Centre Number

Student Number

CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NEW SOUTH WALES

2003

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

Morning Session
Monday 11 August 2003

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided separately
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–10
- All questions are of equal value

Disclaimer

Every effort has been made to prepare these 'Trial' Higher School Certificate Examinations in accordance with the Board of Studies documents, *Principles for Setting HSC Examinations in a Standards-Referenced Framework* (BOS Bulletin, Vol 8, No 9, Nov/Dec 1999), and *Principles for Developing Marking Guidelines Examinations in a Standards Referenced Framework* (BOS Bulletin, Vol 9, No 3, May 2000). No guarantee or warranty is made or implied that the 'Trial' Examination papers mirror in every respect the actual HSC Examination question paper in any or all courses to be examined. These papers do not constitute 'advice' nor can they be construed as authoritative interpretations of Board of Studies intentions. The CSSA accepts no liability for any reliance use or purpose related to these 'Trial' question papers. Advice on HSC examination issues is only to be obtained from the NSW Board of Studies.

2602-1

Total marks – 120
Attempt Questions 1–10
All questions are of equal value

Answer each question in a SEPARATE writing booklet.

Question 1 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Evaluate correct to one decimal place $\dfrac{2.1^2 \times 4.5^2}{2.1^2 + 4.5^2}$. **2**

(b) Factorise fully $128x - 16x^4$. **2**

(c) Graph on a number line the solution to $|2x+1| \leq 5$. **2**

(d) Rationalise the denominator of $\dfrac{\sqrt{5}}{3\sqrt{2}-1}$. **2**

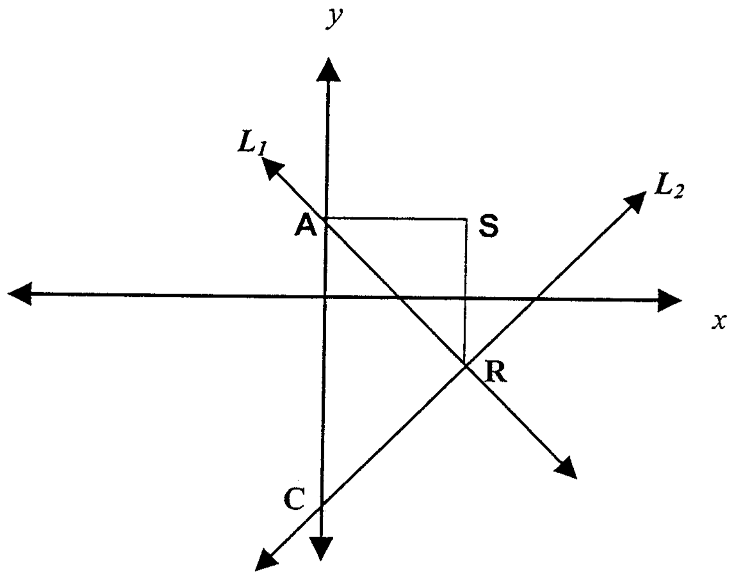
(e) Find the exact value of $\tan\dfrac{\pi}{3} + \operatorname{cosec}\dfrac{\pi}{4}$. **2**

(f) Sketch the curve $y = 2e^{-x}$, clearly showing where the curve cuts the y-axis. **2**

Question 2 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Consider the quadratic function $x^2 - (k+2)x + 4 = 0$.
For what value of k does the quadratic function have real roots? **2**

(b)



NOT TO SCALE

Line L_1 has equation $x + y = 2$ and intersects the y-axis at point A.
Line L_2 has equation $x - y = 4$ and intersects the y-axis at point C.
Line L_1 and line L_2 intersect at point R.

The horizontal line through A intersects the vertical line through R, at S.

(i) Find the coordinates of point A and C. **2**

(ii) Show that R has coordinates (3, -1). **1**

(iii) State the equation of the line SR. **1**

(iv) Find the gradient of line L_1. **1**

(v) Find the distance AR. **1**

(vi) Show that triangle ARC is a right-angled isosceles triangle. **2**

(vii) Find the equation of the circle with centre R, passing through the points A and C. **2**

Question 3 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Differentiate with respect to x:

(i) $\sqrt{x}$ **1**

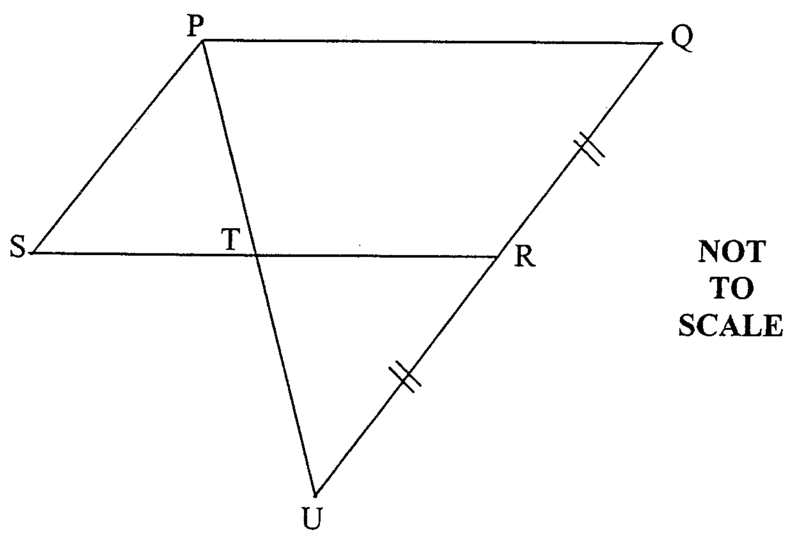
(ii) $x^3 e^{-3x}$ **2**

(iii) $\dfrac{\tan x}{2x+1}$ **2**

(b) Find $\displaystyle\int \frac{e^{2x}}{e^{2x}+4}\,dx$. **2**

(c) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \left(\frac{1}{2}x + \cos 2x\right)\,dx$. **2**

(d) In the diagram, PQRS is a parallelogram. QR is produced to U so that QR = RU.
Copy this diagram into your answer booklet.



