Centre Number

Student Number

CSSA

CATHOLIC SECONDARY SCHOOLS ASSOCIATION OF NSW

2011
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Morning Session
Monday, 8 August 2011

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided as a separate page
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–8
- All questions are of equal value

6400-1

Total marks – 120
Attempt Questions 1–8
All questions are of equal value

Answer each question in a SEPARATE writing booklet.

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\int \frac{e^x}{\sqrt{e^x+1}}\,dx$. **2**

(b) Find $\int \cos^3 x\,dx$. **2**

(c) Find $\int \frac{x^2}{x^2+4}\,dx$. **2**

(d) (i) Find real numbers a and b such that **2**

$$\frac{1}{x^2+6x-7} \equiv \frac{a}{(x-1)} + \frac{b}{(x+7)}.$$

(ii) Hence, or otherwise, find $\int \frac{dx}{x^2+6x-7}$. **1**

(e) Find $\int \frac{dx}{\sqrt{1+2x-x^2}}$. **2**

(f) Use the substitution $t = \tan\frac{x}{2}$, or otherwise, to evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{dx}{1-\cos x}$. **4**

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z = 2 - i$ and $w = 1 + 3i$. Find, in the form $x + iy$,

(i) $w - z$ **1**

(ii) $z\overline{w}$. **1**

(b) Find the square roots of $-5 + 12i$. **3**

(c) (i) Sketch the locus of z if $|z - 3 + 3i| \le 2$. **2**

(ii) Find the maximum value of $|z|$. **1**

(d) The complex number z has $|z| = 1$ and $\arg z = \dfrac{\pi}{6}$.

(i) Find a quadratic equation for which z and $\overline{z}$ are the roots. **2**

(ii) The points represented by the complex numbers z, 0 and ω form an equilateral triangle on the Argand plane. **1**

Find a possible complex number ω.

(e) (i) Factorise the cubic polynomial $z^3 + 8$ over the field of real numbers. **1**

Let w be one of the non-real roots of the equation $z^3 + 8 = 0$.

(ii) Show that $w^2 = 2w - 4$. **1**

(iii) Hence find the value of $(2w - 4)^6$. **2**

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the hyperbola with equation

$$\frac{y^2}{16} - \frac{x^2}{9} = 1.$$

(i) Find the points of intersection of the hyperbola with the y-axis. **1**

(ii) Find the equations of the directrices and the asymptotes. **3**

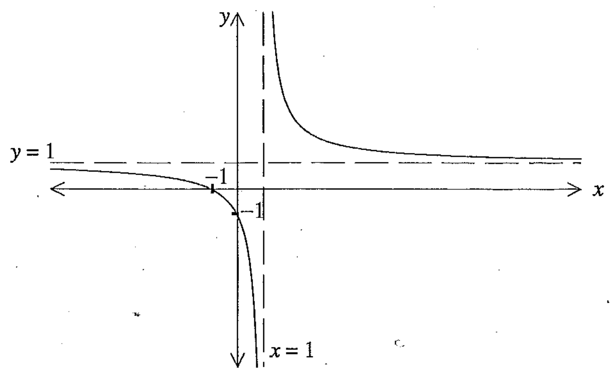
(iii) Sketch the hyperbola. **1**

(b) Consider the curve that is defined by $4x^2 - 2xy + y^2 - 6x = 0$.

(i) Show that $\dfrac{dy}{dx} = \dfrac{3 - 4x + y}{y - x}$. **3**

(ii) Find the x-coordinates of all points where the tangent is vertical. **2**

(c) The diagram shows the graph of $y = f(x)$, with asymptotes at $x = 1$ and $y = 1$.



Draw separate one-third page sketches of the graphs of the following:

(i) $y = |f(x)|$ **1**

(ii) $y^2 = f(x)$ **2**

(iii) $y = \ln[f(x)]$. **2**

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) The roots of $x^3+7x^2-4x+2=0$ are α, β and γ.

(i) Find a cubic polynomial equation whose roots are α^2, β^2 and γ^2. **2**

(ii) Find the value of $\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2$. **1**

(b) Let $z_1=\frac{1}{\sqrt{2}}(1+i)$ and $z_2=\frac{1}{2}\left(1+\sqrt{3}\,i\right)$.

(i) Find $\frac{z_2}{z_1}$ in the form $a+ib$. **2**

(ii) Express $\frac{z_2}{z_1}$ in modulus-argument form. **2**

(iii) Hence, or otherwise, write down the exact value of $\sin\frac{\pi}{12}$. **1**

(c) (i) Given that $\sin(A+B)-\sin(A-B)=2\sin B\cos A$, express $2\sin\theta\cos 6\theta$ as the difference of two sines. **1**

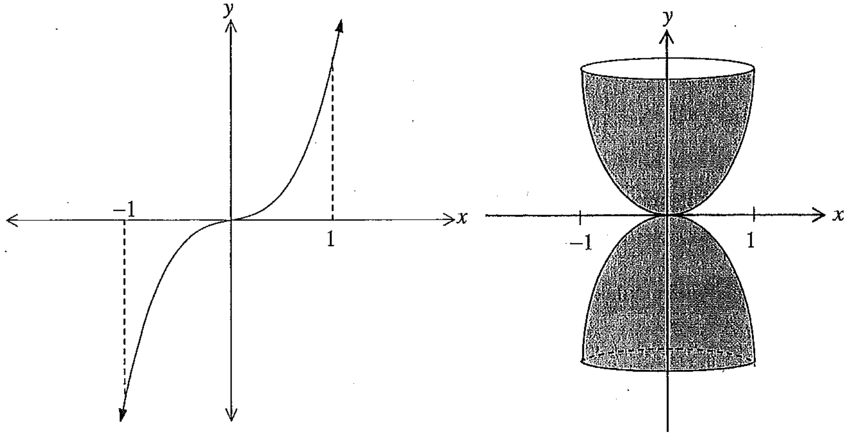
(ii) Show that $2\sin\theta(\cos 6\theta+\cos 4\theta+\cos 2\theta)=\sin 7\theta-\sin\theta$. **1**

(iii) Hence, deduce that $\cos\frac{12\pi}{7}+\cos\frac{8\pi}{7}+\cos\frac{4\pi}{7}=-\frac{1}{2}$. **2**

Question 4 continues on page 6

Question 4 (continued)

(d) The diagram shows the curve $y=x+6x^3$. The section of the curve from $x=-1$ to $x=1$ is rotated about the y-axis to form an hourglass. **3**

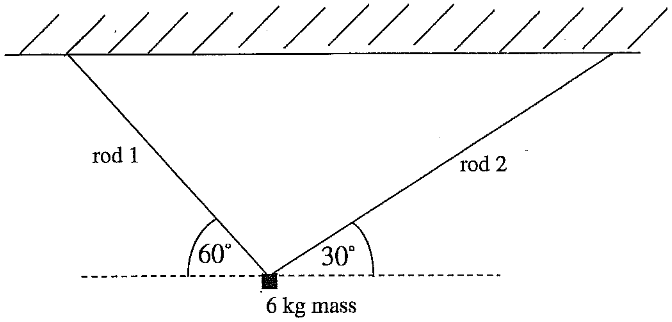


Use the method of cylindrical shells to find the volume of the hourglass formed.

