	1

Sample Answer:

Differentiate both sides of the expression in (i)

$LHS = 1 + 2x + 3x^2 + \ldots + (n-1)x^{n-2}$

$RHS = \dfrac{(1-x)\times -nx^{n-1} - (1-x^n)\times -1}{(1-x)^2}$

$= \dfrac{-nx^{n-1} + nx^n + 1 - x^n}{(1-x)^2}$

$= \dfrac{(n-1)x^n - nx^{n-1} + 1}{(1-x)^2}$

$\therefore\ 1 + 2x + 3x^2 + \ldots + (n-1)x^{n-2} = \dfrac{(n-1)x^n - nx^{n-1} + 1}{(1-x)^2}$

(b) (iii) (3 marks)

Outcomes assessed: E2, E9

Targeted Performance Bands: E3-E4

Criteria	Marks
• uses correct substitution	1
• progress towards simplifying expression	1
• gives correct solution	1

Sample Answer:

Let $x=\frac{1}{2}$ in (ii)

$$LHS=1+1+\frac{3}{4}+\frac{4}{8}+\frac{5}{16}+\ldots+\frac{n-1}{2^{n-2}}$$

$$RHS=\frac{\frac{(n-1)}{2^n}-\frac{n}{2^{n-1}}+1}{\left(\frac{1}{2}\right)^2}$$

$$=4\left(\frac{n}{2^n}-\frac{1}{2^n}-\frac{n}{2^{n-1}}+1\right)$$

$$=4\left(1-\frac{n}{2^{n-1}}\left(1-\frac{1}{2}\right)-\frac{1}{2^n}\right)$$

$$=4\left(1-\frac{n}{2^n}-\frac{1}{2^n}\right)$$

$$=4\left(1-\frac{(n+1)}{2^n}\right)$$

$$\therefore\ 1+1+\frac{3}{4}+\frac{4}{8}+\frac{5}{16}+\ldots+\frac{n-1}{2^{n-2}}=4\left(1-\frac{(n+1)}{2^n}\right)$$

Since $0<\frac{(n+1)}{2^n}<1$ for $n>1$

then $0<1-\frac{(n+1)}{2^n}<1$

$$\therefore\ 1+1+\frac{3}{4}+\frac{4}{8}+\frac{5}{16}+\ldots+\frac{n-1}{2^{n-2}}<4$$

Question 5 (15 marks)

(a) (i) (2 marks)

Outcomes assessed: E7

Targeted Performance Bands: E2-E3

Criteria	Marks
• finds correct expression for radius	1
• derives correct expression	1

Sample Answer:

$$A(x)=\frac{1}{2}\pi r^2$$

$$=\frac{1}{2}\pi\left(\frac{2\sin x-\tan x}{2}\right)^2$$

$$=\frac{\pi}{8}\left(4\sin^2 x-4\sin x\tan x+\tan^2 x\right)\qquad\text{as required}$$

(a) (ii) (4 marks)

Outcomes assessed: E7, E8

Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes correct expression for volume or significant progress towards solution	1
• gives correct substitutions	1
• integrates correctly	1
• finds the volume (correct numerical equivalence)	1

Sample Answer:

$$V=\frac{\pi}{8}\int_0^{\frac{\pi}{3}}\left(4\sin^2 x-4\sin x\tan x+\tan^2 x\right)dx$$

$$=\frac{\pi}{8}\int_0^{\frac{\pi}{3}}\left(4\times\frac{1}{2}(1-\cos 2x)-\frac{4\sin^2 x}{\cos x}+\sec^2 x-1\right)dx$$

$$=\frac{\pi}{8}\int_0^{\frac{\pi}{3}}\left(1-2\cos 2x+\sec^2 x-\frac{4(1-\cos^2 x)}{\cos x}\right)dx$$

$$=\frac{\pi}{8}\left[x-\sin 2x+\tan x\right]_0^{\frac{\pi}{3}}-\frac{\pi}{2}\int_0^{\frac{\pi}{3}}(\sec x-\cos x)\,dx$$