(i) Giving clear reasons, show that the triangles PST and URT are congruent. **2**

(ii) Hence, or otherwise, show that T is the midpoint of SR. **1**

Question 4 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Evaluate $\sum_{k=4}^{20} 2k-5$. **2**

(b) The third term of a geometric series is $\frac{3}{4}$ and the seventh term is 12.
Find the 14th term of this series. **2**

(c) Consider the function $f(x)=|4-x|$.

(i) Sketch the function $f(x)$. **1**

(ii) Evaluate $\int_0^6 f(x)\ dx$. **1**

(d) Solve for x the equation $\sqrt[x]{m}=n^3$. **2**

(e) Sketch a graph of $y=3\cos x$ for $0\le x\le 2\pi$. **2**

(f) The population (P) of a coastal town is increasing at a decreasing rate.

Comment on $\frac{dP}{dt}$ and $\frac{d^2P}{dt^2}$ for this function. **2**

Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) In a bag there are 20 marbles. The bag consists of 7 red marbles, 9 gold marbles and 4 blue marbles. One marble is drawn from the bag and not replaced, and then a second marble is drawn.

With the aid of a tree diagram, or otherwise, find the probability of choosing:

(i) two gold marbles **1**

(ii) marbles of different colour **2**

(b) Consider the curve given by $y = 6x^2 - x^3$.

(i) Find the coordinates of the two stationary points. **2**

(ii) Determine the nature of the stationary points. **2**

(iii) Show that there exists a point of inflexion when $x = 2$. **1**

(iv) Sketch the curve for the domain $-2 \le x \le 6$. **2**

(c) Given that $y = 3e^{-2x}$, show that $2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 2y = 0$. **2**

Question 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

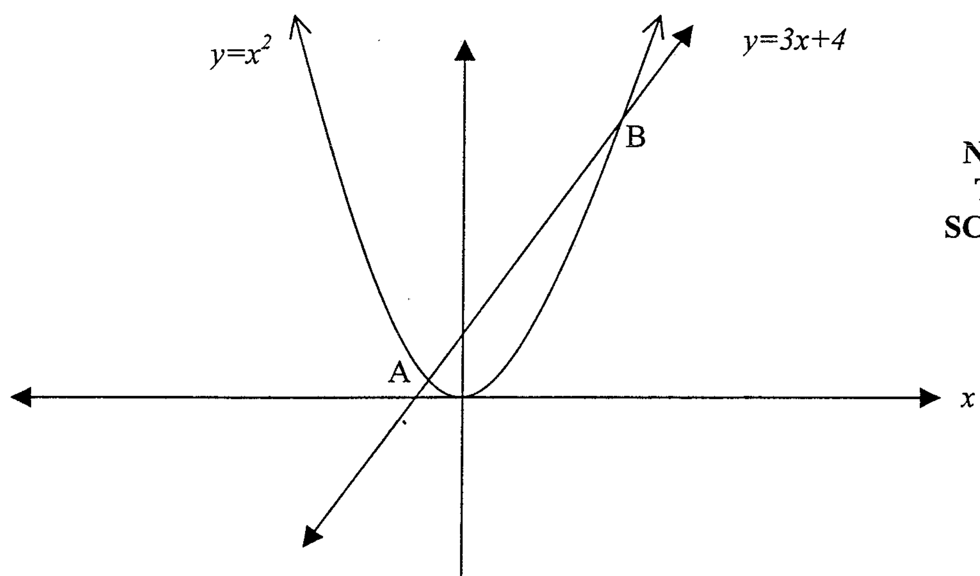
(a) Find the equation of the normal to the curve $y = x\sin x$ at the point where $x = \frac{\pi}{2}$. **3**

(b) The table shows the values of a function $f(x)$ for five values of x.

x	0	1	2	3	4
$f(x)$	2	3	12	35	80

Use Simpson's rule with these five values to find an approximation to $\int_0^4 f(x)\ dx$. **2**

(c)



NOT TO SCALE

(i) The curve $y = x^2$ and the line $y = 3x + 4$ intersect at the points A and B as shown in the diagram above.
Find the x coordinates of the points A and B. **2**

(ii) Find the area bounded by the curve $y = x^2$ and the line $y = 3x + 4$. **2**

(d) Find the volume generated when the curve $y = \sqrt{\cot x}$ is rotated about the $x - axis$ between $x = \frac{\pi}{3}$ and $x = \frac{\pi}{4}$. Leave your answer in exact form. **3**

Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find $\frac{dy}{dx}$ given that $y = \log_e\left(\frac{2x+1}{3x-7}\right)$. **2**

(b) A particle moves in a straight line so that its velocity, v metres per second, at time t is given by $v = 3 - \frac{2}{1+t}$.

The particle is initially 1 metre to the right of the origin.

(i) Find an expression for the position x, of the particle at time t. **2**

(ii) Explain why the velocity of the particle is never 3 metres per second. **1**

(iii) Find the acceleration of the particle when $t = 2$ seconds. **2**

(c) (i) Show that $(\text{cosec}^2 A - 1)\sin^2 A = \cos^2 A$. **2**

(ii) Hence, or otherwise, solve $(\text{cosec}^2 A - 1)\sin^2 A = \frac{3}{4}$ for $-\pi \le A \le \pi$. **3**

Question 8 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Cristina borrows \$480 000 from a finance company to buy a house. She pays interest at 6% per annum, calculated quarterly on the balance still owing. The loan is to be repaid at the end of 20 years with equal quarterly repayments of \$P.

Let A_n = the amount owing after the n th repayment.

(i) Show that after the first quarterly repayment of \$P Cristina owes an amount equivalent to $A_1 =$ \$ 487 200 - \$P. **1**

(ii) Find an expression for the amount still owing after 3 repayments of \$P. **2**

(iii) Find the value of \$P to the nearest cent. **2**

(b) Water is draining from a storage tank at a rate which is proportional to the volume of water contained in the tank. When full, the storage tank holds 1 000 litres of water. On inspection the tank was found to be full but 40 minutes later it was found to contain only 800 litres of water.