End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) A six kilogram mass is to be suspended from a ceiling by two light inextensible rods. At the place where the particle hangs, rod 1 makes an angle of 60° from the horizontal and rod 2 makes an angle of 30° from the horizontal, but on the opposite side of the mass. **3**



Find the tensions T_1 and T_2 in rods 1 and 2, respectively.
(Give your answers in terms of g ms^{-2} where g is the acceleration due to gravity)

(b) Let $I_n = \int_0^M x^n e^{-x} dx$ where $n \geq 0$ and n is an integer.

(i) Find I_0. **1**

(ii) Show that $I_n = nI_{n-1} - \frac{M^n}{e^M}$ where $n \geq 1$. **2**

(iii) Show that $\lim_{M \to \infty} I_n = n!$ **2**

Question 5 continues on page 8

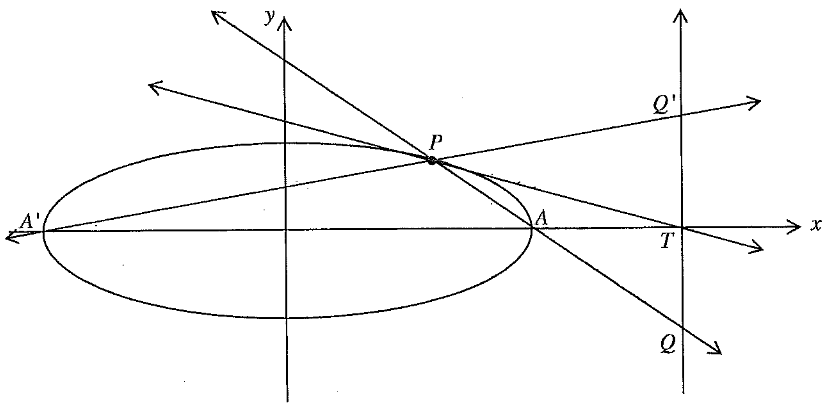
Question 5 (continued)

(c) Prove that $\frac{\sec\theta+1}{\cos\theta+1}+\frac{\sec\theta-1}{\cos\theta-1}\equiv 0$. **1**

(d) The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ intersects the x-axis at $A(a,0)$ and $A'(-a,0)$.

The point $P(a\cos\theta,\ b\sin\theta)$ is on the ellipse. The tangent at P meets the x-axis at T.

The secants AP and $A'P$ meet the perpendicular to the x-axis at T in Q and Q' respectively, as shown in the diagram.



(i) Show that the equation of the tangent at P is $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$. **2**

(ii) Show that T has coordinates $\left(\frac{a}{\cos\theta},\ 0\right)$. **1**

(iii) Show that $QT = Q'T$. **3**

End of Question 5

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Prove for integers $n \geq r > 1$, **1**

$$\binom{n}{r} < n\binom{n-1}{r-1}.$$

(ii) Given the expansion of $(a+b)^n$ can be written as **2**

$$\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n}b^n$$

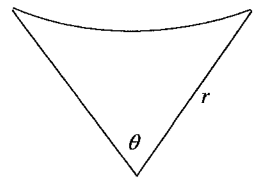
show that for $a, b > 0$ and integers $n > 1$

$$(a+b)^n - a^n < nb(a+b)^{n-1}.$$

Question 6 continues on page 10

Question 6 (continued)

(b) (i) A sector is a shape made up of a triangle and a segment of a circle. The diagram shows a "re-entrant" sector which is made by subtracting the area of the segment from the triangle. **1**



Show that the area of a "re-entrant" sector is given by $A = r^2\left(\sin\theta - \frac{\theta}{2}\right)$.

(ii) The diagram below shows a dome tent. **4**

SIDE VIEW TOP VIEW

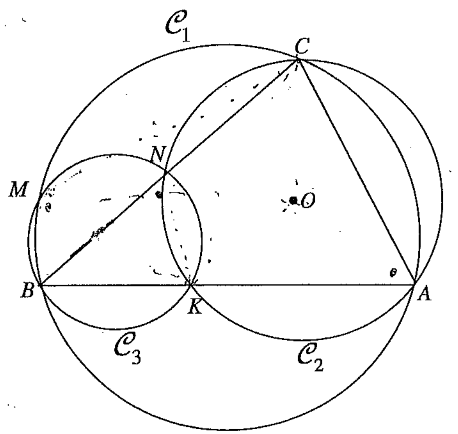
When erected, the base is made up of six congruent "re-entrant" sectors, measuring two metres between each adjacent corner. The tent is supported by flexible exterior poles extended between opposite corners in semi-circular arcs.

By taking slices parallel to the base of the tent, find the volume enclosed by the tent.

Question 6 continues on page 11

(c) A triangle, ABC, has its vertices on the circle $\mathscr{C}_1$. Another circle $\mathscr{C}_2$, has its centre O lying inside triangle ABC. This circle passes through A and C and cuts AB and BC at K and N respectively.

A third circle $\mathscr{C}_3$ through B, K and N cuts circle $\mathscr{C}_1$ at M.



Let $\angle BMK = \alpha$.

(i) Show that $\angle KAC = \alpha$. **2**

(ii) State why $\angle BMC = 180 - \alpha$. **1**

(iii) Show that $MKOC$ is a cyclic quadrilateral. **2**

(iv) Deduce that OM is perpendicular to BM. **2**

End of Question 6

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) The nth Fermat number, F_n, is defined by $F_n = 2^{2^n} + 1$ for $n = 0, 1, 2, \ldots$ where 2^{2^n} means 2 raised to the power of 2^n. **3**

Prove, by induction, that for all positive integers n, $F_0 F_1 F_2 \ldots F_{n-1} = F_n - 2$.

(b) A particle of mass m kilograms starts falling from rest having been initially projected vertically upwards from the ground. It experiences air resistance of magnitude mkv^2 on both the upward and downward motion of the journey, where k is a positive constant and v is the velocity of the particle at any instant.

(i) Show that the terminal velocity, V, of the particle is given by **1**

$$V = \sqrt{\frac{g}{k}}.$$

(ii) If W is the velocity of the particle when it hits the ground, show that the distance, D, fallen is given by **3**

$$D = -\frac{1}{2k}\ln\left(1 - \frac{W^2}{V^2}\right).$$

(iii) The maximum height attained by the particle, in terms of its initial velocity, U, and its eventual terminal velocity, V, is given by **1**

$$\frac{1}{2k}\ln\left(1 + \frac{U^2}{V^2}\right). \text{ (Do NOT prove this.)}$$

Show that $\dfrac{1}{W^2} = \dfrac{1}{U^2} + \dfrac{1}{V^2}$.