$$=\frac{\pi}{8}\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}+\sqrt{3}-(0-0+0)\right)-\frac{\pi}{2}\left[\ln(\sec x+\tan x)-\sin x\right]_0^{\frac{\pi}{3}}$$

$$=\frac{\pi^2}{24}+\frac{\sqrt{3}}{16}\pi-\frac{\pi}{2}\left(\ln\left(\sec\frac{\pi}{3}+\tan\frac{\pi}{3}\right)-\sin\frac{\pi}{3}-(\ln(\sec 0+\tan 0)-\sin 0)\right)$$

$$=\frac{\pi^2}{24}+\frac{5\sqrt{3}}{16}\pi-\frac{\pi}{2}\ln\left(2+\sqrt{3}\right)\ \text{cubic units}$$

(b) (i) (2 marks)
Outcomes assessed: E3, E4
Targeted Performance Bands: E2-E3

Criteria	Mark
• establishes correct expansion	1
• gives correct conclusion	1

Sample Answer:

Let $z = \cos\theta + i\sin\theta$

Then $z^3 = \cos 3\theta + i\sin 3\theta$ by De Moivre's Theorem (1)

Also $z^3 = (\cos\theta + i\sin\theta)^3$

$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta i\sin\theta + 3\cos\theta i^2\sin^2\theta + i^3\sin^3\theta$

$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$ (2)

Equating real and imaginary parts from (1) and (2) gives:

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

(b) (ii) (2 marks)
Outcomes assessed: E3, E4
Targeted Performance Bands: E2-E3

Criteria	Marks
• progress towards result	1
• deduces expression	1

Sample Answer:

$$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta}$$

$$= \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta} \quad \text{divide by } \cos^3\theta$$

$$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

$$= \frac{\tan^3\theta - 3\tan\theta}{3\tan^2\theta - 1}$$

(b) (iii) (3 marks)
Outcomes assessed: E4
Targeted Performance Bands: E3-E4

Criteria	Marks
• recognises to solve for unity	1
• solves correctly	1
• recognises first solution and gives conclusion	1

Sample Answer:

$\tan 3\theta = 1 \quad \Rightarrow \quad \dfrac{\tan^3\theta - 3\tan\theta}{3\tan^2\theta - 1} = 1$

$3\theta = \dfrac{\pi}{4} + 2k\pi$

$\theta = \dfrac{\pi + 8k\pi}{12}$

$\theta = \dfrac{\pi}{12}, \dfrac{9\pi}{12}, \dfrac{17\pi}{12}, \ldots$

$\tan^3\theta - 3\tan\theta = 3\tan^2\theta - 1$

$\tan^3\theta - 3\tan^2\theta - 3\tan\theta + 1 = 0$

Puting $\tan\theta = x$ gives $x^3 - 3x^2 - 3x + 1 = 0$

$\therefore\ x = \tan\dfrac{\pi}{12}$ is a solution of $x^3 - 3x^2 - 3x + 1 = 0$

(b) (iv) (2 marks)
Outcomes assessed: E4
Targeted Performance Bands: E3-E4

Criteria	Marks
• writes down the solutions	1
• uses sum of roots to get desired result	1

Sample Answer:

Solutions of $x^3 - 3x^2 - 3x + 1 = 0$ are

$\tan\dfrac{\pi}{12}, \tan\dfrac{9\pi}{12}, \tan\dfrac{17\pi}{12}$

i.e. $\tan\dfrac{\pi}{12}, -1, \tan\dfrac{5\pi}{12}$

Sum of roots: $\tan\dfrac{\pi}{12} - 1 + \tan\dfrac{5\pi}{12} = 3$

Thus $\tan\dfrac{\pi}{12} + \tan\dfrac{5\pi}{12} = 4$.