(i) How much water (to the nearest litre) will the tank hold after 1 hour? **2**

(ii) How long, in hours and minutes, will it take to reach 1 litre of water in the tank? **2**

(c) Consider the series $\sin^2 x + \sin^4 x + \sin^6 x + ...$ $\quad 0 < x < \frac{\pi}{2}$.

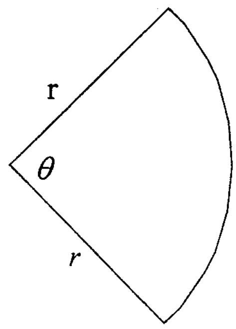
(i) Show that a limiting sum exists. **1**

(ii) Find the limiting sum expressing the answer in simplest form. **2**

Question 9 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Arcsec Landscaping Company are designing a garden bed for a local park in the shape of a sector with radius r and sector angle θ.

They have a total of 375 metres of garden edging materials to use as the perimeter of the garden bed.



NOT TO SCALE

(i) Show that the area A of the garden bed is given by $A = \frac{r}{2}(375 - 2r)$. **2**

(ii) Find the greatest area of garden bed which can be made using 375 metres of edging material. **3**

(iii) After inspecting the location for the garden bed the designers calculate that the sector angle for the garden must be less than $110°$.
Can they still create the garden bed with maximum area found in (ii)?
Justify your answer. **2**

(b) A train is travelling at a constant velocity of 80 kilometres per hour as it passes through the railway station at a town. At the same time, a second train commences its journey from rest at the railway station. The second train accelerates uniformly for 15 minutes until it reaches 100 kilometres per hour and maintains this velocity for a further 5 minutes.

At this time each of the trains then begins to slow down at a constant rate, arriving at the next station at the same time.

(i) By illustrating graphically the relationship between velocity and time, calculate the time taken for the trains to travel between the two stations. **3**

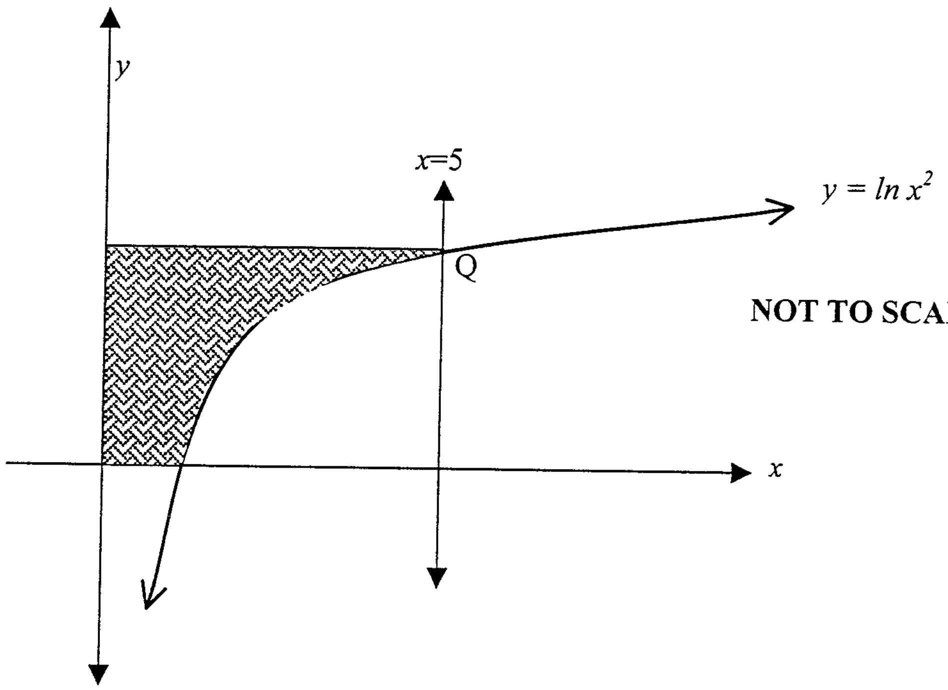
(ii) How far apart are the stations? **2**

Question 10 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Show that $\frac{d}{dx}(x\ln x - x) = \ln x$. **1**

(ii) Hence, or otherwise, find $\int \ln x^2 \, dx$. **1**

(iii) The graph shows the curve $y = \ln x^2$, ($x > 0$) which meets the line $x = 5$ at Q. Using your answers from (i) and (ii), or otherwise, find the area of the shaded region. **3**



NOT TO SCALE

(b) Consider the function $f(x) = e^{-x}\cos x$ for $0 \le x \le 2\pi$.

(i) Find the x values where the stationary points occur. **2**

(ii) Determine the nature of the stationary points. **2**

(iii) Sketch the curve showing the coordinates of the stationary points in exact form and the intercepts with the axes. **2**

(iv) Find the number of solutions to the equation $e^{-x}\cos x - \frac{1}{2}x = 0$.

In the domain $0 \le x \le 2\pi$. Justify your answer. **1**

End of paper

SUGGESTED SOLUTIONS TO MATHEMATICS CSSA TRIAL 2003

Question 1

(a) $\dfrac{2.1^2 \times 4.5^2}{2.1^2 + 4.5^2} = \dfrac{89.3}{24.7} = 3.6$

(b) $128x - 16x^4 = 16x(8 - x^3)$

$= 16x(2 - x)(4 + 2x + x^2)$

(c) $|2x+1| \le 5$

$2x + 1 \le 5$	$-2x - 1 \le 5$
$2x \le 5 - 1$	$-2x \le 5 + 1$
$2x \le 4$	$-2x \le 6$
$x \le 2$	$x \ge -3$

(d) $\dfrac{\sqrt{5}}{3\sqrt{2} - 1} = \dfrac{\sqrt{5}}{3\sqrt{2} - 1} \times \dfrac{3\sqrt{2} + 1}{3\sqrt{2} + 1}$