(iv) The particle hits the ground with a velocity of $\dfrac{3V}{5}$. What initial velocity, U, was needed to achieve this? Give your answer in terms of V. **1**

Question 7 continues on page 13

Question 7 (continued)

(c) Pairs of numbers from the list 1, 2, 3, 4, …, n are chosen and their product is calculated. The same number may be chosen twice (e.g. $3\times3=9$) but the order of the selection does not matter (e.g. 3×2 and 2×3 are considered the same pair).

(i) Show that there are $\frac{n^2+n}{2}$ ways that the pairs may be chosen. **2**

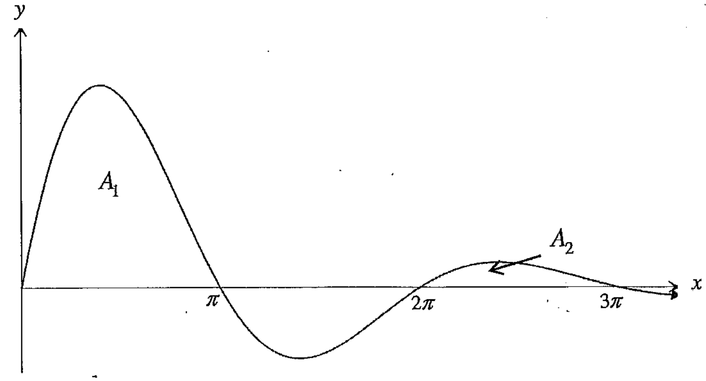
(ii) Show that in the case when n is even, there will be $\frac{3n^2+2n}{8}$ pairs that result in an even product. **3**

(iii) Deduce that when n is even, and $n\to\infty$, three quarters of all the possible pairs will result in an even product. **1**

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a)



The diagram shows a sketch of part of the curve with equation

$$y=e^{-x}\sin x,\quad x\geq 0.$$

(i) Use integration by parts to show that: **3**

$$\int e^{-x}\sin x\,dx=-\frac{1}{2}e^{-x}(\sin x+\cos x).$$

(ii) The terms $A_1, A_2, \ldots, A_n$ represent successive areas above the x-axis and bounded by the curve $y=e^{-x}\sin x$. **2**

The area A_n is bounded by the curve $y=e^{-x}\sin x$ and the x-axis between $x=(2n-2)\pi$ and $x=(2n-1)\pi$.

The areas represented by A_1 and A_2 are shown in the diagram.

Show that $A_n=\frac{1}{2}\left(e^{(1-2n)\pi}+e^{(2-2n)\pi}\right)$.

(iii) Show that $A_1+A_2+A_3+\ldots$ is a geometric series and that $S_\infty=\frac{e^\pi}{2(e^\pi-1)}$. **3**

(iv) Given that $\lim_{n\to\infty}\int_0^n e^{-x}\sin x\,dx=\frac{1}{2}$, find the exact value of **2**

$$\lim_{n\to\infty}\int_0^n \left|e^{-x}\sin x\right|dx.$$

Question 8 continues on page 15

Question 8 (continued)

(b) The numbers x, y and z satisfy

$$x+y+z=5$$
$$x^2+y^2+z^2=8$$
$$x^3+y^3+z^3=13.$$

(i) Show that $yz+xz+xy=\dfrac{17}{2}$. **1**

(ii) Show that $x^2y+x^2z+xy^2+xz^2+y^2z+yz^2=27$. **1**

(iii) Hence, show that $xyz=\dfrac{31}{6}$. **1**

Let $S_n=x^n+y^n+z^n$.

(iv) Use the above results to find numbers a, b and c such that **2**

$S_{n+1}=aS_n+bS_{n-1}+cS_{n-2}$ holds for all integral n, $n\geq 2$.

End of paper

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CATHOLIC SECONDARY SCHOOLS ASSOCIATION OF NSW
2011 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION
MATHEMATICS EXTENSION 2

Question 1 (15 marks)
(a) (2 marks)
Outcomes assessed: E8, HE6

Sample Answer:

$$\int \frac{e^x}{\sqrt{e^x+1}}\,dx = \int \frac{\frac{d}{dx}\left(e^x+1\right)}{\left(e^x+1\right)^{\frac{1}{2}}}\,dx$$
$$= \int \left[\frac{d}{dx}\left(e^x+1\right)\right]\left(e^x+1\right)^{-\frac{1}{2}}\,dx$$
$$= 2\left(e^x+1\right)^{\frac{1}{2}}+c$$

(b) (2 marks)

Sample Answer:

$$\int \cos^3 x\,dx = \int \cos x\,\left(1-\sin^2 x\right)dx$$
$$= \int \cos x - \cos x \sin^2 x\,dx$$
$$= \sin x - \frac{\sin^3 x}{3}+c$$

(c) (2 marks)

Sample Answer:

$$\int \frac{x^2}{x^2+4}\,dx = \int \frac{x^2+4-4}{x^2+4}\,dx$$
$$= \int \left[1-\frac{4}{x^2+4}\right]dx$$
$$= x - 2\tan^{-1}\frac{x}{2}+c$$

(d) (i) (2 marks)

Sample Answer:

$$\frac{1}{x^2+6x-7} \equiv \frac{a}{(x-1)}+\frac{b}{(x+7)}$$
$$\equiv \frac{a(x+7)+b(x-1)}{(x-1)(x+7)}$$
$$\therefore \left.\begin{array}{r} a+b=0 \\ 7a-b=1 \end{array}\right\} a=\frac{1}{8}, b=\frac{-1}{8}$$

(d) (ii) (1 mark)

Sample Answer:

$$\therefore \int \frac{dx}{x^2+6x-7} = \int \frac{1}{8(x-1)}-\frac{1}{8(x+7)}\,dx$$
$$= \frac{1}{8}\int \left[\frac{1}{(x-1)}-\frac{1}{(x+7)}\right]dx$$
$$= \frac{1}{8}\left[\ln(x-1)-\ln(x+7)\right]+c$$
$$= \frac{1}{8}\ln\left|\frac{x-1}{x+7}\right|+c$$

(e) (2 marks)

Sample Answer:

$$\int \frac{dx}{\sqrt{1+2x-x^2}} = \int \frac{dx}{\sqrt{2-(x-1)^2}}$$
$$= \sin^{-1}\frac{(x-1)}{\sqrt{2}}+c$$

(f) (4 marks)