Question 6 (15marks)
(a) (i) (1 mark)
Outcomes assessed: E3, E4
Targeted Performance Bands: E2-E3

Criteria	Marks
• shows correct gradient	1

Sample Answer:

Given $x = a\cos\theta$ and $y = b\sin\theta$

$$\frac{dx}{d\theta} = -a\sin\theta \text{ and } \frac{dy}{d\theta} = b\cos\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= b\cos\theta \times \frac{1}{-a\sin\theta}$$

$$= \frac{-b}{a} \times \frac{\cos\theta}{\sin\theta}$$

$$= \frac{-b}{a}\cot\theta$$

OR

From $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad \frac{2x}{a^2} + \frac{2y}{b^2} \times \frac{dy}{dx} = 0$

$$\therefore \frac{dy}{dx} = \frac{-b^2 x}{a^2 y}$$

At $P(a\cos\theta,\ b\sin\theta)$

$$\frac{dy}{dx} = \frac{-b^2 \times a\cos\theta}{a^2 \times b\sin\theta}$$

$$= \frac{-b}{a} \times \frac{\cos\theta}{\sin\theta}$$

$$= \frac{-b}{a}\cot\theta$$

(a) (ii) (2 marks)
Outcomes assessed: E3, E4
Targeted Performance Bands: E2-E3

Criteria	Marks
• solves equations simultaneously or significant progress towards solution	1
• shows correct coordinates	1

Sample Answer:

OQ is parallel to the tangent at P

$\therefore$ the equation of OQ is $y = \frac{-bx}{a}\cot\theta$

and Q lies on the ellipse

$$\therefore \frac{x^2}{a^2} + \frac{1}{b^2} \times \frac{b^2 x^2 \cos^2\theta}{a^2 \sin^2\theta} = 1$$

i.e. $x^2(\sin^2\theta + \cos^2\theta) = a^2\sin^2\theta$

$$\therefore x = -a\sin\theta$$

$$\therefore y = \frac{-b}{a} \times (-a\sin\theta) \times \cot x$$

$$= b\cos\theta$$

i.e. Q is $(-a\sin\theta,\ b\cos\theta)$

(a) (iii) (3 marks)
Outcomes assessed: E3, E4
Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes expression for distance	1
• considers case for angles	1
• gives correct conclusion	1

Sample Answer:

$P(a\cos\theta,\ b\sin\theta)$ and $Q(-a\sin\theta,\ b\cos\theta)$

$$PQ = \sqrt{a^2(\cos\theta + \sin\theta)^2 + b^2(\sin\theta - \cos\theta)^2}$$

$$= \sqrt{a^2(\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta) + b^2(\sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta)}$$

$$= \sqrt{a^2(1 + \sin 2\theta) + b^2(1 - \sin 2\theta)}$$

$$= \sqrt{a^2 + b^2 + (a^2 - b^2)\sin 2\theta}$$

maximum for PQ when $\sin 2\theta = 1$ i.e. when $\theta = \frac{\pi}{4}$

$$\therefore PQ = \sqrt{2a^2} = a\sqrt{2}$$

minimum for PQ when $\sin 2\theta = -1$ i.e. when $\theta = \frac{-\pi}{4}$

$$\therefore PQ = \sqrt{2b^2} = b\sqrt{2}$$

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(b) (i) (1 mark)
Outcomes assessed: E2, E9
Targeted Performance Bands: E3-E4

Criteria	Marks
• shows correct result	1

Sample Answer:

$$\left(\sqrt{x}-\sqrt{y}\right)^2 \geq 0 \text{ i.e. } x-2\sqrt{xy}+y\geq 0$$
$$\therefore x+y\geq 2\sqrt{xy}$$

(b) (ii) (2 marks)
Outcomes assessed: E2, E9
Targeted Performance Bands: E3-E4

Criteria	Marks
• applies result from (i) correctly	1
• shows correct result	1

Sample Answer:

From (i) $x+y\geq 2\sqrt{xy}$, i.e. $\dfrac{x+y}{2}\geq\sqrt{xy}$

$\therefore \dfrac{w+z}{2}\geq\sqrt{wz}$

Now $\dfrac{x+y}{2}+\dfrac{w+z}{2}\geq\sqrt{xy}+\sqrt{wz}$

i.e. $\dfrac{1}{2}(x+y+w+z)\geq\sqrt{xy}+\sqrt{wz}$

Using (i) again $\sqrt{xy}+\sqrt{wz}\geq 2\sqrt{\sqrt{xy}\times\sqrt{wz}}=2\sqrt[4]{xywz}$

So $\dfrac{1}{2}(x+y+w+z)\geq 2\sqrt[4]{xywz}$

i.e. $x+y+z+w\geq 4\sqrt[4]{xyzw}$

(b) (iii) (2 marks)
Outcomes assessed: E2, E9
Targeted Performance Bands: E3-E4

Criteria	Marks
• applies result from (ii) correctly	1
• shows correct result	1

Sample Answer:

From (ii) $x+y+z+w\geq 4\sqrt[4]{xyzw}$

$\therefore\ x+y+z+\dfrac{x+y+z}{3}\geq 4\sqrt[4]{xyzw}$

i.e. $\dfrac{4(x+y+z)}{3}\geq 4\sqrt[4]{xyzw}$

$w\geq\sqrt[4]{xyzw}$

So $w^4\geq xyzw$ i.e. $w^3\geq xyz$

$\therefore\ w\geq\sqrt[3]{xyz}$

i.e. $\dfrac{x+y+z}{3}\geq\sqrt[3]{xyz}$

(c) (i) (3 marks)
Outcomes assessed: E8
Targeted Performance Bands: E3-E4

Criteria	Marks
• uses the substitution	1
• simplifies the integral	1
• uses integration result to show correct conclusion	1

Sample Answer:

$I=\displaystyle\int_{-1}^{1}\frac{x^2e^x}{e^x+1}\,dx$ let $u=-x$ $\quad\dfrac{du}{dx}=-1$

$=-\displaystyle\int_{1}^{-1}\frac{u^2e^{-u}}{e^{-u}+1}\,du$ when $x=1,\ u=-1$ and when $x=-1,\ u=1$

$=\displaystyle\int_{-1}^{1}u^2e^{-u}\div\left(\frac{1+e^u}{e^u}\right)du$

$=\displaystyle\int_{-1}^{1}\frac{u^2}{1+e^u}\,du$

$=\displaystyle\int_{-1}^{1}\frac{x^2}{e^x+1}\,dx$ using the result $\displaystyle\int_a^b f(x)\,dx=\int_a^b f(u)\,du$

$=J$

(c) (ii) (1 mark)
Outcomes assessed: E8
Targeted Performance Bands: E3-E4

Criteria	Marks
• evaluates correctly	1

Sample Answer:

$$I=\int_{-1}^{1}\frac{x^2e^x}{e^x+1}dx \text{ and } J=\int_{-1}^{1}\frac{x^2}{e^x+1}dx$$

$$\therefore I+J=\int_{-1}^{1}\left(\frac{x^2e^x}{e^x+1}+\frac{x^2}{e^x+1}\right)dx$$

$$=\int_{-1}^{1}\frac{x^2(e^x+1)}{e^x+1}dx$$

$$=\int_{-1}^{1}x^2\,dx$$

$$=\left[\frac{x^3}{3}\right]_{-1}^{1}$$

$$=\frac{2}{3}$$

From (i) $I=J$

$$\therefore I=\frac{1}{3}$$

Question 7 (15 marks)
(a) (i) (1 mark)
Outcomes assessed: E2
Targeted Performance Bands: E2-E3

Criteria	Marks
• shows correct result	1

Sample Answer:

$$RHS=\sin(A+B)-\sin(A-B)$$
$$=\sin A\cos B+\cos A\sin B-(\sin A\cos B-\cos A\sin B)$$
$$=2\cos A\sin B$$