$= \dfrac{3\sqrt{10} + \sqrt{5}}{18 - 1}$

$= \dfrac{3\sqrt{10} + \sqrt{5}}{17}$

(e) $\tan\dfrac{\pi}{3} + \operatorname{cosec}\dfrac{\pi}{4}$

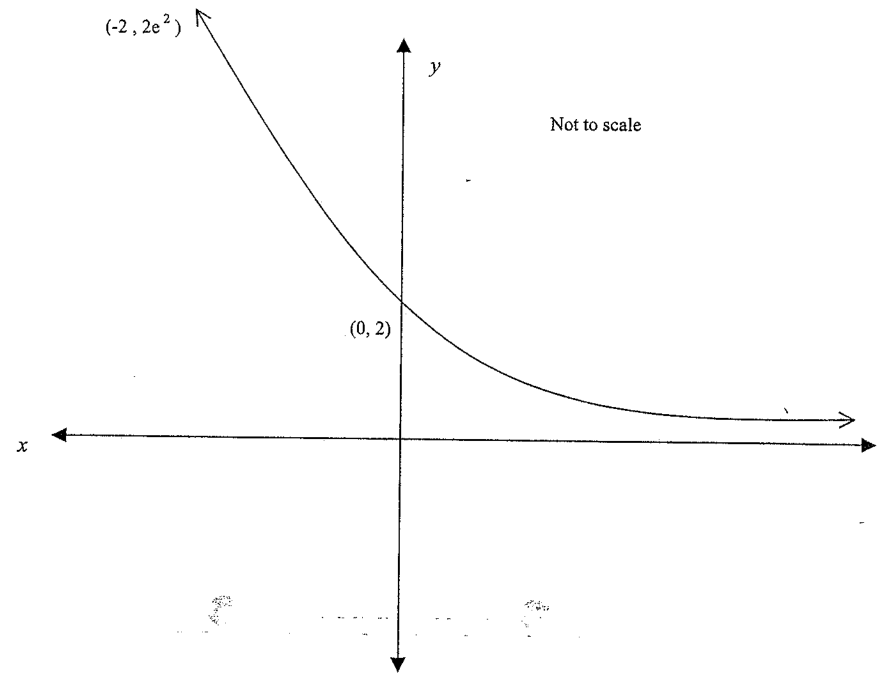
Using the exact triangles $\tan\dfrac{\pi}{3} = \sqrt{3}$

$\operatorname{cosec}\dfrac{\pi}{4} = \dfrac{1}{\sin\frac{\pi}{4}} = \dfrac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$

$\tan\dfrac{\pi}{3} + \operatorname{cosec}\dfrac{\pi}{4} = \sqrt{3} + \sqrt{2}$

Question 1 (continued)

(f)



Notice the curve passes through the y-axis at the point (0, 2)
As $x \to \infty$ $y \to 0$

Question 2

(a) A quadratic function has real roots when $b^2 - 4ac \geq 0$.

$x^2 - (k+2)x + 4 = 0$ $\quad a = 1;\ b = -k-2;\ c = 4$

$(-k-2)^2 - 4 \times 1 \times 4 \geq 0$

$k^2 + 4k + 4 - 16 \geq 0$

$k^2 + 4k - 12 \geq 0$

$(k+6)(k-2) \geq 0$ $\quad\therefore$ the quadratic has real roots when

$k \leq -6$ and $k \geq 2$

(b) (i) Point A is where L_1 intersects the y axis A (0, 2)
Point C is where L_2 intersects the y axis C (0, -4)

(ii) Can solve equations L_1 and L_2 simultaneously or show that the point R(3, -1) satisfies L_1 and L_2 by direct substitution.

Solving simultaneously

L_1 $\quad x + y = 2$

L_2 $\quad x - y = 4$

Adding L_1 and L_2

$2x = 6$

$x = 3$ $\quad$ Substituting $x = 3$ into L_1 or L_2 results in $y = -1$ $\therefore$ R(3, -1)

(iii) Line SR is parallel to the y axis and passes through the point R(3, -1)
$\therefore$Line SR has equation $x = 3$

(iv) Line L_1 has equation $x + y = 2$ $\quad m = = \dfrac{-a}{b} = -1$

Using the gradient formula with two points or even $m = \dfrac{rise}{run}$ with the diagram will also generate the answer to the gradient as -1.

(v) Using $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ with A(0, 2), R(3, -1)

$d = \sqrt{(3-0)^2 + (-1-2)^2}$

$d = \sqrt{9+9}$

$d = \sqrt{18} = 3\sqrt{2}$ units

Question 2 (continued)

(vi) From above (iv) the gradient of L_1 =-1

Line L_2 has equation $x - y = 4$. Using $m = \dfrac{-a}{b}$ or other methods it can be seen that the gradient of line L_2 is 1.

As $m_{L1} \times m_{L2} = -1$ $\quad \Delta$ARC is a right angled triangle.

We know from above (v) the distance of AR = $3\sqrt{2}$ units.
Finding the distance of CR, using the distance formula
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ with C(0, -4) and R(3, -1)

$d = \sqrt{(3-0)^2 + (-1+4)^2}$

$d = \sqrt{9+9}$

$d = \sqrt{18} = 3\sqrt{2}$ units.

The distance of AC along the y axis is 6 units.
As two sides of ΔARC are equal the triangle is isosceles.
$\therefore \Delta$ARC is an isosceles, right angled triangle.

(vii) Centre (3, -1) and radius $3\sqrt{2}$ units.