Sample Answer:

$$t=\tan\frac{x}{2} \to x = 2\tan^{-1} t$$
$$\frac{dx}{dt}=\frac{2}{1+t^2}$$
$$x=\frac{\pi}{3}, t=\frac{1}{\sqrt{3}}$$
$$x=\frac{\pi}{2}, t=1$$
$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{dx}{1-\cos x} = \int_{\frac{1}{\sqrt{3}}}^{1} \frac{1}{1-\frac{1-t^2}{1+t^2}}\cdot\frac{2\,dt}{1+t^2}$$
$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{1}{\frac{1+t^2-1+t^2}{1+t^2}}\cdot\frac{2\,dt}{1+t^2} = \int_{\frac{1}{\sqrt{3}}}^{1}\frac{1}{t^2}\,dt$$
$$= \left[\frac{-1}{t}\right]_{\frac{1}{\sqrt{3}}}^{1}$$
$$= \sqrt{3}-1$$

Question 2 (15 marks)
(a) (i) (1 mark)

Sample Answer:

$$w-z=1+3i-(2-i)=-1+4i$$

(a) (ii) (1 mark)

Sample Answer:

$$z\overline{w}=(2-i)(1-3i)=-1-7i$$

(b) (3 marks)

Sample Answer:

$$\text{Let } (a+ib)^2=-5+12i$$

$$a^2-b^2=-5 \quad \text{and} \quad 2ab=12 \to b=\frac{6}{a}$$

$$a^2-\left(\frac{6}{a}\right)^2=-5$$

$$a^4+5a^2-36=0$$

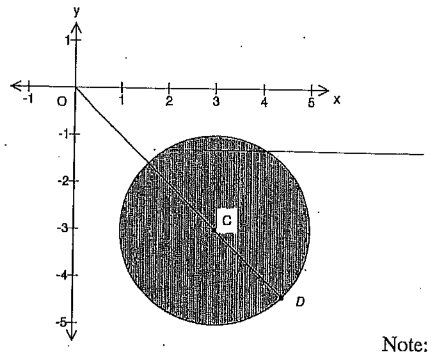
$$(a^2-4)(a^2+9)=0 \to a=\pm 2\,(a \text{ real})$$

$$b=\pm 3$$

$$\therefore \sqrt{-5+12i}=\pm(2+3i)$$

(c) (i) (2 marks)

Sample Answer:



Note: radius = 2 units

(c) (ii) (1 mark)

Sample Answer:

Maximum value of $|z|=OC+CD$ $\quad OC=\sqrt{3^2+3^2}=3\sqrt{2}$

$CD=2$

$$\therefore Max|z|=3\sqrt{2}+2$$

(d) (i) (2 marks)

Sample Answer:

Sum of the roots: $z+\overline{z}=2\text{Re}(z)=2\cos\frac{\pi}{6}=\sqrt{3}$

Product of the roots: $z\overline{z}=|z|^2=1$

Equation is: $x^2-\sqrt{3}x+1=0$

(d) (ii) (1 mark)

Sample Answer:

$$\omega=i \quad or \quad \omega=cis\left(-\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}-\frac{1}{2}i\ [=\overline{z}]$$

(e) (i) (1 mark)

Sample Answer:

$$z^3+8=(z+2)(z^2-2z+4)$$

(e) (ii) (1 mark)

Sample Answer:

Since $z=w$ is a solution and $w\neq -2$ (w non-real) then $w^2-2w+4=0 \to w^2=2w-4$

(e) (iii) (2 marks)

Sample Answer:

$$(2w-4)^6=(w^2)^6=w^{12}=(w^3)^4$$
$$=(-8)^4$$
$$=4096$$

Question 3 (15 marks)
(a) (i) (1 mark)

Sample Answer:

$$\frac{y^2}{16}-\frac{x^2}{9}=1$$

$$When\ x=0, \frac{y^2}{16}=1 \to y=\pm 4$$

(a) (ii) (3 marks)

Sample Answer:

$$b^2=a^2(e^2-1) \to \quad 9=16(e^2-1)$$

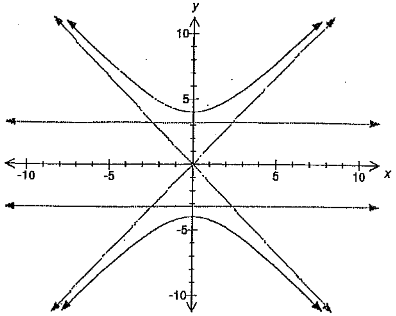
$$e^2=\frac{25}{16}$$

$$e=\frac{5}{4}, a=4, b=3 \qquad Directrices: y=\pm\frac{a}{e}=\pm\frac{16}{5}$$

$$Asymptotes: y=\pm\frac{ax}{b}=\pm\frac{4x}{3}$$

(a) (iii) (1 mark)

Sample Answer:



(b) (i) (3 marks)

Sample Answer:

$$4x^2-2xy+y^2-6x=0$$

$$8x-\left[2x\frac{dy}{dx}+2y\right]+2y\frac{dy}{dx}-6=0$$

$$2y\frac{dy}{dx}-2x\frac{dy}{dx}=6-8x+2y$$

$$\frac{dy}{dx}=\frac{6-8x+2y}{2y-2x}=\frac{2(3-4x+y)}{2(y-x)}$$

$$\therefore\frac{dy}{dx}=\frac{3-4x+y}{y-x}$$

(b) (ii) (2 marks)

Sample Answer:

Tangent is vertical when $\frac{dy}{dx}$ is undefined, ie when $y=x$

$$\therefore 4x^2-2xy+y^2-6x=0\rightarrow 4x^2-2x^2+x^2-6x=0$$

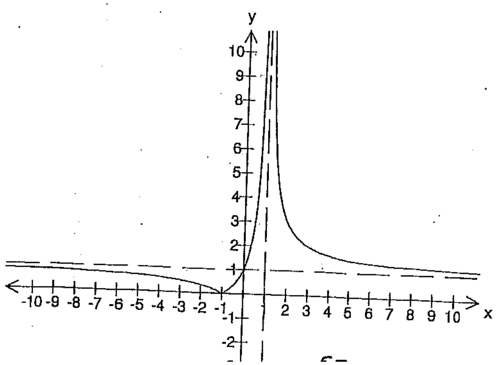
$$3x^2-6x=0$$

$$3x(x-2)=0$$

$$x=0, x=2$$

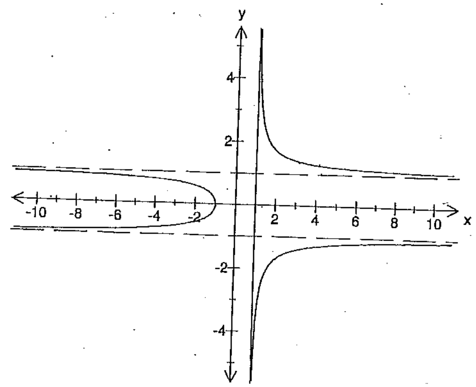
(c) (i) (1 mark)

Sample Answer:



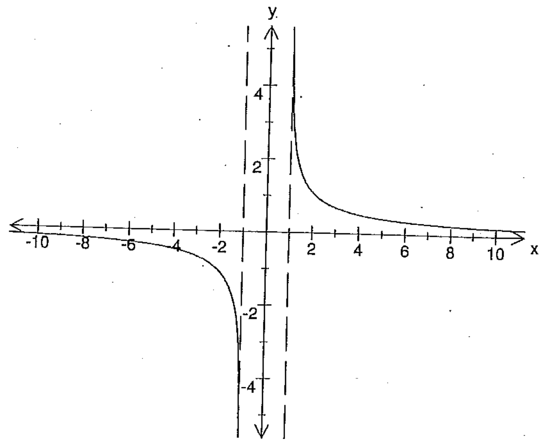
(c) (ii) (2 marks)

Sample Answer:



(c) (iii) (2 marks)

Sample Answer:



Question 4 (15 marks)

(a) (i) (2 marks)

Sample Answer:

$$x^3+7x^2-4x+2=0$$

$$Let\ y=x^2\rightarrow x=y^{\frac{1}{2}}$$

$$y^{\frac{3}{2}}+7y-4y^{\frac{1}{2}}+2=0$$

$$y^{\frac{3}{2}}-4y^{\frac{1}{2}}=-7y-2$$

$$y^3-8y^2+16y=49y^2+28y+4$$

$$y^3-57y^2-12y-4=0$$

(a) (ii) (1 mark)

Sample Answer:

$$\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2=\frac{c}{a}=-12 \text{ using (i)}$$

(b) (i) (2 marks)

Sample Answer:

$$\frac{z_2}{z_1}=\frac{\frac{1}{2}(1+\sqrt{3}\,i)(1-i)}{\frac{1}{\sqrt{2}}(1+i)(1-i)}=\frac{\sqrt{2}\left[(1+\sqrt{3})+i(\sqrt{3}-1)\right]}{2\times 2}$$

$$=\frac{\sqrt{2}+\sqrt{6}}{4}+\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)i$$

(b) (ii) (2 marks)

Sample Answer:

$$z_1=\frac{1}{\sqrt{2}}(1+i)=cis\frac{\pi}{4} \qquad z_2=\frac{1}{2}(1+\sqrt{3}\,i)=cis\frac{\pi}{3}$$

$$\frac{z_2}{z_1}=cis\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=cis\frac{\pi}{12}$$

(b) (iii) (1 mark)

Sample Answer:

$$\sin\frac{\pi}{12}=\frac{\sqrt{6}-\sqrt{2}}{4}$$

(c) (i) (1 mark)

Sample Answer:

$$2\sin\theta\cos 6\theta=\sin 7\theta-\sin 5\theta$$

(c) (ii) (1 mark)

Sample Answer:

$$2\sin\theta(\cos 6\theta+\cos 4\theta+\cos 2\theta)=2\sin\theta\cos 6\theta+2\sin\theta\cos 4\theta+2\sin\theta\cos 2\theta$$

$$=\sin 7\theta-\sin 5\theta+\sin 5\theta-\sin 3\theta+\sin 3\theta-\sin\theta$$

$$=\sin 7\theta-\sin\theta$$

(c) (iii) (2 marks)

Sample Answer:

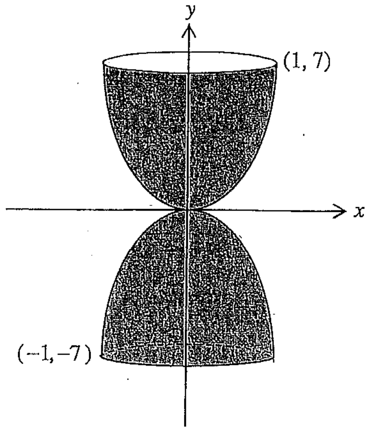
From (ii) $2\sin\theta(\cos 6\theta+\cos 4\theta+\cos 2\theta)=\sin 7\theta-\sin\theta$, and using $\theta=\frac{2\pi}{7}$

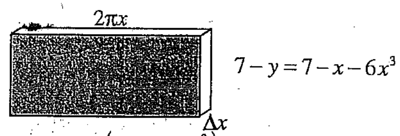
$$2\sin\frac{2\pi}{7}\left(\cos 6\left(\frac{2\pi}{7}\right)+\cos 4\left(\frac{2\pi}{7}\right)+\cos 2\left(\frac{2\pi}{7}\right)\right)=\sin 7\left(\frac{2\pi}{7}\right)-\sin\left(\frac{2\pi}{7}\right)$$

$$\cos\left(\frac{12\pi}{7}\right)+\cos\left(\frac{8\pi}{7}\right)+\cos\left(\frac{4\pi}{7}\right)=\frac{\sin 2\pi-\sin\frac{2\pi}{7}}{2\sin\frac{2\pi}{7}}=\frac{-\sin\frac{2\pi}{7}}{2\sin\frac{2\pi}{7}}=\frac{-1}{2}$$

(d) (3 marks)

Sample Answer:





$$A=2\pi x\times(7-x-6x^3)$$

$$\Delta V=2\pi x(7-x-6x^3)\Delta x$$

$$V=2\times\lim_{\Delta x\to 0}\sum_{x=0}^{1}2\pi x(7-x-6x^3)\Delta x$$

$$=4\pi\int_0^1(7x-x^2-6x^4)\,dx$$

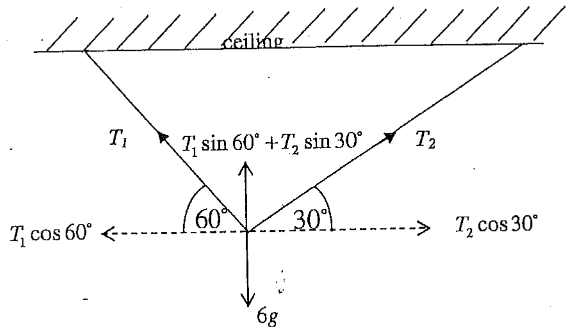
$$=4\pi\left[\frac{7}{2}x^2-\frac{1}{3}x^3-\frac{6}{5}x^5\right]_0^1$$

$$=4\pi\left(\frac{7}{2}-\frac{1}{3}-\frac{6}{5}\right)=\frac{118\pi}{15}\,units^3$$

Question 5 (15 marks)

(a) (3 marks)

Sample Answer:



Resolving forces vertically:

$$T_1\sin 60^\circ+T_2\sin 30^\circ=6g$$

$$\frac{\sqrt{3}T_1}{2}+\frac{T_2}{2}=6g$$

$$\sqrt{3}T_1+T_2=12g$$

horizontally:

$$T_1\cos 60^\circ=T_2\cos 30^\circ$$

$$\frac{T_1}{2}=\frac{\sqrt{3}T_2}{2}$$

$$T_1=\sqrt{3}T_2$$

$$\therefore 3T_2+T_2=12g$$

$$T_2=3g \qquad T_1=3\sqrt{3}g$$

Sample Answer:

$$I_0=\int_0^M e^{-x}dx=\left[-e^{-x}\right]_0^M=1-e^{-M}$$

(b) (ii) (2 marks)