(a) (ii) (2 marks)
Outcomes assessed: E2
Targeted Performance Bands: E3-E4

Criteria	Marks
• uses result from (i)	1
• shows correct result	1

Sample Answer:

$$LHS=\frac{\sin\left(\frac{3x}{2}\right)-\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}$$

$$=\frac{2\cos x\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)} \quad \text{using (i)}$$

$$=\cos x$$
$$=RHS$$

(a) (iii) (4 marks)
Outcomes assessed: E2
Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes the truth of $S(1)$	1
• uses assumption correctly in $S(k+1)$	1
• shows progress toward correct expression	1
• deduces the required result	1

Sample Answer:

Let $S(n)$ be the statement $\cos x+\cos 2x+\cos 3x+\ldots+\cos nx=\dfrac{\sin\left(\dfrac{2n+1}{2}x\right)}{2\sin\left(\dfrac{x}{2}\right)}-\dfrac{1}{2}$

Consider $S(1)$: Prove result true when $n=1$

$LHS=\cos x$

$$RHS=\frac{\sin\left(\frac{3x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}-\frac{1}{2}$$

$$=\frac{\sin\left(\frac{3x}{2}\right)-\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}$$

$=\cos x$ from (ii)

$=LHS$

Hence $S(1)$ is true

Assume $S(k)$ is true i.e. result is true for $n=k$

i.e. $\cos x+\cos 2x+\cos 3x+\ldots+\cos kx=\dfrac{\sin\left(\dfrac{2k+1}{2}x\right)}{2\sin\left(\dfrac{x}{2}\right)}-\dfrac{1}{2}$

Prove $S(k+1)$ is true i.e. prove result is true for $n=k+1$

i.e. $\cos x+\cos 2x+\cos 3x+\ldots+\cos(k+1)x=\dfrac{\sin\left(\dfrac{2k+3}{2}x\right)}{2\sin\left(\dfrac{x}{2}\right)}-\dfrac{1}{2}$

$S(k+1)$:

$LHS=S(k)+\cos(k+1)x$

$$=\frac{\sin\left(\frac{2k+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)}-\frac{1}{2}+\cos(k+1)x$$

$$=\frac{\sin\left(\frac{2k+1}{2}x\right)+2\sin\left(\frac{x}{2}\right)\cos(k+1)x}{2\sin\left(\frac{x}{2}\right)}-\frac{1}{2}$$

$$=\frac{\sin\left(\frac{2k+1}{2}x\right)+\sin\left(\frac{2k+3}{2}x\right)-\sin\left(\frac{2k+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)}-\frac{1}{2}$$ from (i)

$$=\frac{\sin\left(\frac{2k+3}{2}x\right)}{2\sin\left(\frac{x}{2}\right)}-\frac{1}{2}$$

$=RHS$

Hence if $S(k)$ is true then $S(k+1)$ is true.

Thus since $S(1)$ is true it follows by induction that $S(n)$ is true for positive integral n.

(b) (i) (2 marks)
Outcomes assessed: E5
Targeted Performance Bands: E2-E3

Criteria	Marks
• draws correct diagram indicating all resolved forces	1
• derives correct expressions	1

Sample Answer:

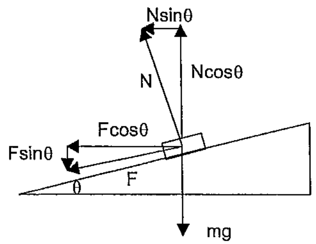
Resolving vertically:

$F\sin\theta+mg=N\cos\theta$

$\therefore\ N\cos\theta-F\sin\theta=mg$

Resolving horizontally:

$F\cos\theta+N\sin\theta=\dfrac{mv^2}{r}$



(b) (ii) (2 marks)
Outcomes assessed: E5
Targeted Performance Bands: E3-E4

Criteria	Marks
• finds squares of both expressions	1
• finds correct expression	1

Sample Answer:

Squaring equations in (i)

$$m^2g^2 = N^2\cos^2\theta - 2FN\cos\theta\sin\theta + F^2\sin^2\theta \quad (1)$$

$$\frac{m^2v^4}{r^2} = F^2\cos^2\theta + 2FN\cos\theta\sin\theta + N^2\sin^2\theta \quad (2)$$