The equation of the circle is $(x-3)^2 + (y+1)^2 = (3\sqrt{2})^2$

$(x-3)^2 + (y+1)^2 = 18$ Or $x^2 - 6x + y^2 + 2y - 8 = 0$

Question 3

(a) (i) $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right)$

$= \frac{1}{2}x^{\frac{-1}{2}}$

$= \frac{1}{2\sqrt{x}}$

(ii) $\frac{d}{dx}\left(x^3 e^{-3x}\right) = \left(x^3\right)\left(-3e^{-3x}\right) + \left(3x^2\right)\left(e^{-3x}\right)$

$= 3x^2 e^{-3x}(1-x)$

(iii) $\frac{d}{dx}\left(\frac{\tan x}{2x+1}\right) = \frac{(2x+1)(\sec^2 x) - (\tan x)(2)}{(2x+1)^2}$

$= \frac{2x\sec^2 x + \sec^2 x - 2\tan x}{(2x+1)^2}$

(b) $\int \frac{e^{2x}}{e^{2x}+4}\, dx = \frac{1}{2}\ln\left(e^{2x}+4\right) + C$

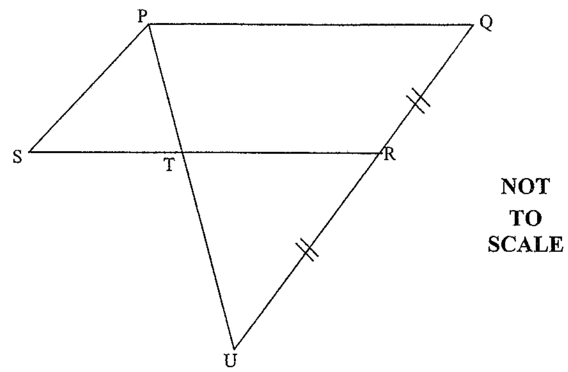
(c) $\int_0^{\frac{\pi}{4}} \frac{1}{2}x + \cos 2x \; dx = \left[\frac{x^2}{4} + \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}}$

$= \left(\frac{\frac{\pi^2}{16}}{4} + \frac{1}{2}\right) - (0)$

$= \frac{\pi^2 + 32}{64}$

Question 3 (continued)

(d)



(i) In triangles PST and URT:

PS = RU (RU = QR (given) and PS = QR, opposite sides of parallelogram PQRS.)

∠PST = ∠URT(Alternate angles are equal PS || QU)

∠PTS = ∠RTU (Vertically opposite angles are equal)

∴ ΔPTS ≡ ΔURT (two angles and one side)

(ii) Since ΔPST ≡ ΔURT, ST = TR because corresponding sides in congruent triangles are equal.

∴ T is the midpoint of SR.

Question 4

(a) $\sum_{k=4}^{20} 2k-5 = 3+5+7+\ldots\ldots\ldots+35$

This represents an Arithmetic series with $a = 3$, $l = 35$ and $n = 17$.

Using $S_n = \frac{n}{2}(a+l)$ $\quad S_{17} = \frac{17}{2}(3+35)$

$= 323$

(b) In a Geometric series, $T_n = ar^{n-1}$ so :

$ar^2 = \frac{3}{4}$

$ar^6 = 12$

Solving simultaneously gives : $\frac{ar^6}{ar^2} = \frac{12}{\frac{3}{4}}$

$\therefore r^4 = 16$ $\quad$ so, $\quad r = \pm 2$

For both r = 2 and r = -2, $a = \frac{3}{16}$

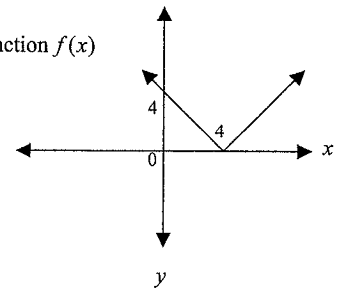
The fourteenth term of the series, $\quad T_{14} = ar^{13}$

$= \frac{3}{16}(\pm 2)^{13}$

$= \pm 1536$

(c) (i)

The required function $f(x)$



(ii) $\int_0^6 f(x)\ dx$ = Area under the curve between $x = 0$ and $x = 6$

$= (\frac{1}{2}\times 4\times 4) + \left(\frac{1}{2}\times 2\times 2\right) = 10$

(d) $\sqrt[x]{m} = n^3$

$\therefore m = n^{3x}$

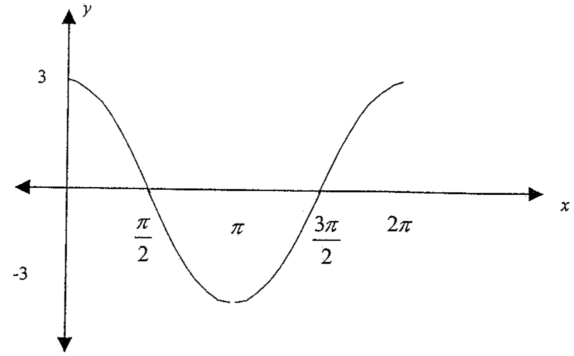
$\log m = \log n^{3x}$

$\log m = 3x \log n$

$x = \frac{\log m}{3\log n}$

Question 4 (continued)

(e)



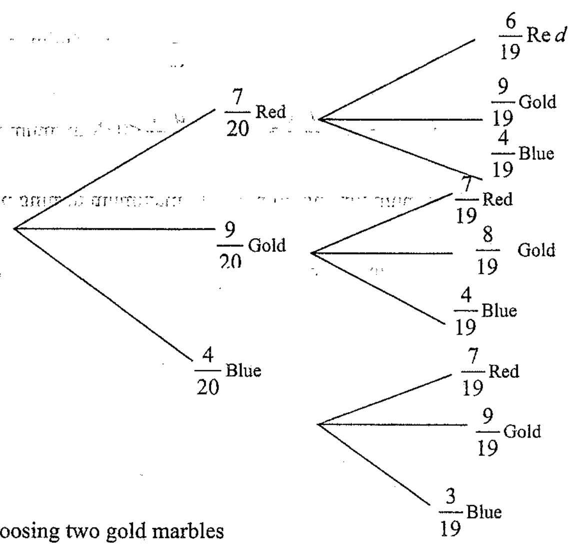
$y = 3\cos x$

(f) $\frac{dP}{dt} > 0$ because the function is increasing

$\frac{d^2P}{dt^2} < 0$ because the function is increasing at a *decreasing* rate

Question 5

(a)



(i) Probability of choosing two gold marbles

$P(\text{Gold, Gold}) = \frac{9}{20} \times \frac{8}{19} = \frac{18}{95}$

(ii) Probability of choosing marbles of different colour

P(marbles with different colour) = 1 – P(Same colour)