Sample Answer:

$$I_n=\int_0^M x^n e^{-x}dx$$

Let $u=x^n$ $\quad dv=e^{-x}$

$du=nx^{n-1}$ $\quad v=-e^{-x}$

$$I_n=\left[-x^n e^{-x}\right]_0^M+n\int_0^M x^{n-1}e^{-x}dx$$

$$=\frac{-M^n}{e^M}+nI_{n-1}$$

(b) (iii) (2 marks)

Sample Answer:

$$\lim_{M\to\infty}I_n=\lim_{M\to\infty}\left[nI_{n-1}-\frac{M^n}{e^M}\right]$$

$$=n\lim_{M\to\infty}I_{n-1}-0 \qquad \lim_{M\to\infty}\frac{M^n}{e^M}=0$$ (as the exponential has dominance over the polynomial for very large numbers.)

$$=n(n-1)\lim_{M\to\infty}I_{n-2}$$

$$=n(n-1)(n-2)\lim_{M\to\infty}I_{n-3}$$

$$=n(n-1)(n-2)\ldots\times2\times1\times\lim_{M\to\infty}I_0$$

$$=n(n-1)(n-2)\ldots\times2\times1\times1=n! \qquad as\ \lim_{M\to\infty}I_0=1$$

(c) (1 mark)

Sample Answer:

$$\frac{\sec\theta+1}{\cos\theta+1}+\frac{\sec\theta-1}{\cos\theta-1}=\frac{(\sec\theta+1)(\cos\theta-1)+(\sec\theta-1)(\cos\theta+1)}{\cos^2\theta-1}$$

$$=\frac{1-\sec\theta+\cos\theta-1+1+\sec\theta-\cos\theta-1}{-\sin^2\theta}$$

$$=0$$

(d) (i) (2 marks)

Sample Answer:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{2x}{a^2}+\frac{2y}{b^2}\cdot\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{b^2x}{a^2y}$$

gradient of tangent at $P=-\frac{ab^2\cos\theta}{a^2b\sin\theta}$

$$=-\frac{b\cos\theta}{a\sin\theta}$$

$$y-b\sin\theta=-\frac{b\cos\theta}{a\sin\theta}(x-a\cos\theta)$$

$$ay\sin\theta-ab\sin^2\theta=-bx\cos\theta+ab\cos^2\theta$$

$$bx\cos\theta+ay\sin\theta=ab$$

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

(d) (ii) (1 mark)

Sample Answer:

x intercept occurs when $y=0$

i.e. $\frac{x\cos\theta}{a}+0=1$

$x=\frac{a}{\cos\theta}$ $\quad\therefore T$ has coordinates $\left(\frac{a}{\cos\theta},0\right)$

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(d) (iii) (3 marks)

Sample Answer:

$$m_{AQ}=m_{AP}$$

$$\frac{y_Q-0}{\frac{a}{\cos\theta}-a}=\frac{b\sin\theta-0}{a\cos\theta-a}$$

$$y_Q=\frac{b\sin\theta}{a(\cos\theta-1)}\times\frac{a-a\cos\theta}{\cos\theta}$$

$$y_Q=\frac{-b\sin\theta}{\cos\theta}$$

$$m_{AQ'}=m_{AP}$$

$$\frac{y_{Q'}-0}{\frac{a}{\cos\theta}+a}=\frac{b\sin\theta-0}{a\cos\theta+a}$$

$$y_{Q'}=\frac{b\sin\theta}{a(\cos\theta+1)}\times\frac{a+a\cos\theta}{\cos\theta}$$

$$y_{Q'}=\frac{b\sin\theta}{\cos\theta}$$

$$|y_Q|=|y_{Q'}|$$

$$\therefore QT=Q'T$$

Question 6 (15marks)

(a) (i) (1 mark)

Sample Answer:

$$\binom{n}{r}=\frac{n!}{r!(n-r)!}$$

$$=\frac{n(n-1)!}{r(r-1)!(n-r)!}$$

$$=\frac{1}{r}\times n\times\frac{(n-1)!}{(r-1)!(n-r)!}$$

$$=\frac{1}{r}\times n\binom{n-1}{r-1}$$

$$<n\binom{n-1}{r-1} \qquad since\ r>1\Rightarrow\frac{1}{r}<1$$

(a) (ii) (2 marks)

Sample Answer:

$$(a+b)^n - a^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n}b^n - a^n$$

$$< n\binom{n-1}{0}a^{n-1}b + n\binom{n-1}{1}a^{n-2}b^2 + \ldots + n\binom{n-1}{n-1}b^n$$

$$= nb\left[\binom{n-1}{0}a^{n-1} + \binom{n-1}{1}a^{n-2}b + \ldots + \binom{n-1}{n-1}b^{n-1}\right]$$

$$= nb(a+b)^{n-1}$$

$$\therefore (a+b)^n - a^n < nb(a+b)^{n-1}$$

(b) (i) (1 mark)

Sample Answer:

$$A_{\text{sector}} = \frac{1}{2}r^2\theta \qquad A_{\text{segment}} = \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$$

$$A_{\text{re-entrant segment}} = A_{\text{sector}} - 2A_{\text{segment}}$$

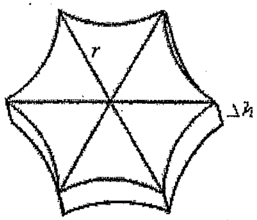
$$= \frac{1}{2}r^2\theta - 2\left(\frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta\right)$$

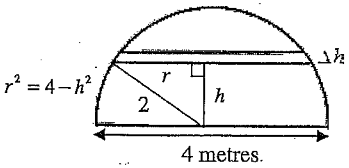
$$= r^2\sin\theta - \frac{1}{2}r^2\theta$$

$$= r^2\left(\sin\theta - \frac{\theta}{2}\right)$$

(b) (ii) (4 mark)

Sample Answer:





$$A(r) = 6\times r^2\left(\sin\frac{\pi}{3} - \frac{\pi}{6}\right)$$

$$= r^2\left(3\sqrt{3} - \pi\right)$$

$$A(h) = \left(3\sqrt{3} - \pi\right)\left(4 - h^2\right)$$

$$\Delta V = \left(3\sqrt{3} - \pi\right)\left(4 - h^2\right)\Delta h$$

$$V = \lim_{\Delta h \to 0}\sum_{h=0}^{2}\left(3\sqrt{3} - \pi\right)\left(4 - h^2\right)\Delta h$$