$(1)+(2)$ $$m^2g^2 + \frac{m^2v^4}{r^2} = N^2\cos^2\theta + N^2\sin^2\theta + F^2\sin^2\theta + F^2\cos^2\theta$$

$$\therefore N^2 + F^2 = m^2g^2 + \frac{m^2v^4}{r^2}$$

(b) (iii) (2 marks)
Outcomes assessed: E5
Targeted Performance Bands: E3-E4

Criteria	Marks
• substitutes in the correct expression for F	1
• finds correct expression	1

Sample Answer:

From (ii) $$N^2 + F^2 = m^2g^2 + \frac{m^2v^4}{r^2}$$

Since $F = \frac{1}{6}N$ $$N^2 + \frac{1}{36}N^2 = m^2g^2 + \frac{m^2v^4}{r^2}$$

$$\frac{37}{36}N^2 = m^2\left(g^2 + \frac{v^4}{r^2}\right)$$

$$N^2 = \frac{36m^2}{37}\left(g^2 + \frac{v^4}{r^2}\right)$$

Given speed is 72 km/h $\Rightarrow v = 20$ m/s

i.e. $$N = \frac{6m}{\sqrt{37}}\sqrt{g^2 + \frac{160000}{r^2}}$$

(b) (iv) (2 marks)
Outcomes assessed: E5
Targeted Performance Bands: E2-E3

Criteria	Marks
• establishes correct equations	1
• finds the correct answer (correct numerical equivalence)	1

Sample Answer:

From (i) $$F\cos\theta + N\sin\theta = \frac{mv^2}{r}$$

From (ii) $$N^2 + F^2 = m^2g^2 + \frac{m^2v^4}{r^2}$$

If $F = 0$ $$N = \frac{mv^2}{r\sin\theta} \text{ and } N^2 = m^2g^2 + \frac{m^2v^4}{r^2}$$

$$\therefore \frac{m^2v^4}{r^2\sin^2\theta} = m^2g^2 + \frac{m^2v^4}{r^2}$$

i.e. $$\frac{v^4}{90000\sin^2 5^\circ} = 100 + \frac{v^4}{90000}$$

$$\frac{v^4}{90000\sin^2 5^\circ} - \frac{v^4}{90000} = 100$$

$$v^4\left(\frac{1}{90000\sin^2 5^\circ} - \frac{1}{90000}\right) = 100$$

$$v^4 = 68888.39621\ldots$$

$$v = 16.2 \text{ m/s}$$
$$= 58.3 \text{ km/h}$$

Question 8 (15 marks)
(a) (i) (1 mark)
Outcomes assessed: E5
Targeted Performance Bands: E2-E3

Criteria	Marks
• finds the correct primitive	1

Sample Answer:

$$ma = -mkv(c+v)$$
$$\therefore a = -k(c+v) \qquad \text{as required}$$

(a) (ii) (4 marks)
Outcomes assessed: E5
Targeted Performance Bands: E3-E4

Criteria	Marks
• integrates correctly or some progress towards solution	1
• further progress towards solution	1
• significant progress towards solution	1
• derives the correct solution	1

Sample Answer:

$$a = -k(c+v)$$

$$\frac{dv}{dt} = -k(c+v)$$

$$\frac{dv}{(c+v)} = -k\,dt$$

$$\ln(c+v) = -kt + b$$

when $t = 0, v = U \therefore b = \ln(c+U)$

$$\therefore -kt = \ln\left(\frac{c+v}{c+U}\right)$$

$$e^{-kt} = \frac{c+v}{c+U}$$

when $t = T,\ v = 0 \quad \Rightarrow e^{-kT} = \dfrac{c}{(c+U)}$ (1)

when $t = \frac{1}{2}T,\ v = \frac{1}{4}U \Rightarrow e^{\frac{-kT}{2}} = \dfrac{c+\frac{U}{4}}{(c+U)}$ (2)