$= 1 - \left[\left(\frac{7}{20}\times\frac{6}{19}\right)+\left(\frac{9}{20}\times\frac{8}{19}\right)+\left(\frac{4}{20}\times\frac{3}{19}\right)\right]$

$= \frac{127}{190}$

(b) $y = 6x^2 - x^3$

(i) Stationary points occur when $\frac{dy}{dx} = 0$

$\frac{dy}{dx} = 12x - 3x^2 = 0$

$3x(4 - x) = 0$

Stationary points occur when $x = 0$ and $x = 4$

Stationary points are (0, 0) and (4, 32)

Question 5 (continued)

(ii) To determine nature of the stationary points we can use $\frac{d^2y}{dx^2} = 12 - 6x$

When $x = 0$ $\frac{d^2y}{dx^2} = 12$ $\frac{d^2y}{dx^2} > 0$ (Minimum turning point at $x = 0$)

When $x = 4$ $\frac{d^2y}{dx^2} = -12$ $\frac{d^2y}{dx^2} < 0$ (Maximum turning point at $x = 4$)

Minimum turning point (0, 0), maximum turning point (4, 32)

(iii) Point of inflexion when $\frac{d^2y}{dx^2} = 0$ and concavity changes.

$\frac{d^2y}{dx^2} = 12 - 6x = 0$

$6x = 12$

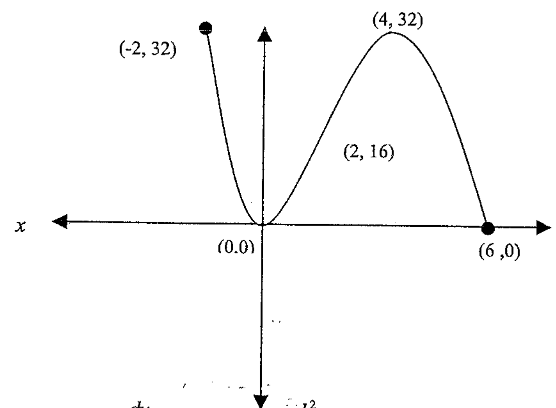
$x = 2$

Change in concavity and point of inflexion when $x = 2$

Point of inflexion is (2 , 16)

x	<2	2	>2
$\frac{d^2y}{dx^2}$	+	0	-

(iv)



(c) If $y = 3e^{-2x}$ then $\frac{dy}{dx} = -6e^{-2x}$ and $\frac{d^2y}{dx^2} = 12e^{-2x}$

So, $2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 2y = 2(12e^{-2x}) + 3(-6e^{-2x}) - 2(3e^{-2x})$

$= 24e^{-2x} - 18e^{-2x} - 6e^{-2x}$

$= 0$ as required.

Question 6

(a) $y = x \sin x$ $\qquad \frac{dy}{dx} = x \cos x + \sin x$

When $x = \frac{\pi}{2}$ $\qquad m_T = 1 \qquad \therefore m_N = -1$

$y = \frac{\pi}{2}$

$\therefore$ The equation of the normal at $(\frac{\pi}{2}, \frac{\pi}{2})$ is: $y - \frac{\pi}{2} = -1\,(x - \frac{\pi}{2})$

$$y - \frac{\pi}{2} = -x + \frac{\pi}{2}$$

$$x + y - \pi = 0$$

(b) Using Simpson's rule with h = 1

$$\int_0^4 f(x)\ dx \approx \frac{h}{3}[f(0) + 4\{f(1) + f(3)\} + 2\{f(2)\} + f(4)]$$

$$\approx \frac{1}{3}[2 + 4\{3 + 35\} + 2\{12\} + 80]$$

$$\approx 86$$

(c) (i) Solving simultaneously to find the points of intersection between $y = x^2$ and $y = 3x + 4$.

$$x^2 = 3x + 4$$
$$x^2 - 3x - 4 = 0$$
$$(x+1)(x-4) = 0$$
$$x = -1,\ x = 4$$

$\therefore$ At A, x = -1 and at B, x = 4

(ii) Area = $\int_{-1}^{4} 3x + 4\ dx - \int_{-1}^{4} x^2\ dx$

$$= \left[\frac{3x^2}{2} + 4x - \frac{x^3}{3}\right]_{-1}^{4}$$

$$= \left(\frac{3\times 4^2}{2} + 4\times 4 - \frac{4^3}{3}\right) - \left(\frac{3\times -1^2}{2} + 4\times -1 - \frac{-1^3}{3}\right)$$

$$= 18\frac{2}{3} + 2\frac{1}{6}$$

$$= 20\frac{5}{6}\ \text{units}^2 \qquad (20.8\dot{3}\ \text{units}^2)$$

Question 6 (continued)

(d) Volume = V = $\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} y^2\ dx \quad = \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cot x\ \ dx$

$$= \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos x}{\sin x}\ dx$$

$$= \pi[\ln(\sin x)]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \pi\left[\ln(\sin\frac{\pi}{3}) - \ln(\sin\frac{\pi}{4})\right]$$

$$= \pi\left[\ln(\frac{\sqrt{3}}{2}) - \ln(\frac{1}{\sqrt{2}})\right]$$

$$= \pi\left[\ln\frac{\sqrt{6}}{2}\right]\ \text{units}^3.$$

Question 7

(a) $\log_e\left(\dfrac{2x+1}{3x-7}\right) = \log_e(2x+1) - \log_e(3x-7)$

$\dfrac{dy}{dx} = \dfrac{2}{2x+1} - \dfrac{3}{3x-7}$

(b) (i) $x = 3t - 2\ln(1+t) + c$ *where c is a constant*

Since the particle is initially 1 metre to the right of the origin, when $t = 0$, $x = 1$

$\therefore \quad 1 = 3t - 2\ln(1+t) + c$

$\therefore \quad c = 1$

$\therefore \quad x = 3t - 2\ln(1+t) + 1$

(ii) Since $\dfrac{2}{1+t}$ can never be 0, v will never be 3.