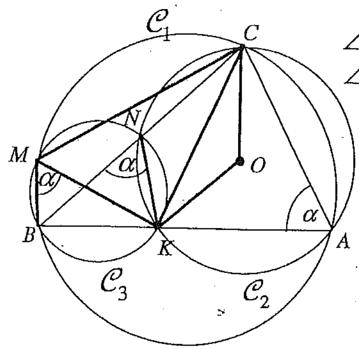
$$= \left(3\sqrt{3} - \pi\right)\int_0^2\left(4 - h^2\right)dh$$

$$= \left(3\sqrt{3} - \pi\right)\left[4h - \frac{1}{3}h^3\right]_0^2$$

$$= \left(3\sqrt{3} - \pi\right)\left(\frac{16}{3}\right) = \left(16\sqrt{3} - \frac{16\pi}{3}\right) \text{ units}^3$$

(c) (i) (2 marks)

Sample Answer:



$\angle BMK = \alpha = \angle BNK$ *(angles on same arc BK)*

$\angle BNK = \alpha = \angle KAC$ *(ext∠ of cyclic quad ACNK)*

(c) (ii) (1 mark)

Sample Answer:

$\angle BMC = 180 - \alpha$ (since it is opposite angle to $\angle KAC$ in cyclic quad $ABMC$)

(c) (iii) (2 marks)

Sample Answer:

$\angle KOC = 2\alpha$ *(angle at centre on same arc KC is double angle at circumference)*

$\angle CMK = \angle BMC - \angle BMK$

$= 180 - \alpha - \alpha$

$= 180 - 2\alpha$ $\quad\therefore$ *MKOC is a cyclic quad as opposite angles are supplementary*

(c) (iv) (2 marks)

Sample Answer:

Angle sum of $\Delta KOC = 180^\circ$ $\therefore \angle OCK = \angle OKC = 90^\circ - \alpha$ *(isosceles Δ)*

$\angle OCK = \angle OMK = 90^\circ - \alpha$ *(angles on the same arc OK)*

$\angle OMB = \angle OMK + \angle BMK = 90^\circ - \alpha + \alpha = 90^\circ$ $\qquad\therefore OM \perp BM$

Question 7 (15 marks)
(a) (3 marks)

Sample Answer:

When $n=1$;

$LHS = F_0 = 2^{2^0}+1 = 2^1+1 = 3$

$RHS = F_1 - 2 = 2^{2^1}-2 = 2^2-1 = 3$

$LHS = RHS$

Hence the result is true for $n=1$

Assume the result is true for $n=k$, i.e. $F_0F_1F_2\ldots F_{k-1} = F_k - 2$

Prove the result is true for $n=k+1$, i.e. $F_0F_1F_2\ldots F_k = F_{k+1}-2$

$$\begin{aligned}F_0F_1F_2\ldots F_k &= F_0F_1F_2\ldots F_{k-1}F_k\\ &= (F_k-2)F_k\\ &= \left(2^{2^k}-1\right)\left(2^{2^k}+1\right)\\ &= \left(2^{2^k}\right)^2-1\\ &= 2^{2^k\times 2}-1\\ &= 2^{2^{k+1}}+1-2\\ &= F_{k+1}-2\end{aligned}$$

Hence the result is true for $n=k+1$, if it is true for $n=k$.
Since the result is true for $n=1$, then it is true for all $n \geq 1$ by induction.

(b) (i) (1 mark)

Sample Answer:

$$m\ddot{x} = mg - mkv^2$$
$$\ddot{x} = g - kv^2$$

Terminal velocity V occurs when $\ddot{x}=0, \rightarrow g-kV^2=0$

$$V^2 = \frac{g}{k}$$
$$V = \sqrt{\frac{g}{k}}$$

(b) (ii) (3 marks)

Sample Answer:

$$\ddot{x} = g - kv^2$$
$$v\frac{dv}{dx} = g - kv^2$$
$$\frac{dv}{dx} = \frac{g-kv^2}{v}$$
$$\frac{dx}{dv} = \frac{v}{g-kv^2}$$

$$\int_0^D dx = -\frac{1}{2k}\int_0^W \frac{-2kv}{g-kv^2}\,dv$$

$$\begin{aligned}D &= -\frac{1}{2k}\left[\ln\left(g-kv^2\right)\right]_0^W\\ &= -\frac{1}{2k}\left[\ln\left(g-kW^2\right)-\ln g\right]\\ &= -\frac{1}{2k}\left[\ln\left(\frac{g-kW^2}{g}\right)\right] = -\frac{1}{2k}\left[\ln\left(1-\frac{kW^2}{g}\right)\right]\\ &= -\frac{1}{2k}\ln\left(1-\frac{W^2}{V^2}\right) \quad as\ \frac{1}{V^2}=\frac{k}{g} \quad from\ (i)\end{aligned}$$

(b) (iii) (1 mark)

Sample Answer:

$$\frac{1}{2k}\ln\left(1+\frac{U^2}{V^2}\right) = -\frac{1}{2k}\ln\left(1-\frac{W^2}{V^2}\right)$$
$$\ln\left(\frac{V^2+U^2}{V^2}\right) = \ln\left(\frac{V^2}{V^2-W^2}\right)$$
$$\left(V^2+U^2\right)\left(V^2-W^2\right) = V^4$$
$$-V^2W^2+U^2V^2-U^2W^2=0$$

Dividing by $U^2V^2W^2$, $-\frac{1}{U^2}+\frac{1}{W^2}-\frac{1}{V^2}=0$

$$\therefore \frac{1}{W^2} = \frac{1}{U^2}+\frac{1}{V^2}$$

(b) (iv) (1 mark)

Sample Answer:

$$\begin{aligned}\frac{1}{U^2} &= \frac{1}{W^2}-\frac{1}{V^2}\\ &= \frac{V^2-W^2}{W^2V^2}\end{aligned}$$

$$\therefore \quad U^2 = \frac{W^2V^2}{V^2-W^2}$$

$$W=\frac{3V}{5}, \qquad U^2 = \frac{\left(\frac{3V}{5}\right)^2V^2}{V^2-\left(\frac{3V}{5}\right)^2} = \frac{\frac{9V^2}{25}\times V^2}{V^2-\frac{9V^2}{25}} = \frac{9V^4}{16V^2} = \frac{9V^2}{16}$$

$$\therefore \quad U = \frac{3V}{4}$$

(c) (i) (2 marks)

Sample Answer:

There are n ways of selecting the same number.

There are ${}^nC_2$ ways of selecting two different numbers.

Total number of ways: $n+{}^nC_2=n+\frac{n(n-1)}{2}=\frac{n^2+n}{2}$

(c) (ii) (3 marks)

Sample Answer:

Total number of even products=total number of products –total number of odd products.

There are $\frac{n}{2}$ ways of choosing the same odd number.

There are ${}^{\frac{n}{2}}C_2$ ways of selecting two different odd numbers.