$$(2)^2 \to (1) \quad \frac{\left(c+\frac{U}{4}\right)^2}{(c+U)^2} = \frac{c}{(c+U)}$$

$$c^2 + \frac{Uc}{2} + \frac{U^2}{16} = c^2 + Uc$$

$$\frac{U^2}{16} = \frac{Uc}{2}$$

$\therefore c = \frac{1}{8}U$ as required

(a) (iii) (2 marks)
Outcomes assessed: E5
Targeted Performance Bands: E3-E4

Criteria	Marks
• substitutes into correct equation	1
• derives the correct solution	1

Sample Answer:

$$e^{-kt} = \frac{c+v}{c+U}$$

$$e^{-kt} = \frac{\frac{U}{8}+v}{\frac{U}{8}+U}$$

$$\frac{9U}{8}e^{-kt} = \frac{U}{8}+v$$

$$v = \frac{U}{8}(9e^{-kt}-1)$$

$\therefore \frac{8v}{U} = 9e^{-kt} - 1$ as required

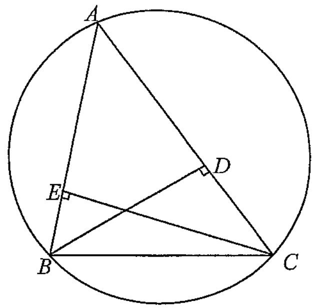
(b) (i) (2 marks)

Outcomes assessed: E2

Targeted Performance Bands: E2-E3

Criteria	Marks
• establishes cyclic quadrilateral using angles in the same segment	1
• uses angle in a semicircle to establish that BC is a diameter	1

Sample Answer:



$\angle BEC = 90^\circ$ $\quad EC \perp AB$

$\angle BDC = 90^\circ$ $\quad BD \perp AC$

$\therefore BCDE$ is a cyclic quadrilateral (interval BC subtends two equal angles on the same side)

Also BC is a diameter (angles in a semicircle are right angles)

(b) (ii) (3 marks)

Outcomes assessed: E2

Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes $\angle BAC$ is constant	1
• deduces that $\angle ABD$ is constant	1
• uses property of equal chords to deduce that ED is constant	1

Sample Answer:

BC is a constant length $\therefore$ angle at circumference is constant $\Rightarrow \angle BAC$ is constant

$\therefore \angle ABD = 90^\circ - \angle BAD$ $\quad BD \perp AC$

i.e. $\angle EBD$ is constant.

Similarly $\angle DCE$ is constant.

$\angle DCE = \angle EBD$ (angles in a segment are equal; $BCDE$ is a cyclic quadrilateral)

So chord ED subtends equal, constant, angles on circumference

$\therefore$ ED must be constant (equal chords subtend equal angles)

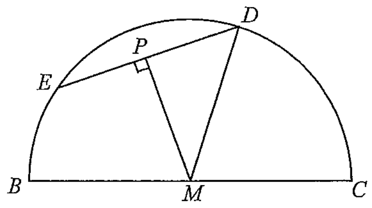
(b) (iii) (3 marks)

Outcomes assessed: E2

Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes $MP \perp ED$	1
• uses Pythagoras to establish that MP is a constant	1
• deduces final result	1

Sample Answer:



Let P and M be the midpoints of ED and BC respectively.

Join MP and MD

BC is the diameter of a circle $\therefore$ M is the centre of the circle.

$\therefore MP \perp ED$ (line from centre to the midpoint of the chord is perpendicular to the chord)

By Pythagoras $MP^2 = MD^2 - PD^2$

$MD = \frac{1}{2}BC$ $\quad$ BC fixed chord

$PD = \frac{1}{2}ED$ $\quad$ ED fixed chord

$MP^2 = \frac{1}{4}BC^2 - \frac{1}{4}ED^2$

i.e. MP^2 is a constant

$\therefore$ locus of P is part of a circle, centre is the midpoint of BC and radius is $\frac{1}{2}\sqrt{BC^2 - ED^2}$

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