(iii) $v = 3 - 2(1+t)^{-1}$ $\quad a = \dfrac{dv}{dt}$

$a = 2(1+t)^{-2}$

$a = \dfrac{2}{(1+t)^2}$

When $t = 2$ seconds, $a = \dfrac{2}{(1+2)^2} = \dfrac{2}{9}$ m/s^2

(c) (i) LHS $= (\text{cosec}^2 A - 1)\sin^2 A$

$= \left(\dfrac{1}{\sin^2 A} - 1\right)\sin^2 A$

$= 1 - \sin^2 A$

$= \cos^2 A$

$=$ RHS

(ii) $(\text{cosec}^2 A - 1)\sin^2 A = \frac{3}{4}$

$\cos^2 A = \frac{3}{4}$

$\cos A = \pm\dfrac{\sqrt{3}}{2} \qquad -\pi \le A \le \pi$

$A = \pm\dfrac{\pi}{6}, \pm\dfrac{5\pi}{6}.$

Question 8

(a) (i) 6 % p.a = 1.5 % per quarter.

After 1 quarter $\quad A_1 = 480000\left(1 + \dfrac{1.5}{100}\right)^1 - \$P.$

$= \$487\,200 - \$P$

(ii) $A_1 = \$487\,200 - \$P$

$A_2 = A_1 \times \left(1 + \dfrac{1.5}{100}\right) - \$P.$

$A_2 = \$480000(1.015)^2 - \$P(1 + 1.015)$

$A_3 = A_2\left(1 + \dfrac{1.5}{100}\right) - \$P.$

$A_3 = \$480000(1.015)^3 - \$P(1 + 1.015 + 1.015^2)$

(iii) 20 years = 80 repayments.

Pattern continues.....

$A_{80} = \$480000(1.015)^{80} - \$P(1 + 1.015 + 1.015^2 + 1.015^3 + ...1.015^{79})$

$A_{80} = 0$ (Loan repaid)

$\$480000(1.015)^{80} - \$P(1 + 1.015 + 1.015^2 + 1.015^3 ....+ 1.015^{79}) = 0$

$\$480000(1.015)^{80} = \$P(1 + 1.015 + 1.015^2 + 1.015^3 ....+ 1.015^{79})$

$\$P = \dfrac{\$480000(1.015)^{80}}{(1 + 1.015 + 1.015^2 + 1.015^3 + ...1.015^{79})}$

The denominator is the sum of a geometric series where

a = 1 $\quad$ r = 1.015 $\quad$ n = 80 $\therefore$ $S = \dfrac{1(1.015^{80} - 1)}{1.015 - 1}$

$\$P = \dfrac{\$480000(1.015)^{80}}{\dfrac{1(1.015^{80} - 1)}{1.015 - 1}}$

$\$P = \$\,10\,343.20$ (nearest cent)

(b) (i) Since the volume is changing at a rate proportional to the present volume, $\dfrac{dV}{dt} = kV$

and $V = V_0 e^{-kt}$ can be used.

Since initial volume is 1 000 L, $V_0 = 1\,000$.

$\therefore V = 1\,000\, e^{-kt}$

When $t = 40$ minutes, $V = 800$ L, so :

$800 = 1\,000\, e^{-40k}$

$0.8 = e^{-40k}$

$\ln 0.8 = \ln e^{-40k}$

$k = \dfrac{\ln 0.8}{-40} = 5.5786 \times 10^{-3}$

When $t = 60$, $\quad V = 1\,000\, e^{-60k}$

$(k = 5.5786 \times 10^{-3})$

$V = 715.54175$.....litres

$V = 716$ L (to the nearest litre)

Question 8 (continued)

(ii) When $V = 1$ then $1\,000e^{-kt} = 1$ $(k = 5.5786 \times 10^{-3})$

$e^{-kt} = 0.001$

$\ln e^{-kt} = \ln 0.001$

$kt = \ln 0.001$

$t = 1238.2621\ldots$ minutes

The storage tank will reach the last litre after 20 hours and 38 minutes.

(c) (i) A limiting sum exists as $|r| < 1$

$r = \sin^2 x \qquad 0 < x < \frac{\pi}{2}$

Note $|\sin^2 x| < 1$ does hold and a limiting sum exists.

(ii) Using $S = \frac{a}{1-r}$ where $|r| < 1 \qquad r = \sin^2 x\,,\ a = \sin^2 x$

$\therefore \quad S = \frac{\sin^2 x}{1 - \sin^2 x}$

$\therefore \quad S = \frac{\sin^2 x}{\cos^2 x}$

$\therefore \quad S = \tan^2 x$

Question 9

(a) (i) The area of a sector is given by $Area = A = \frac{1}{2}r^2\theta$

The perimeter of the sector is given by $r + r + r\theta$ (Where length of arc $= r\theta$)
The perimeter is given to be 375 metres.

$\therefore 2r + r\theta = 375$

$\theta = \frac{375 - 2r}{r}$

Substituting $\theta = \frac{375-2r}{r}$ into $Area = A = \frac{1}{2}r^2\theta$

gives $A = \frac{1}{2}r^2(\frac{375-2r}{r})$

$A = \frac{r}{2}(375 - 2r)$

(ii) Greatest Area occurs when $\frac{dA}{dr} = 0$ and $\frac{d^2A}{dr^2} < 0$

$\frac{dA}{dr} = \frac{375}{2} - 2r = 0 \qquad r = 93.75$ metres

$\frac{d^2A}{dr^2} = -2$

As $\frac{d^2A}{dr^2} < 0$ $\therefore$ maximum area occurs when $r = 93.75$ metres.

Maximum area is $A = \frac{93.75}{2}(375 - 2 \times 93.75)$

$= 8789.06\ m^2$ (2 decimal places)

(iii) The maximum area is $8789.06\ m^2$

Using $Area = \frac{1}{2}r^2\theta$

$8789.06 = \frac{1}{2} \times 93.75^2 \times \theta$

$\theta =$ 2 radians.