Using (i) the total number of odd products = $\frac{\left(\frac{n}{2}\right)^2+\frac{n}{2}}{2}=\frac{n^2+2n}{8}$

Therefore the total number of even products = $\frac{n^2+n}{2}-\frac{n^2+2n}{8}=\frac{3n^2+2n}{8}$

(c) (iii) (1 mark)

Sample Answer:

$$\lim_{n\to\infty}\frac{\frac{3n^2+2n}{8}}{\frac{n^2+n}{2}}=\lim_{n\to\infty}\frac{3n^2+2n}{4n^2+4n}$$

$$=\lim_{n\to\infty}\frac{3+\frac{2}{n}}{4+\frac{4}{n}}=\frac{3}{4}$$

Question 8 (15 marks)

(a) (i) (3 marks)

Sample Answer:

$$\int e^{-x}\sin x\,dx \qquad u=e^{-x}\quad v'=\sin x \qquad u'=-e^{-x}\quad v=-\cos x$$

$$\int e^{-x}\sin x\,dx=-e^{-x}\cos x-\int e^{-x}\cos x\,dx \qquad u=e^{-x}\quad v'=\cos x \qquad u'=-e^{-x}\quad v=\sin x$$

$$=-e^{-x}\cos x-e^{-x}\sin x-\int e^{-x}\sin x\,dx$$

$$\therefore 2\int e^{-x}\sin x\,dx=-e^{-x}(\cos x+\sin x)+c$$

$$\int e^{-x}\sin x\,dx=-\frac{1}{2}e^{-x}(\cos x+\sin x)+c$$

(a) (ii) (2 marks)

Sample Answer:

$$A_1=\int_0^{\pi}e^{-x}\sin x\,dx,\quad A_2=\int_{2\pi}^{3\pi}e^{-x}\sin x\,dx,\quad A_3=\int_{4\pi}^{5\pi}e^{-x}\sin x\,dx$$

$$A_n=\int_{(2n-2)\pi}^{(2n-1)\pi}e^{-x}\sin x\,dx$$

$$=-\frac{1}{2}\left[e^{-x}(\cos x+\sin x)\right]_{(2n-2)\pi}^{(2n-1)\pi}$$

$$=-\frac{1}{2}\left(-e^{-(2n-1)\pi}-e^{-(2n-2)\pi}\right)$$

$$=\frac{1}{2}\left(e^{(1-2n)\pi}+e^{(2-2n)\pi}\right)$$

(a) (iii) (3 marks)

Sample Answer:

$$A_1+A_2+A_3+\ldots$$

$$=\frac{1}{2}\left(e^{-\pi}+1\right)+\frac{1}{2}\left(e^{-3\pi}+e^{-2\pi}\right)+\frac{1}{2}\left(e^{-5\pi}+e^{-4\pi}\right)+\ldots$$

$$=\frac{1}{2}\left(1+e^{-\pi}+e^{-2\pi}+e^{-3\pi}+e^{-4\pi}+e^{-5\pi}+\ldots\right)$$

$\therefore$ geometric series with $a=\frac{1}{2}, r=e^{-\pi}$

$$S_\infty=\frac{\frac{1}{2}}{1-e^{-\pi}}=\frac{1}{2\left(1-e^{-\pi}\right)}\times\frac{e^{\pi}}{e^{\pi}}=\frac{e^{\pi}}{2\left(e^{\pi}-1\right)}$$

(a) (iv) (2 marks)

Sample Answer:

$A_n=$ areas above x axis with $A_1+A_2+A_3+\ldots=X$

and $a_n=$ areas below x axis with $a_1+a_2+a_3+\ldots=Y$

$$\therefore \lim_{n\to\infty}\int_0^n e^{-x}\sin x\,dx=X-Y$$

$$\frac{1}{2}=\frac{e^{\pi}}{2\left(e^{\pi}-1\right)}-Y$$

$$Y=\frac{e^{\pi}-\left(e^{\pi}-1\right)}{2\left(e^{\pi}-1\right)}$$

$$=\frac{1}{2\left(e^{\pi}-1\right)}$$

$$\lim_{n\to\infty}\int_0^n\left|e^{-x}\sin x\right|dx=X+Y=\frac{e^{\pi}}{2\left(e^{\pi}-1\right)}+\frac{1}{2\left(e^{\pi}-1\right)}$$

$$=\frac{e^{\pi}+1}{2\left(e^{\pi}-1\right)}$$

(b) (i) (1 mark)

Sample Answer:

$$2(xy+xz+yz)=(x+y+z)^2-(x^2+y^2+z^2)$$
$$2(xy+xz+yz)=5^2-8$$
$$=17$$
$$xy+xz+yz=\frac{17}{2}$$

(b) (ii) (1 mark)

Sample Answer:

$$(x^2+y^2+z^2)(x+y+z)=x^3+y^3+z^3+x^2y+x^2z+xy^2+xz^2+y^2z+yz^2$$
$$x^2y+x^2z+xy^2+xz^2+y^2z+yz^2=(x^2+y^2+z^2)(x+y+z)-(x^3+y^3+z^3)$$
$$=(8)(5)-13$$
$$=27$$

(b) (iii) (1 mark)

Sample Answer:

$$(x+y+z)^3=x^3+y^3+z^3+3(x^2y+x^2z+xy^2+xz^2+y^2z+yz^2)+6xyz$$
$$6xyz=(x+y+z)^3-x^3-y^3-z^3-3(x^2y+x^2z+xy^2+xz^2+y^2z+yz^2)$$
$$=(5)^3-(13)-3(27)$$
$$=31$$
$$xyz=\frac{31}{6}$$

(b) (iv) (2 marks)

Sample Answer:

$$x+y+z=5$$
$$xy+xz+yz=\frac{17}{2}$$
$$xyz=\frac{31}{6}$$

Hence x, y & z satisfy the cubic equation

$$m^3-5m^2+\frac{17}{2}m-\frac{31}{6}=0$$

Thus

$$x^3-5x^2+\frac{17}{2}x-\frac{31}{6}=0$$
$$y^3-5y^2+\frac{17}{2}y-\frac{31}{6}=0$$
$$z^3-5z^2+\frac{17}{2}z-\frac{31}{6}=0$$

multiplying by x^{n-2}, y^{n-2} and z^{n-2} respectively;

$$x^{n+1}-5x^n+\frac{17}{2}x^{n-1}-\frac{31}{6}x^{n-2}=0$$
$$y^{n+1}-5y^n+\frac{17}{2}y^{n-1}-\frac{31}{6}y^{n-2}=0$$
$$z^{n+1}-5z^n+\frac{17}{2}z^{n-1}-\frac{31}{6}z^{n-2}=0$$

$$S_{n+1}-5S_n+\frac{17}{2}S_{n-1}-\frac{31}{6}S_{n-2}=0$$

$$S_{n+1}=5S_n-\frac{17}{2}S_{n-1}+\frac{31}{6}S_{n-2}$$
$$\therefore a=5, b=-\frac{17}{2}, c=\frac{31}{6}$$