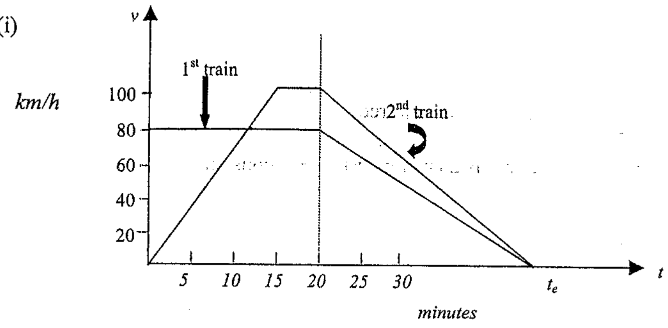
$\theta =$ 115° (nearest degree)

This is the angle required to produce max area from (ii).

The maximum area found in part (ii) created with a radius of 93.75 m would not be possible with an angle less than 110°. θ is required to be 2 radians. (115° to the nearest degree).

Question 9 (continued)

(b) (i)



Let t_e be the time at which the trains stop at the next station.
Since both trains cover the same distance, the areas between each velocity-time graph and the time axis are equal. We use this to find the time taken for the journey, t_e.

$$1\,600 + 40\,t_e - 800 = \tfrac{1}{2}(20 + 5) \times 100 + \tfrac{1}{2} \times 100(t_e - 20)$$
$$10\,t_e = 550$$
$$t_e = 55 \text{ minutes}$$

(ii) Converting time to hours because velocity is measured in km/h we can calculate the distance as the area under either velocity graph.

$$\text{Distance between the two stations} = 80 \times \frac{20}{60} + \tfrac{1}{2} \times 80 \times \left(\frac{55-20}{60}\right)$$

$$= 50 \text{ kilometres.}$$

The stations are 50 kilometres apart.

Question 10

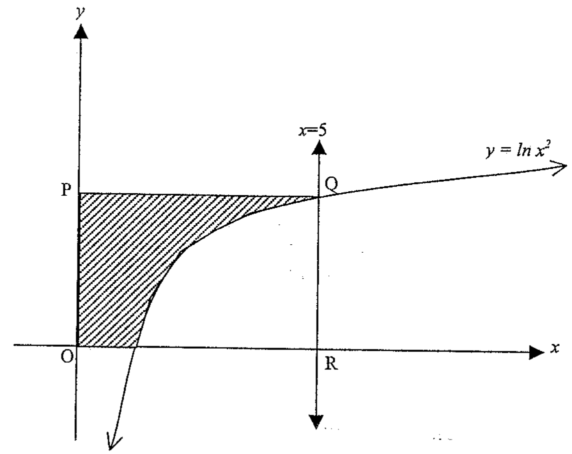
(a) (i) $\frac{d}{dx}(x\ln x - x) = x.\frac{1}{x} + 1.\ln x - 1$

$= 1 + \ln x - 1$

$= \ln x$ as required.

(ii) $\ln x^2 = 2\ln x$

$\therefore$ a primitive of $2\ln x$ is $2(x\ln x - x)$ [+ a constant]



(iii) The shaded area in the diagram is found by taking the area under the curve $y = \ln x^2$ between the line $x = 5$ and the x-axis from the area of the rectangle OPQR.

Area OPQR : P has co-ordinates (0, ln25)

$\therefore$ the rectangle has dimensions $5 \times \ln 25$

Area OPQR $= 5\ln 25$

$= 5\ln 5^2$

$= 10\ln 5$ units^2

The curve crosses the x-axis at (0, 1)

Area under the curve is found by evaluating $\int_1^5 \ln x^2\,dx$

$\int_1^5 \ln x^2\,dx = 2[x\ln x - x]_1^5$

$= 2\{5\ln 5 - 5 - (\ln 1 - 1)\}$

$= 10\ln 5 - 8$

Shaded area $= 10\ln 5 - (10\ln 5 - 8) = 8$ *units*^{2}

Question 10 (continued)

(b) (i) $f(x) = e^{-x}\cos x \quad 0 \le x \le 2\pi$

Stationary points occur when $f'(x) = 0$

$f'(x) = (e^{-x})(-\sin x) + (-e^{-x})(\cos x) = 0$

$= -e^{-x}(\sin x + \cos x) = 0$

Stationary points occur when $\sin x = -\cos x$

$\therefore$ when $\tan x = -1$

$x = \frac{3\pi}{4}, \frac{7\pi}{4}$

(ii) To determine nature of the stationary points we can use a before and after test with the first derivative.

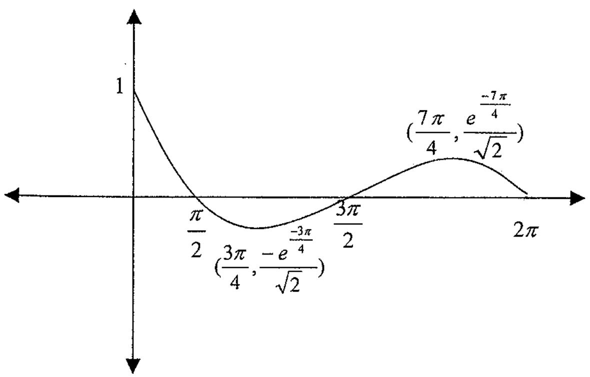
x	$\frac{3\pi}{4} - \varepsilon$	$\frac{3\pi}{4}$	$\frac{3\pi}{4} + \varepsilon$
$f'(x)$	-	0	+

Minimum turning point at $x = \frac{3\pi}{4}$

x	$\frac{7\pi}{4} - \varepsilon$	$\frac{7\pi}{4}$	$\frac{7\pi}{4} + \varepsilon$
$f'(x)$	+	0	-

Maximum turning point at $x = \frac{7\pi}{4}$

(iii)



Question 10 (continued)

(iv) The equation $e^{-x}\cos x - \frac{1}{2}x = 0$ can be solved graphically

Sketching $f(x) = e^{-x}\cos x$ and $f(x) = \frac{1}{2}x$

As $e^{-x}\cos x = \frac{1}{2}x$. The curve and the line intersect at only one point.

$\therefore$ One solution exists for the equation $e^{-x}\cos x - \frac{1}{2}x = 0 \qquad (0 \le x \le 2\pi)$